Layers and stability

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Abstract

The hierarchy associated to clusters in the HDBSCAN algorithm has layers, which are defined by cardinality. The layers define a subposet of the HDBSCAN hierarchy, which is a strong deformation retract and admits a stability analysis. That stability analysis is introduced here. Cardinality arguments lead to sharper stability results than one sees for branch points.

Introduction

Every finite metric space $X = (X, d)$ has an associated system of partially ordered sets $P_s(X)$, where s is a non-negative real number. This system is filtered by the systems $P_{s,k}(X)$ where k is a positive integer.

The poset $P_s(X)$ consists of those subsets σ of X such that $d(x, y) \leq s$ for all $x, y \in \sigma$.

The poset $P_{s,k}(X)$ consists of those subsets τ of X such that each $x \in \tau$ has at least k distinct neighbours $y \in X$ such that $d(x, y) \leq s$. We also require that $d(x, x') \leq s$ for any two members x, x' of τ.

The poset $P_s(X)$ is the poset of simplices for the Vietoris-Rips complex $V_s(X)$, and the poset $P_{s,k}(X)$ is the poset of simplices of the degree Rips comples $L_{s,k}(X)$.

Observe that $P_{s,0}(X) = P_s(X)$, so that the complex $L_{s,0}(X)$ is the Vietoris-Rips complex $V_s(X)$.

For a fixed density parameter k, the path component functor π_0 defines an assignment $s \mapsto \pi_0 L_{s,k}(X)$, giving a functor defined on the poset $[0, \infty]$ that

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takes values in sets. The sets of path components $\pi_0 L_{s,k}(X)$ are commonly called clusters.

This functor defines a graph $\Gamma_k(X)$, with vertices consisting of pairs $(s,[x])$ with $[x] \in \pi_0 L_{s,k}(X)$. There is an edge $(s,[x]) \to (t,[y])$ if $s \leq t$ and $[x]=[y]$ in $\pi_0 L_{t,k}(X)$. This graph is a hierarchy, or tree, which is commonly called the HDBSCAN hierarchy. It is also a poset because the edges can be composed, and this poset has a terminal object. I write $(s,[x]) \leq (t,[y])$ for edges (or morphisms) of $\Gamma_k(X)$ to reflect the poset structure.

A vertex $(s,[x])$ of $\Gamma_k(X)$ is a *branch point* if either $(s,[x])$ has no antecedents $(t,[y]) \leq (s,[x])$, or if $(s,[x])$ has distinct antecedents $(t,[y_1])$ and $(t,[y_2])$ for sufficiently close $t < s$.

A vertex $(t,[y])$ is a *layer point* if it has no antecedents, or if for all antecedents $(s,[z]) \leq (t,[y])$ with $s < t$, the set $[z]$ is strictly smaller than $[y]$ as a subset of X.

Every branch point is a layer point, but the converse assertion does not hold in general. Layer points and branch points do coincide for the Vietoris-Rips system $V_s(X) = L_{s,0}(X)$, since the underlying system of vertex sets is constant.

The branch points and layer points, respectively, define subposets $\mathrm{Br}_k(X)$ and $\mathrm{L}_k(X)$ of the tree $\Gamma_k(X)$, and there are poset inclusions

$$\mathrm{Br}_k(X) \subseteq \mathrm{L}_k(X) \subset \Gamma_k(X).$$

These subposets are themselves hierarchies.

The purpose of this note is to describe stability properties of the layer poset $\mathrm{L}_k(X)$. There is a similar investigation of stability properties of branch points in [2] — the posets $\mathrm{Br}_k(X)$ of branch points and $\mathrm{L}_k(X)$ of layer points have similar properties, but the cardinality counts associated with layer points are sharper tools.

