Metric spaces and homotopy types

J.F. Jardine*

Department of Mathematics
University of Western Ontario
London, Ontario, Canada

jardine@uwo.ca

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Abstract

By analogy with methods of Spivak, there is a realization functor which takes a persistence diagram Y in simplicial sets to an extended pseudo-metric space (or ep-metric space) $\mathrm{Re}(Y)$. The functor Re has a right adjoint, called the singular functor, which takes an ep-metric space Z to a persistence diagram $S(Z)$. We give an explicit description of $\mathrm{Re}(Y)$, and show that it depends only on the 1-skeleton $\mathrm{sk}_1\, Y$ of Y. If X is a totally ordered ep-metric space, then there is an isomorphism $\mathrm{Re}(V_*(X)) \cong X$, between the realization of the Vietoris-Rips diagram $V_*(X)$ and the ep-metric space X. The persistence diagrams $V_*(X)$ and $S(X)$ are section-wise equivalent for all such X.

Introduction

An extended pseudo-metric space, here called an ep-metric space, is a set X together with a function $d : X \times X \to [0, \infty]$ such that the following conditions hold:

1) $d(x, x) = 0$,

2) $d(x, y) = d(y, x)$,

3) $d(x, z) \leq d(x, y) + d(y, z)$.

There is no condition that $d(x, y) = 0$ implies x and y coincide — this is where the adjective "pseudo" comes from, and the gadget is "extended" because we are allowing an infinite distance.

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A metric space is an ep-metric space for which $d(x,y) = 0$ implies $x = y$, and all distances $d(x,y)$ are finite.

The traditional objects of study in topological data analysis are finite metric spaces X, and the most common analysis starts by creating a family of simplicial complexes $V_s(X)$, the Vietoris-Rips complexes for X, which are parameterized by a distance variable s.

To construct the complex $V_s(X)$, it is harmless at the outset is to list the elements of X, or give X a total ordering — one can always do this without damaging the homotopy type. Then $V_s(X)$ is a simplicial complex (and a simplicial set), with simplices given by strings

$$x_0 \leq x_1 \leq \cdots \leq x_n$$

of elements of X such that $d(x_i, x_j) \leq s$ for all i, j. If $s \leq t$ then there is an inclusion $V_sX \subset V_t(X)$, and varying the distance parameter s gives a diagram (functor) $V_*(X) : [0,\infty] \to s\mathbf{Set}$, taking values in simplicial sets.

Following Spivak [4] (sort of), one can take an arbitrary diagram $Y : [0,\infty] \to s\mathbf{Set}$, and produce an ep-metric space $\mathrm{Re}(Y)$, called its realization. This realization functor has a right adjoint S, called the singular functor, which takes an ep-metric space Z and produces a diagram $S(Z) : [0,\infty] \to s\mathbf{Set}$ in simplicial sets.

One needs good cocompleteness properties to construct the realization functor Re. Ordinary metric spaces are not well behaved in this regard, but it is shown in the first section (Lemma 3) that the category of ep-metric spaces has all of the colimits one could want. Then $\mathrm{Re}(Y)$ can be constructed as a colimit of finite metric spaces U^n_s, one for each simplex $\Delta^n \to Y_s$ of some section of Y.

The metric space U^n_s is the set $\{0, 1, \dots, n\}$, equipped with a metric d, where $d(i,j) = s$ for $i \neq j$. A morphism $U^n_s \to Z$ of ep-metric spaces is a list $(x_0, x_1, \dots, x_n)$ of elements of Z such that $d(x_i, x_j) \leq s$ for all i, j. Such lists have nothing to with orderings on Z, and could have repeats.

With a bit of categorical homotopy theory, one shows (Proposition 7) that $\mathrm{Re}(Y)$ is the set of vertices of the simplicial set Y_∞ (evaluation of Y at ∞), equipped with a metric that is imposed by the proof of Lemma 3.

One wants to know about the homotopy properties of the counit map $\eta : Y \to S(\mathrm{Re}(Y))$, especially when Y is an old friend such as the Vietoris-Rips system $V_*(X)$. But $\mathrm{Re}(V_*(X))$ is the original metric space X (Example 13), the object $S(X)$ is the diagram $[0,\infty] \to s\mathbf{Set}$ with $(S(X)_t)_n = \hom(U^n_t, X)$, and the counit $\eta : V_t(X) \to S_t(X)$ in simplicial sets takes an n-simplex $\sigma : \Delta^n \to V_t(X)$ to the list $(\sigma(0), \sigma(1), \dots, \sigma(n))$ of its vertices.

We show in Section 3 (Theorem 16, the main result of this paper) that the map $\eta : V_t(X) \to S_t(X)$ is a weak equivalence for all distance parameter values t. The proof proceeds in two main steps, and involves technical results from the theory of simplicial approximation. The steps are the following:

1) We show (Lemma 14) that the map η induces a weak equivalence $\eta_* : BNV_t(X) \to BNS_t(X)$, where $\eta_* : NV_t(X) \to NS_t(X)$ is the induced comparison of posets of non-degenerate simplices. Here, $V_t(X)$ is a simplicial complex,

so that $BNV_t(X)$ is a copy of the subdivision $\mathrm{sd}(V_t(X))$, and is therefore weakly equivalent to $V_t(X)$.

2) There is a canonical map $\pi : \mathrm{sd}\, S_t(X) \to BNS_t(X)$, and the second step in the proof of Theorem 16 is to show (Lemma 15) that this map π is a weak equivalence.

It follows that the map η induces a weak equivalence $\mathrm{sd}(V_t(X)) \to \mathrm{sd}(S_t(X))$, and Theorem 16 is a consequence.