The first section of this paper establishes the formal properties of the poset $L_k(X)$ of layer points. The most important feature of $L_k(X)$ is that it has a calculus of least upper bounds (Lemma 4), which mirrors the theory of least upper bounds for the branch point poset $\mathrm{Br}_k(X)$ that appears in [2]. The inclusion $\mathrm{Br}_k(X) \subset L_k(X)$ preserves least upper bounds. The poset morphisms

$$\mathrm{Br}_k(X) \subseteq L_k(X) \subset \Gamma_k(X)$$

define both $\mathrm{Br}_k(X)$ and $\mathrm{L}_k(X)$ as strong deformation retracts of the poset $\Gamma_k(X)$, in a way that is consistent with the inclusion $\mathrm{Br}_k(X) \subset L_k(X)$ — see Lemma 5 and Lemma 6. The retraction map $\max : \Gamma_k(X) \to L_k(X)$ is defined by setting $\max(t,[x])$ to be the maximal layer point below $(t,[x])$. The layer point $\max(t,[x])$ can also be defined to be the minimal point $(s,[z]) \leq (t,[x])$ such that $[z]=[x]$ as subsets of the set X.

If $i : X \subset Y$ is an inclusion of finite metric spaces, then there is an induced poset map $i_* : L_k(X) \to L_k(Y)$, where $i_*(t,[x])$ is defined to be the maximal layer point below $(t,[i(x)])$ in $\Gamma_k(Y)$.

The degree Rips stability theorem (Theorem 4 of [3]) says that there are homotopy commutative diagrams

$$\begin{array}{ccc} L_{s,k}(X) & \xrightarrow{\sigma} & L_{s+2r,k}(X) \\ {\scriptstyle i}\downarrow & \nearrow^{\theta} & \downarrow{\scriptstyle i} \\ L_{s,k}(Y) & \xrightarrow[\sigma]{} & L_{s+2r,k}(Y) \end{array} \tag{1}$$

in the presence of a condition $d_H(X^k_{dis}, Y^k_{dis}) < r$ on Hausdorff distance between spaces of $k+1$ distinct points in X and Y. Theorem 8 of Section 2 says that the diagram (1) induces a homotopy commutative diagram

$$\begin{array}{ccc} L_k(X) & \xrightarrow{\sigma_*} & L_k(X) \\ {\scriptstyle i_*}\downarrow & \nearrow^{\theta_*} & \downarrow{\scriptstyle i_*} \\ L_k(Y) & \xrightarrow[\sigma_*]{} & L_k(Y) \end{array} \tag{2}$$

The map i_* in (2) has already been defined, and all other maps in (2) are defined analogously. For example, the shift homomorphism σ_* is defined, for a layer point $(t, [x])$, by taking $\sigma_*(t, [x])$ to be the maximal layer point below $(t + 2r, [x])$. The homotopy commutativity of (2) amounts to the existence of natural relations

$$\theta_* \cdot i_* \leq \sigma_* \text{ and } i_* \cdot \theta_* \leq \sigma_*.$$

These relations have to be interpreted a bit carefully. If, for example, $(s.[x])$ is a layer point of $\Gamma_k(X)$, then $(t, [x])$ is a layer point below $(t+2r, [x])$ so that $(t, [x]) \leq \sigma_*(t, [x])$. This means that $\sigma_*(t, [x])$ is a common upper bound for $(t, [x])$ and $\theta_* i_*(t, [x])$, while $\sigma_*(t, [x])$ has the form $(u, [x])$ for some parameter value $t \leq u \leq t + 2r$.

I have not yet found a good way to estimate the corresponding parameter value of $\theta_* i_*(t, [x])$ without some extra assumptions. At this level of generality, we have the same issues with locating parameter values for the points $i_*(t, [x])$ and $\theta_*(s, [y])$, relative to t and s, respectively.

We can sharpen these relations if the layer points are sufficiently sparse. The *layer parameters* are the parameters t associated to the layer points $(t, [x])$ of $\Gamma_k(X)$. A layer parameter t can have a successor t_+ and a predecessor t_-. Lemma 14 of this paper says that, if $r < t < t_+ - 2r$, then $i_*(t, [x]) = (s, [y])$, where $t - 2r \leq s \leq t$. Under the same assumptions, Corollary 15 further says that $\theta_* i_*(t, [x]) = (t, [x])$.

Lemma 14 and Corollary 15 deal with layer points $(t, [x])$ of $\Gamma_k(X)$ which have enough room "above" them. If r is sufficiently small such that $r < t < t_+ - 2r$ for all layer parameters t of X, then $\theta_* i_*(t, [x]) = (t, [x])$ for all layer points $(t, [x])$ of X, so that $\Gamma_k(X)$ is a retract of $\Gamma_k(Y)$.

This can be achieved, for example, if $X \subset Z$ is an inclusion of metric spaces, where X is interpreted as a set of marked points, r is chosen sufficently small that $r < t < t_+ - 2r$ for all layer parameters t of X, and the points of $Y \subset Z$ are chosen such that $d_H(X^k_{dis}, Y^k_{dis}) < r$ in Z^k_{dis}.

The analysis simplifies for Vietoris-Rips complexes. In that case, X and Y are the vertex sets of $V_s(X)$ and $V_s(Y)$, respectively, for all s. Then Lemma 17 says that if $(s,[y])$ is a layer point of $\Gamma_0(Y)$ and $(t,[x])$ is a maximal layer point below $(s+2r,[\theta(y)])$, then $s \leq t \leq s+2r$. This means, for example, that every layer parameter s of $\Gamma_0(Y)$ satisfies $t-2r \leq s \leq t$ for some layer parameter t of $\Gamma_0(X)$. Lemma 14 and Lemma 17 together say that the layer parameters of $\Gamma_0(X)$ and $\Gamma_0(Y)$ for the respective Vietoris-Rips systems are very tightly bound, in a predictable way.

1 Layer points

Suppose that X is a finite metric space and that k is a positive integer. The functor $s \mapsto \pi_0 L_{s,k}(X)$ has a homotopy colimit $\Gamma_k(X)$ having objects $(s,[x])$ with $[x] \in \pi_0 L_{s,k}(X)$ and morphisms $(s,[x]) \to (t,[x])$ with $s \leq t$. Here, the distance parameters s are positive real numbers, and hence members of the interval $[0,\infty]$.

This category $\Gamma_k(X)$ is a partially ordered set, and has the structure of a tree, and one writes $(s,[x]) \leq (t,[y])$ for its morphisms. The spaces $L_{s,k}(X)$ are connected for s sufficiently large, say $s \geq R$, since X is a finite set.

I often write $[x]_s$ for $[x] \in \pi_0 L_{s,k}$. The path component $[x]_s$ is a subset of the vertices of $L_{s,k}(X)$. There is a relation $(s,[x]) \leq (t,[y])$ if and only if $s \leq t$ and $[x]_s \subset [y]_t$ as subsets of X.

A *branch point* in the tree $\Gamma_k(X)$ is a vertex $(t,[x])$ such that either of following two conditions hold:

1) there is an $s_0 < t$ such that for all $s_0 \leq s < t$ there are distinct vertices $(s,[x_0])$ and $(s,[x_1])$ with $(s,[x_0]) \leq (t,[x])$ and $(s,[x_1]) \leq (t,[x])$, or

2) there is no relation $(s,[y]) \leq (t,[x])$ with $s < t$.

The second condition means that the path component $[x]$ does not have a representative in $L_{s,k}(X)$ for $s < t$. Write $\mathrm{Br}_k(X)$ for the subposet of $\Gamma_k(X)$, which is defined by the branch points.

A *layer point* of $\Gamma_k(X)$ is a vertex $(t,[x])$ such that one of the following two conditions hold:

1) if there is a relation $(s,[y]) \leq (t,[x])$ with $s < t$, then $[y]_s$ is a proper subset of $[x]_t$, equivalently there is a proper inequality $|[y]_s| < |[x]_t|$ in cardinality, or

2) there is no relation $(s,[y]) \leq (t,[x])$ with $s < t$.