The fact that the space $S_t(X)$ is weakly equivalent to $V_t(X)$ for each t means that we have yet another system of spaces $S_*(X)$ that models persistent homotopy invariants for a data set X.

One should bear in mind, however, that $S_t(X)$ is an infinite complex. To see this, observe that if x_0 and x_1 are distinct points in X with $d(x_0, x_1) \leq t$, then all of the lists

$$(x_0, x_1, x_0, x_1, \ldots, x_0, x_1)$$

define non-degenerate simplices of $S_t(X)$.

Contents

1 ep-metric spaces

An *extended pseudo-metric space* [2] (or an *uber metric space* [4]) is a set Y, together with a function $d : Y \times Y \to [0, \infty]$, such that the following conditions hold:

a) $d(x, x) = 0$,

b) $d(x, y) = d(y, x)$,

c) $d(x, z) \leq d(x, y) + d(y, z)$.

Following [3], I use the term *ep-metric spaces* for these objects, which will be denoted by (Y, d) in cases where clarity is required for the metric.

Every metric space (X, d) is an ep-metric space, by composing the distance function $d : X \times X \to [0, \infty)$ with the inclusion $[0, \infty) \subset [0, \infty]$.

A morphism between ep-metric spaces (X, d_X) and (Y, d_Y) is a function $f : X \to Y$ such that

$$d_Y(f(x), f(y)) \leq d_X(x, y).$$

These morphisms are sometimes said to be non-expanding [2].

I shall use the notation $ep - \mathbf{Met}$ to denote the category of ep-metric spaces and their morphisms.

Example 1 (Quotient ep-metric spaces)**.** Suppose that (X, d) is an ep-metric space and that $p : X \to Y$ is a surjective function.

For $x, y \in Y$, set

$$D(x, y) = \inf_P \sum_i d(x_i, y_i), \tag{1}$$

where each P consists of pairs of points (x_i, y_i) with $x = x_0$ and $y_k = y$, such that $p(y_i) = p(x_{i+1})$.

Certainly $D(x, x) = 0$ and $D(x, y) = D(y, x)$. One thinks of each P in the definition of $D(x, y)$ as a "polygonal path" from x to y. Polygonal paths concatenate, so that $D(x, z) \leq D(x, y) + D(y, z)$, and D gives the set Y an ep-metric space structure. This is the *quotient* ep-metric space structure on Y.

If x, y are elements of X, the pair (x, y) is a polygonal path from x to y, so that $D(p(x), p(y)) \leq d(x, y)$. It follows that the function p defines a morphism $p : (X, d) \to (Y, D)$ of ep-metric spaces.

Example 2 (Dividing by zero)**.** Suppose that (X, d) is an ep-metric space. There is an equivalence relation on X, with $x \sim y$ if and only if $d(x, y) = 0$. Write $p : X \to X/\sim =: Y$ for the corresponding quotient map.

Given a polygonal path $P = \{(x_i, y_i)\}$ from x to y in X as above, $d(y_i, x_{i+1}) = 0$, so the sum corresponding to P in (1) can be rewritten as

$$d(x, y_0) + d(y_0, x_1) + d(x_1, y_1) + \cdots + d(x_k, y).$$

It follows that $d(x, y) \leq D(p(x), p(y))$, whereas $D(p(x), p(y)) \leq d(x, y)$ by construction.

Thus, if $D(p(x), p(y)) = 0$, then $d(x, y) = 0$ so that $p(x) = p(y)$.

Lemma 3. *The category $ep - \mathbf{Met}$ of ep-metric spaces is cocomplete.*

Proof. The empty set is the initial object for this category,

Suppose that $(X_i, d_i), i \in I$, is a list of ep-metric spaces. Form the set theoretic disjoint union $X = \sqcup_i X_i$, and define a function

$$d : X \times X \to [0, \infty]$$

by setting $d(x, y) = d_i(x, y)$ if x, y belong to the same summand X_i and $d(x, y) = \infty$ otherwise. Any collection of morphisms $f_i : X_i \to Y$ in $ep - \mathbf{Met}$ defines a unique function $f = (f_i) : X \to A$, and this function is a morphism of $ep - \mathbf{Met}$ since

$$d(f(x), f(y)) = d(f_i(x), f_j(y)) \leq \infty = d(x, y)$$

if $x \in X_i$ and $y \in X_j$ with $i \neq j$.

Suppose given a pair of morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} X$$

in $ep - \mathbf{Met}$, and form the set theoretic coequalizer $\pi : X \to C$. The function p is the canonical map onto a set of equivalence classes of X, which classes are defined by the relations $f(a) \sim g(a)$ for $a \in A$. We give C the quotient ep-metric space structure, as in Example 1.

Suppose that $\alpha : (X, d_X) \to (Z, d_Z)$ is an morphism of ep-metric spaces such that $\alpha \cdot f = \alpha \cdot g$. Write $\alpha_* : C \to Z$ for the unique function such that $\alpha_* \cdot p = \alpha$.

Suppose given a polygonal path $P = \{(x_i, y_i)\}$ from x to y in X. Then $\alpha(y_i) = \alpha(x_{i+1})$, so that

$$d_Y(\alpha(x), \alpha(y)) \leq \sum_i d_Y(\alpha(x_i), \alpha(y_i)) \leq \sum_i d_X(x_i, y_i).$$

This is true for every polygonal path from x to y in X, so that

$$d_Y(\alpha_* p(x), \alpha_* p(y)) \leq d_C(p(x), p(y)).$$

It follows that $\alpha_* : (C, d_C) \to (Z, d_Z)$ is a morphism of ep-metric spaces. □

Example 4 ("Bad" filtered colimit)**.** If one starts with a diagram of metric spaces, the colimit C that is produced by Lemma 3 is an ep-metric space, and it may be that $d(x, y) = 0$ in the coequalizer C for some elements x, y with $x \neq y$.

In particular, suppose that $X_s = \{(\frac{1}{s\sqrt{2}}, 0), (0, \frac{1}{s\sqrt{2}})\} \subset \mathbb{R}^2$ for $0 < s < \infty$. Write $p_s = (\frac{1}{s\sqrt{2}}, 0)$ and $q_s = (0, \frac{1}{s\sqrt{2}})$ in X_s. Then $d(p_s, q_s) = \frac{1}{s}$. For $s \leq t$ there is an ep-metric space map $X_s \to X_t$ which is defined by $p_s \mapsto p_t$ and $q_s \mapsto q_t$.

The filtered colimit $\varinjlim_s X_s$ has two distinct points, namely p_∞ and q_∞, and $d(p_\infty, q_\infty) \leq d(p_s, q_s) = \frac{1}{s}$ for all $s > 0$. It follows that $d(p_\infty, q_\infty) = 0$, whereas $p_\infty \neq q_\infty$.

Lemma 5. *Suppose that X is an ep-metric space. Then there is an isomorphism of ep-metric spaces*

$$\psi : \varinjlim_F F \xrightarrow{\cong} X,$$

where F varies over the finite subsets of X, with their induced ep-metric space structures.

Proof. The collection of finite subsets of X is filtered, and the set X is a filtered colimit of its finite subsets, so the function defining the ep-metric space map ψ is a bijection. Write d_∞ for the metric on the filtered colimit.

If $x, y \in X$ and $d(x, y) = s \leq \infty$ in X, then there is a finite subset F with $x, y \in F$ such that $d(x, y) = s$ in F. The list (x, y) is a polygonal path from x to y in F, so that $d_\infty(x, y) \leq d(x, y)$. It follows that $d(x, y) = d_\infty(x, y)$, and so ψ is an isomorphism. □

An ep-metric space (X, d) has an associated system of posets $P_*(X) : [0, \infty] \to s\mathbf{Set}$, where $P_s(X)$ is the collection of finite subsets F of X such that $d(x, y) \leq s$ for any two members x, y of X.

This construction defines a system of abstract simplicial complexes $V_*(X)$, which can be constructed entirely within simplicial sets when X has a total ordering. In that case, the n-simplices of the simplicial set $V_s(X)$ are the strings $x_0 \leq x_1 \leq \cdots \leq x_n$ such that $d(x_i, x_j) \leq s$. The diagram $V_*(X) : [0, \infty] \to s\mathbf{Set}$ is the Vietoris-Rips system. The spaces $V_s(X)$ are independent up to weak equivalence of the ordering on X, because there is a canonical weak equivalence (a "last vertex map") $\gamma : BP_s(X) \to V_s(X)$ of systems, while the spaces $BP_s(X)$ are defined independently from the ordering. In classical terms, the nerve $BP_s(X)$ of the poset $P_s(X)$ (non-degenerate simplices of the Vietoris-Rips complex $V_s(X)$) is the barycentric subdivision of $V_s(X)$.

Example 6 (Excision for path components)**.** Suppose that X and Y are finite subsets of an ep-metric space Z, with the induced ep-metric space structures. Consider the inclusions of finite ep-metric spaces

$$\begin{array}{ccc} X \cap Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup Y \end{array}$$

inside Z. Write $X \cup_m Y$ for the corresponding pushout in the category of ep-metric spaces. The unique map

$$X \cup_m Y \to X \cup Y$$

of ep-metric spaces is the identity on the underlying point set. Write d_m for the metric on $X \cup_m Y$. Then $d_m(x, y)$ is the minimum of sums

$$\sum d(x_i, x_{i+1}), \tag{2}$$

indexed over paths

$$P : \ x = x_0, x_1, \dots .x_n = y,$$

such that for each i the points x_i, x_{i+1} are either both in X or both in Y.

All sums in (2) are finite, and $d_m(x, y)$ is realized by a particular path P since X and Y are finite. Note that $d(x, y) \leq d_m(x, y)$, by construction, and that $d(x, y) = d_m(x, y)$ if x, y are both in either X or Y.

There are induced simplicial set maps

$$V_s(X) \cup V_s(Y) \to V_s(X \cup_m Y) \to V_s(X \cup Y),$$

all of which are the identity on vertices. There is a 1-simplex $\sigma = \{x, y\}$ of $V_s(X \cup_m Y)$ if and only if there is a path

$$P : \ x = x_0, x_1, \dots, x_n = y$$

consisting of 1-simplices in either X or Y, such that

$$\sum d(x_i, x_{i+1}) \leq s.$$

Then all $d(x_i, x_{i+1}) \leq s$, so that x and y are in the same path component of $V_s(X) \cup V_s(Y)$. It follows that there is an induced isomorphism

$$\pi_0(V_s(X) \cup V_s(Y)) \cong \pi_0 V_s(X \cup_m Y). \tag{3}$$

The isomorphisms (3) induce isomorphisms

$$\pi_0(V_s(X) \cup V_s(Y)) \cong \pi_0 V_s(X \cup_m Y). \tag{4}$$

for arbitrary subsets X and Y of an ep-metric space Z, by an application of Lemma 5.