The layer points form a subposet $\mathrm{L}_k(X)$ of $\Gamma_k(X)$.

Remark 1. There is a maximal finite subsequence

$$0 \neq t_1 < \cdots < t_p$$

of positive real numbers t_j, which are the distances between vertices of

$$L_{k,t_p}(X) = L_{k,\infty}(X).$$

The numbers t_i are the *phase change* numbers for the system $L_{*,k}(X)$. Observe that the vertices of $L_{k,t_i}(X)$ and $L_{k,t_{i+1}}(X)$ could coincide.

We can find the layer points for $\Gamma_k(X)$ by induction on i, starting with the observation that all points $(t_1, [z])$ are layer points. If $[x] \in \pi_0 L_{t_i,k}(X)$, then $[x] \cap L_{t_{i-1},k}(X)_0$ is a disjoint union of path components $[y]$. This intersection could be empty, in which case $(t_i, [x])$ is a layer point. Otherwise, $(t_i, [x])$ is a layer point if all $[y] \subset [x] \cap L_{t_{i-1},k}(X)_0$ satisfy $|[y]| < |[x]|$.

Lemma 2. *All branch points are layer points, and so there are poset inclusions*

$$\mathrm{Br}_k(X) \subseteq \mathrm{L}_k(X) \subset \Gamma_k(X).$$

Proof. Suppose that condition 1) holds for the branch point $(t, [x])$: there is an $s_0 < t$ that for all $s_0 \leq s < t$ there are distinct points $(s, [x_0])$ and $(s, [x_1])$ such that $(s, [x_i]) \leq (t, [x])$.

If $(s, [z]) \leq (t, [x])$ then $[z]$ is one of multiple path components $[v]_s$ of $L_{s,k}(X)$ that map to $[x]_t$ in $L_{t,k}(X)$. All such components are proper subsets of $[x]_t$. □

Recall that $L_{0,s}(X)$ is the Vietoris-Rips complex $V_s(X)$, and that the elements of X are the vertices of the Vietoris-Rips complex $V_s(X)$. All complexes $V_s(X)$ have the same vertices, namely the set X.

Lemma 3. *Every layer point of* $\Gamma_0(X)$ *is a branch point, so that* $\mathrm{Br}_0(X) = L_0(X)$.

Proof. The underlying sets of vertices for $V_s(X)$ and $V_t(X)$ coincide. Thus, if $(t, [x])$ is a layer point of $\Gamma_0(X)$ and $s < t$, then the collection $[y]$ of components of $V_s(X)$ that map to $[x]$ in $V_t(X)$ is non-empty and satisfies $\sqcup\, [y]_s = [x]_t$. There are multiple such summands $[y]_s$, since $(t, [x])$ is a layer point, so that all inclusions $[y]_s \subset [x]_t$ are proper. In particular, there are distinct elements $(s, [y])$ and $(s, [y'])$ below $(t, [x])$. □

Suppose that $(s, [x])$ and $(t, [y])$ are vertices of the graph $\Gamma_k(X)$. There is a unique smallest vertex $(u, [z])$ which is an upper bound for both $(s, [x])$ and $(t, [y])$ in $\Gamma_k(X)$. The number u is the smallest parameter (necessarily a phase change number) such that $[x]_u = [y]_u$ in $\pi_0 L_{u,k}(X)$, and so $[z]_u = [x]_u = [y]_u$. In this case, one writes

$$(s, [x]) \cup (t, [y]) = (u, [z]).$$

The vertex $(u, [z])$ is the *least upper bound* (or join) of $(s, [x])$ and $(t, [y])$.