2 Metric space realizations

Write U^n_s for the collection of axis points $x_i = \frac{s}{\sqrt{2}} e_i$, where

$$e_i = (0, \dots, \overset{i+1}{1}, \dots, 0) \in \mathbb{R}^{n+1}.$$

for $0 \leq i \leq n$. Observe that $d(x_i, x_j) = s$ in $\mathbb{R}^{n+1}$ for $i \neq j$. Another way of looking at it: U^n_s is the set $\mathbf{n} = \{0, 1, \dots, n\}$ with $d(i, j) = s$ for $i \neq j$.

An ep-metric space morphism $f : U^n_s \to Y$ consists of points $f(x_i)$, $0 \leq i \leq n$, such that $d_Y(f(x_i), f(x_j)) \leq s$ for all i, j.

Write $s\mathbf{Set}^{[0,\infty]}$ for the category of diagrams (functors) $X : [0, \infty] \to s\mathbf{Set}$ and their natural transformation, which take values in simplicial sets and are defined on the poset $[0, \infty]$. I usually write $s \mapsto X_s$ for such a diagram X. In particular, X_∞ is the value that the diagram X takes at the terminal object of $[0, \infty]$.

Suppose that K is a simplicial set. The representable diagram $L_s K$ satisfies the universal property

$$\hom(L_s K, X) \cong \hom(K, X_s).$$

One shows that

$$(L_s K)_t = \begin{cases} \emptyset & \text{if } t < s, \\ K & \text{if } t \geq s. \end{cases}$$

The set of maps $L_s \Delta^n \to X$ can be identified with the set of n-simplices of the simplicial set X_s.

A morphism $L_t \Delta^m \to L_s \Delta^n$ consists of a relation $s \leq t$ and a simplicial map $\theta : \Delta^m \to \Delta^n$. In the presence of such a morphism, the function $\theta : \mathbf{m} \to \mathbf{n}$ defines an ep-metric space morphism $U^m_t \to U^n_s$, since $s = d(\theta(i), \theta(j)) \leq d(i, j) = t$.

2.1 The realization functor

Suppose that $X : [0, \infty] \to s\mathbf{Set}$ is a diagram. The category $\mathbf{\Delta}/X$ of simplices of X has maps $L_s\Delta^n \to X$ as objects and commutative diagrams

$$\begin{array}{ccc} L_t\Delta^m & \xrightarrow{\theta} & L_s\Delta^n \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & X & \end{array}$$

as morphisms.

Equivalently, a simplex of X is a simplicial set map $\Delta^n \to X_s$, and a morphism of simplices is a diagram

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\theta} & \Delta^n \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X_t & \leftarrow & X_s \end{array} \tag{5}$$

Every simplex $\Delta^n \to X_s$ determines a simplex

$$\Delta^n \to X_s \to X_\infty,$$

and we have a functor $r : \mathbf{\Delta}/X \to \mathbf{\Delta}/X_\infty$, where $\mathbf{\Delta}/X_\infty$ is the simplex category of the simplicial set X_∞.

There is an inclusion $i : \mathbf{\Delta}/X_\infty \to \mathbf{\Delta}/X$, and the composite $r \cdot i$ is the identity. The maps

$$\begin{array}{ccc} \Delta^n & \xrightarrow{1} & \Delta^n \\ {\scriptstyle r(\sigma)}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X_\infty & \leftarrow & X_s \end{array} \tag{6}$$

define a natural transformation $h : i \cdot r \to 1$.

There is a functor

$$\mathbf{\Delta}/X \to s\mathbf{Set} \tag{7}$$

which takes a morphism (5) to the map $\theta : \Delta^m \to \Delta^n$.

The translation category E_X for the functor (7) is a simplicial category that has objects consisting of pairs (σ, x) where $\sigma : \Delta^n \to X_s$ and $x \in \Delta^n$ (of a fixed dimension). A morphism $(\tau, y) \to (\sigma, x)$ of E_X is a morphism $\theta : \tau \to \sigma$ as in (5) such that $\theta(y) = x$. The path component simplicial set $\pi_0 E_X$ of the category E_X is isomorphic to the colimit

$$\varinjlim_{L_s\Delta^n \to X} \Delta^n.$$

There is a correponding translation category E_{X_∞} for the functor which takes the simplex $\Delta^n \to X_\infty$ to the simplicial set Δ^n, and there is an induced functor $i_* : E_{X_\infty} \subset E_X$. The functor $r : \mathbf{\Delta}/X \to \mathbf{\Delta}/X_\infty$ induces a functor $r_* : E_X \to E_{X_\infty}$. The composite $r_* \cdot i_*$ is the identity on E_{X_∞}, and the map (6) defines a natural transformation $i_* \cdot r_* \to 1$ of functors $E_{X_\infty} \to E_{X_\infty}$.

The translation categories E_X and E_{X_∞} are therefore homotopy equivalent, and thus have isomorphic simplicial sets of path components. It follows that there are isomorphisms of simplicial sets

$$X_\infty \xleftarrow{\cong} \varinjlim_{\Delta^n \to X_\infty} \Delta^n \xrightarrow{\cong} \varinjlim_{L_s\Delta^n \to X} \Delta^n \tag{8}$$

Suppose that $X : [0,\infty] \to s\mathbf{Set}$ is a diagram, and set

$$\mathrm{Re}(X) = \varinjlim_{L_s\Delta^n \to X} U^n_s$$

in the category of ep-metric spaces.

It follows from the identifications of (8) that $\mathrm{Re}(X)$ is the set of vertices of X_∞, equipped with an ep-metric space structure.

If x and y are two such vertices, and are the boundary of a 1-simplex

$$\Delta^1 \to X_s \to X_\infty$$

then x and y are in the image of a map $U^1_s \to \mathrm{Re}(X)$, so that $d(x,y) \leq s$. If there is a sequence of 1-simplices $\omega_i : \Delta^1 \to X_{s_i}$ that define a polygonal path

$$P : x = x_0 \leftrightarrows x_1 \leftrightarrows \dots \leftrightarrows x_k = y$$

of 1-simplices in X_∞, then $d(x,y) \leq \sum_i s_i$ by definition. Formally, we set

$$d(x,y) = \inf_P \{\sum_i s_i\}. \tag{9}$$

provided such polygonal paths exist. Otherwise, we set $d(x,y) = \infty$.

The resulting metric d is the metric which is imposed on the set of vertices of X_∞ by the requirement that

$$\mathrm{Re}(X) = \varinjlim_{L_s\Delta^n \to X} U^n_s$$

in the category of ep-metric spaces — see Lemma 3. We have shown the following:

Proposition 7. *Suppose that $X : [0,\infty] \to s\mathbf{Set}$ is a diagram. Then the ep-metric space $\mathrm{Re}(X)$ has underlying set given by the set of vertices of X_∞, with metric defined within path components by (9). Elements x and y that are in distinct path components have $d(x,y) = \infty$.*

Example 8 (Realization of Vietoris-Rips systems)**.** Suppose that X is a finite ep-metric space, and that X is totally ordered.

The realization $\mathrm{Re}(V_*(X))$ has X as its underlying set, and $V_\infty(X) = \Delta^X$ is a finite simplex, which is connected, so that there is a finite polygonal path in X between any two points $x, y \in X$. We have a relation

$$d(x,y) \leq \sum_i \ s_i$$

in X for any polygonal path P which is defined by 1-simplices $\omega_i \in V_{s_i}(X)$. This means that $d(x,y)$ in X coincides with the distance between x and y in the metric space $\mathrm{Re}(X)$. It follows that the identity on the set X induces an isomorphism of ep-metric spaces

$$\phi : \mathrm{Re}(V_*(X)) \xrightarrow{\cong} X.$$

This map ϕ is an isomorphism of ep-metric spaces, by Lemma 5 and the previous paragraphs.

Example 9 (Degree Rips systems)**.** Continue with a finite totally ordered ep-metric space X as in Example 8, let k be a positive integer, and consider the degree Rips system $L_{*,k}(X)$. We choose k such that the system of complexes $L_{*,k}(X)$ is non-empty, ie. such that $k \leq |X|$. Then $L_{t,k}(X) = V_t(X)$ for t sufficiently large, and X is the underlying set of $\mathrm{Re}(L_{*,k}(X))$.

The maps

$$\mathrm{Re}(L_{*,k}(X)) \to \mathrm{Re}(V_*(X)) \to X$$

are isomorphisms of metric spaces, by a cofinality argument.

Here is a special case:

Lemma 10. *Suppose that K is a simplicial complex and that $s > 0$. Then $\mathrm{Re}(L_sK) = \mathrm{Re}(L_s\,\mathrm{sk}_1(K))$, and $\mathrm{Re}(L_sK)$ is the set of vertices K_0 with a metric d defined by*

$$d(x,y) = \begin{cases} \infty & \text{if } [x] \neq [y] \text{ in } \pi_0(K), \\ \min_P \ s \cdot k & \text{if } [x] = [y]. \end{cases}$$

where P varies through the polygonal paths

$$P : x = x_0 \leftrightarrows x_1 \leftrightarrows \cdots \leftrightarrows x_k = y$$

of 1-simplices between x and y.

Proof. Write $\mathrm{Re}(K) = \mathrm{Re}(L_sK)$. The simplicial set K is a colimit of its simplices, and so there is an isomorphism

$$\varinjlim_{\Delta^n \to K} \ L_s\Delta^n \xrightarrow{\cong} L_sK.$$

It follows that there is an isomorphism

$$\varinjlim_{\Delta^n \to K} \ U^n_s \xrightarrow{\cong} \mathrm{Re}(L_sK).$$

Suppose that $n \geq 2$. Then $\partial\Delta^n$ and Δ^n have the same vertices, and any two vertices x, y are on a common face $\Delta^{n-1} \subset \Delta^n$. It follows that $d(x, y) = s$ in $\mathrm{Re}(\partial\Delta^n)$ and $\mathrm{Re}(\Delta^n)$, and the induced map

$$\mathrm{Re}(\partial\Delta^n) \to \mathrm{Re}(\Delta^n)$$

is an isomorphism for $n \geq 2$.

The displayed metric d on the vertices of K defines a metric space $\mathrm{Re}(K)$, with maps $\sigma_* : U^n_s \to \mathrm{Re}(K)$ for all simplices $\sigma : \Delta^n \to K$, which maps are natural with respect to the simplicial structure of K.

Any family of metric space morphisms $f_\sigma : U^n_s \to Y$ determines a unique function $f : K_0 \to Y$. Also, $d(f(x), f(y)) \leq s$ if x, y are in a common simplex $\Delta^1 \to K$. If P is a polygonal path between x and y as above, then $d(f(x), f(y)) \leq k \cdot s$. This is true for all such polygonal paths, so $d(f(x), f(y)) \leq d(x, y)$.

If x and y are in distinct components of K, then $d(f(x), f(y)) \leq d(x, y) = \infty$. □

The following result says that the realization $\mathrm{Re}(X)$ of a diagram $X : [0, \infty] \to s\mathbf{Set}$ depends only on the associated diagram of graphs $\mathrm{sk}_1(X)$.

Lemma 11. *Suppose that $X : [0, \infty] \to s\mathbf{Set}$ is a diagram. Then the inclusion $\mathrm{sk}_1 X \subset X$ induces an isomorphism*

$$\mathrm{Re}(\mathrm{sk}_1 X) \xrightarrow{\cong} \mathrm{Re}(X).$$

Proof. The diagram of 1-skeleta $\mathrm{sk}_1 X$ is a colimit

$$\varinjlim_{L_s\Delta^n \to X} L_s \,\mathrm{sk}_1 \Delta^n,$$

since the functor sk_1 preserves colimits. There are commutative diagrams

$$\begin{array}{ccc} \mathrm{Re}(L_s \,\mathrm{sk}_1 \Delta^n) & \longrightarrow & \mathrm{Re}(\mathrm{sk}_1 X) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \mathrm{Re}(L_s\Delta^n) & \longrightarrow & \mathrm{Re}(X) \end{array}$$

that are natural in the simplices of X, and it follows that the induced map $\mathrm{Re}(\mathrm{sk}_1 X) \to \mathrm{Re}(X)$ is an isomorphism, as required. □

2.2 Partial realizations

Suppose again that $X : [0, \infty] \to s\mathbf{Set}$ is a diagram in simplicial sets. We construct partial realizations by writing

$$\mathrm{Re}(X)_s = \varinjlim_{L_t\Delta^n \to X,\ t \leq s} U^n_t.$$

This is the colimit of a functor taking values in ep-metric spaces, which is defined on the full subcategory $\mathbf{\Delta}/X_{\leq s}$ of $\mathbf{\Delta}/X$ having objects $L_t\Delta^n \to X$ with $t \leq s$. A map $L_t\Delta^n \to X$ can be identified with a simplex $\Delta^n \to X_t$, and the relation $t \leq s$ defines a simplex $\Delta^n \to X_t \to X_s$, so that we have a functor

$$r : \Delta/X_{\leq s} \to \mathbf{\Delta}/X_s,$$

along with an inclusion $i : \mathbf{\Delta}/X_s \subset \mathbf{\Delta}/X_{\leq s}$. The composite $r \cdot i$ is the identity, and the composite $i \cdot r$ is homotopic to the identity, just as before.

There is a functor $\mathbf{\Delta}/X_{\leq s} \to s\mathbf{Set}$ which takes a simplex $\Delta^n \to X_t$ to the simplicial set Δ^n. By manipulating path components of homotopy colimits, one finds isomorphisms

$$X_s \xleftarrow{\cong} \varinjlim_{\Delta^n \to X_s} \Delta^n \xrightarrow{\cong} \varinjlim_{L_t\Delta^n \to X,\ t \leq s} \Delta^n.$$

that are analogous to the isomorphisms of (9). It follows, as in Theorem 7, that the set underlying the metric space $\mathrm{Re}(X)_s$ is the set of vertices of the simplicial set X_s.

The metric d on $(X_s)_0$ is defined as before: $d(x, y) = \infty$ if x and y not in the same path component of X_s. Otherwise

$$d(x, y) = \inf_P \{\sum t_i\}, \tag{10}$$

indexed over all polygonal paths

$$P : x = x_0 \leftrightarrows x_1 \leftrightarrows \cdots \leftrightarrows x_k = y$$

that are defined by 1-simplices $\omega : \Delta^1 \to X_{t_i}$ with $t_i \leq s$.

We then have the following analogue of Proposition 7:

Proposition 12. *Suppose that $X : [0, \infty] \to s\mathbf{Set}$ is a functor. Then the ep-metric space $\mathrm{Re}(X)_s$ has underlying set given by the set of vertices of X_s, with metric defined within path components by (10). Elements x, y in distinct path components have $d(x, y) = \infty$.*

The map $X_s \to X_\infty$ defines a map $\mathrm{Re}(X)_s \to \mathrm{Re}(X)$ and $\mathrm{Re}(X)_\infty = \mathrm{Re}(X)$. There is an isomorphism of ep-metric spaces

$$\varinjlim_s \mathrm{Re}(X)_s \xrightarrow{\cong} \mathrm{Re}(X), \tag{11}$$

since the element ∞ is terminal in $[0, \infty]$ and $\mathrm{Re}(X)_\infty = \mathrm{Re}(X)$.

Example 13 (Partial metrics for Vietoris-Rips complexes)**.** Suppose that X is a finite totally ordered ep-metric space. Consider the associated functor $V_*(X) : [0, \infty] \to s\mathbf{Set}$.

The associated ep-metric space $\mathrm{Re}(X)_s$ has underlying set X. We have $d(x, y) = \infty$ if x, y are in distinct path components of $V_s(X)$. Otherwise

$$d(x, y) = \inf_P \{\sum\ d(x_i, x_{i+1})\},$$

indexed over all polygonal paths

$$P : x = x_0, x_1, \dots, x_n = y,$$

with $d(x_i, x_{i+1}) \leq s$.

If $d(x, y) = t \leq s$ in X then $d(x, y) = t$ in $\mathrm{Re}(X)_s$. Otherwise, the distance between x and y in the same path component of $\mathrm{Re}(X)_s$ is more interesting — it is achieved by a particular path P since X is finite, and $d(x, y)$ is a type of weighted path length.

We see in Example 8 that there is an isomorphism of ep-metric spaces $\phi : \mathrm{Re}(V_*(X)) \xrightarrow{\cong} X$. It follows that there is an ep-metric space map $\phi_s : \mathrm{Re}(X)_s \to X$ which is the identity on the underlying point set X, and compresses distances.

3 The singular functor

The right adjoint S of the realization functor Re is defined for an ep-metric space Y by

$$S(Y)_{s,n} = \hom(U^n_s, Y),$$

where $\hom(U^n_s, Y)$ is the collection of ep-metric space morphisms $U^n_s \to Y$. Equivalently, $S(Y)_{s,n}$ is the set of families of points $(x_0, x_1, \dots, x_n)$ in Y such that $d(x_i, x_j) \leq s$.

A simplex $(x_0, x_1, \dots, x_n)$ is alternatively a function $\mathbf{n} \to Y$ (a "bag of words"), with a distance restriction. There is no requirement that the elements x_i are distinct. This simplex is non-degenerate if and only if $x_i \neq x_{i+1}$ for $0 \leq i \leq n - 1$.

Suppose that an ep-metric space X is totally ordered, as in Example 8 above. Then, in view of the discussion of Example 8, the canonical map $\eta : V_*(X) \to S\,\mathrm{Re}(V_*(X))$ consists of functions $\eta : V_t(X) \to S_t(X)$ which send simplices $\sigma : x_0 \leq x_1 \leq \cdots \leq x_n$ with $d(x_i, x_j) \leq t$ to the list of points $(x_0, x_1, \dots, x_n)$.

If σ is non-degenerate, so that the vertices x_i are distinct, then $\eta(\sigma)$ is a non-degenerate simplex of $S_t(X)$.

The poset NZ of non-degenerate simplices of a simplicial set Z has $\sigma \leq \tau$ if there is a subcomplex inclusion $\langle\sigma\rangle \subset \langle\tau\rangle$, where $\langle\sigma\rangle$ is the subcomplex of Z which is generated by the simplex σ. Equivalently, $\sigma \leq \tau$ if there is an ordinal number map θ such that $\theta^*(\tau) = \sigma$.

The map η induces a morphism $\eta_* : NV_t(X) \to NS_t(X)$ of posets of non-degenerate simplices.

Lemma 14. *Suppose that X is a totally ordered ep-metric space. Then the induced simplicial set map*

$$\eta_* : BNV_t(X) \to BNS_t(X)$$

of associated nerves is a weak equivalence.

Proof. Given a non-degenerate simplex $\sigma \in S_t(X)$, write $L(\sigma)$ for its list of distinct elements.

Suppose that $\langle \tau \rangle \subset \langle \sigma \rangle$, where τ and σ are non-degenerate simplices of $S_t(X)$. Then $\tau = s \cdot d(\sigma)$ for an (iterated) face map d and degeneracy s. Then

$$L(\tau) = L(s \cdot d(\sigma)) = L(d(\sigma)) \subset L(\sigma).$$

It follows that the assignment $\sigma \mapsto L(\sigma)$ defines a poset morphism

$$L : NS_t(X) \to NV_t(X).$$

The composite

$$NV_t(X) \xrightarrow{\eta} NS_t(X) \xrightarrow{L} NV_t(X)$$

is the identity on $NV_t(X)$.

Consider the composite poset morphism

$$NS_t(X) \xrightarrow{L} NV_t(X) \xrightarrow{\eta} NS_t(X). \tag{12}$$

Given a non-degenerate simplex $\tau = (y_0, \dots y_r)$ of $S_t(X)$, write $L(\tau) = (s_0, \dots s_k)$ for the list of distinct elements of τ, in the order specified by the total order for X. Then the list

$$V(\tau) = (y_0, \dots, y_r, s_0, \dots, s_k)$$

is a simplex of $S_t(X)$, since each s_j is some y_{i_j}, and there are relations

$$\langle \tau \rangle \leq \langle V(\tau) \rangle \geq \langle L(\tau) \rangle$$

as subcomplexes of $S_t(X)$.

The simplex $V(\tau)$ has the form $V(\tau) = s(V_*(\tau))$ for a unique iterated degeneracy s and a unique non-degenerate simplex $V_*(\tau)$ (see Lemma 18), and $\langle V(\tau) \rangle = \langle V_*(\tau) \rangle$.

Suppose that γ is non-degenerate in $S_t(X)$ and that $\gamma \in \langle \tau \rangle$. Then $\gamma = d(\tau)$ for some face map d, and $\gamma = (x_0, \dots, x_k)$ is a sublist of $\tau = (y_0, \dots, y_r)$. The ordered list $L(\gamma)$ of distinct elements of γ is a sublist of $L(\tau)$, and $V(\gamma)$ is a sublist of $V(\tau)$. There is a diagram of relations

$$\begin{array}{ccccc} \langle \tau \rangle & \longrightarrow & \langle V_*(\tau) \rangle & \longleftarrow & \langle L(\tau) \rangle \\ \uparrow & & \uparrow & & \uparrow \\ \langle \gamma \rangle & \longrightarrow & \langle V_*(\gamma) \rangle & \longleftarrow & \langle L(\gamma) \rangle \end{array}$$

It follows that the composite (12) is homotopic to the identity on the poset $NS_t(X)$, and the Lemma follows. $\square$

The subdivision $\mathrm{sd}(Z)$ of a simplicial set Z is defined by

$$\mathrm{sd}(Z) = \varinjlim_{\Delta^n \to Z} BN\Delta^n.$$

The poset morphisms $N\Delta^n \to NZ$ that are induced by simplices $\Delta^n \to Z$ together induce a map

$$\pi : \mathrm{sd}(Z) \to BNZ.$$

It is known [1] (and not difficult to prove) that the map π is a bijection for simplicial sets Z that are polyhedral.

A polyhedral simplicial set is a subobject of the nerve of a poset. All oriented simplicial complexes are polyhedral in this sense. Examples include the Vietoris-Rips systems $V_s(X)$ associated to a totally ordered ep-metric space X, since

$$V_s(X) \subset V_\infty(X) = BX,$$

where BX is the nerve of the totally ordered poset X.

Lemma 15. *Suppose that X is an ep-metric space. Then the map*

$$\pi : \mathrm{sd}(S_t(X)) \to BNS_t(X)$$

is a weak equivalence.

Proof. We show that all subcomplexes $\langle\sigma\rangle$ which are generated by non-degenerate simplices σ of $S_t(X)$ are contractible. Then Lemma 4.2 of [1] implies that the map π is a weak equivalence.

A non-degenerate simplex σ has the form $\sigma = (x_0, x_1, \dots, x_k)$ with $x_i \neq x_{i+1}$. The simplices τ of $\langle\sigma\rangle$ have the form

$$\tau = \theta^*\sigma = (x_{\theta(0)}, \dots, x_{\theta(k)}),$$

where $\theta : \mathbf{k} \to \mathbf{n}$ is an ordinal number morphism.

For each such θ, the list

$$(x_0, x_{\theta(0)}, \dots, x_{\theta(k)})$$

defines a simplex τ_* of $\langle\sigma\rangle$, since τ_* is a face of the simplex

$$s_0(\sigma) = (x_0, x_0, \dots, x_k).$$

The simplices τ_* define functors

$$\begin{array}{ccccccc}
x_0 & \longrightarrow & x_0 & \longrightarrow & \dots & \longrightarrow & x_0 \\
\downarrow & & \downarrow & & & & \downarrow \\
x_{\theta(0)} & \longrightarrow & x_{\theta(1)} & \longrightarrow & \dots & \longrightarrow & x_{\theta(m)}
\end{array}$$

or homotopies, that consist of simplices of $\langle\sigma\rangle$ that patch together to give a contracting homotopy $\langle\sigma\rangle \times \Delta^1 \to \langle\sigma\rangle$. □

Theorem 16. *Suppose that X is a totally ordered ep-metric space. Then there is a diagram of weak equivalences*

$$\begin{array}{ccccc} BNV_t(X) & \xleftarrow[\cong]{\pi} & \mathrm{sd}(V_t(X)) & \xrightarrow{\gamma} & V_t(X) \\ \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \eta} \\ BNS_t(X) & \xleftarrow[\pi]{} & \mathrm{sd}(S_t(X)) & \xrightarrow[\gamma]{} & S_t(X) \end{array}$$

In particular, the map $\eta : V_t(X) \to S_t(X)$ is a weak equivalence.

This diagram is natural in t.

Proof. The map $\eta_* : BNV_t(X) \to BNS_t(X)$ is a weak equivalence by Lemma 14. The map $\pi : \mathrm{sd}(S_t(X)) \to BNS_t(X)$ is a weak equivalence by Lemma 15. The instances of the maps γ are weak equivalences [1]. It follows that the maps $\eta_* : \mathrm{sd}(V_t(X)) \to \mathrm{sd}(S_t(X))$ and $\eta : V_t(X) \to S_t(X)$ are weak equivalences. □

Remark 17. The total ordering on the ep-metric space X in Theorem 16 is intimately involved in the definition of the Vietoris-Rips system $V_*(X)$, the morphism

$$\eta : V_t(X) \to S_t(\mathrm{Re}(V_*(X))) = S_t(X),$$

and all induced maps η_*.

The counit $\eta : Z \to S(\mathrm{Re}(Z))$ is not a sectionwise weak equivalence in general. One can show that $S_\infty(\mathrm{Re}(Z))$ is the nerve of the trivial groupoid on the vertex set of Z_∞ (Proposition 7), and is therefore contractible, whereas the space Z_∞ may not be contractible. For example, if K is a simplicial set, then there is an identification $K = (L_s K)_\infty$.

It may be that the map

$$\eta : BP_t(X) \to S_t(\mathrm{Re}(BP_*(X)))$$

is a weak equivalence for arbitrary ep-metric spaces X, but this has not been proved. Such a result would give a non-oriented version of Theorem 16.

The following is a classical result, which is included here for the sake of completeness. This result is usually neither expressed nor proved in the form displayed here.

Lemma 18. *Suppose that σ is an n-simplex of a simplicial set X. Then there is a unique iterated degeneracy and a non-degenerate simplex x such that $\sigma = s(x)$.*

An iterated degeneracy is a surjective ordinal number map $s : \mathbf{n} \to \mathbf{k}$. Such a map induces a function $s : X_k \to X_n$ for a simplicial set X. Lemma 18 says that $\sigma = s(x)$ for some iterated degeneracy s and a non-degenerate simplex x, and that this representation is unique.

Proof of Lemma 18. Suppose that $\sigma = s(x) = s'(x')$ where s, s' are iterated degeneracies and x, x' are non-degenerate.

The $x = d(\sigma)$ for some face map d such that $d \cdot s = 1$, and so $d(s'(x')) = s''(d''(x))$ for some iterated degeneracy s'' and face map d''. But x is non-degenerate, so that $s'' = 1$ and $x = d''(x')$. Similarly, $x' = \tilde{d}(x)$ for some face map $\tilde{d}$. But then x and x' have the same dimension, and $d = 1$, so that $x = x'$.

If $s \neq s'$ there is a face map d such that $d \cdot s = 1$ but $d \cdot s' \neq 1$. Then $\sigma = s(x) = s'(x)$ for $s \neq s'$ and x non-degenerate, then

$$d(\sigma) = x = d(s'(x)) = s''(d''(x))$$

for some degeneracy s'' and face map d'', at least one of which is non-trivial.

But x is non-degenerate, so that $s'' = 1$ and $x = d''(x)$ only if $d'' = 1$. This contradicts the assumption that $s \neq s'$. □

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