Every finite collection of points $(s_1, [x_1]), \dots, (s_p, [x_p])$ has a least upper bound

$$(s_1, [x_1]) \cup \dots \cup (s_p, [x_p])$$

in the tree $\Gamma_k(X)$.

We know from [2] that the least upper bound of two branch points is a branch point, and we have an analogous result for layer points:

Lemma 4. *The least upper bound $(u, [z])$ of layer points $(s, [x])$ and $(t, [y])$ is a layer point.*

Proof. If there is a number v such that $s, t < v < u$, then $(v, [x])$ and $(v, [y])$ are distinct because $(u, [z])$ is a least upper bound. This implies that $L_{v,k}(X)$ has distinct path components $[w]$ which map to $[z]$ in $\pi_0 L_{u,k}(X)$. It follows that $(u, [z])$ is a branch point, and is therefore a layer point by L:emma 2.

Otherwise, $s = u$ or $t = u$, in which case $(u, [z]) = (s, [x])$ or $(u, [z]) = (t, [y])$. In either case, $(u, [z])$ is a layer point. □

Lemma 4 implies that every collection of layer points $(s_1, [x_1]), \dots, (s_p, [x_p])$ has a least upper bound

$$(s_1, [x_1]) \cup \dots \cup (s_p, [x_p])$$

in $\mathrm{L}_k(X)$. The maximal (or terminal) element of $L_k(X)$ is the least upper bound of all members of $L_k(X)$.

It follows from Lemma 4 and the corresponding result for branch points of [2] that the poset inclusions

$$\mathrm{Br}_k(X) \subseteq \mathrm{L}_k(X) \subset \Gamma_k(X)$$

preserve least upper bounds.

Lemma 5. *Every vertex $(s, [x])$ of $\Gamma_k(X)$ has a unique largest layer point $(t, [y])$ such that $(t, [y]) \leq (s, [x])$. In this case, $[y]_t = [x]_s$.*

Proof. There is a smallest phase change number t such that there is a relation $(t, [y]) \leq (s, [x])$ with $[y]_t = [x]_s$. The corresponding point $(t, [y])$ is a layer point, by the minimality of the phase change number t.

The point $(t, [y])$ is also an upper bound on the layer points below $(s, [x])$, since $[y]_t = [x]_s$: if $(u, [z])$ is a layer point below $(s, [x])$, then $z \in [y]_t$ and $u \leq t$ since otherwise $(u, [z])$ is not a layer point. □

The first statement of Lemma 5 is also a corollary of Lemma 4: take the least upper bound of all layer points below $(s, [x])$.

Lemma 6. *The poset inclusion $\mathrm{L}_k(X) \subset \Gamma_k(X)$ has an inverse*

$$\max : \Gamma_k(X) \to \mathrm{L}_k(X),$$

up to homotopy, and $\mathrm{L}_k(X)$ is a strong deformation retract of $\Gamma_k(X)$.

Proof. Every vertex $(s,[x])$ of $\Gamma_k(X)$ has a unique maximal layer point $(s_0,[x_0])$ such that $(s_0,[x_0]) \leq (s,[x])$, by Lemma 5. Set

$$\max(s,[x]) = (s_0,[x_0]).$$

The maximality condition implies that the function max preserves the ordering. The composite $\max \cdot \alpha$ is the identity on $\mathrm{L}_k(X)$, and the relations $(s_0,[x_0]) \leq (s,x)$ define a homotopy $\alpha \cdot \max \leq 1$ that restricts to the identity on $\mathrm{L}_k(X)$. □

Remark 7. Lemma 5 of [2] says that every $(s,[x])$ has a unique maximal branch point $(s_1,[x_1])$ such that $(s_1,[x]_1) \leq (s,[x])$. The branch point $(s_1,[x_1])$ is a layer point by Lemma 2, so that there are relations.

$$(s_1,[x_1]) \leq (s_0,[x_0]) \leq (s,[x]),$$

which are natural in points $(s,[x])$ of $\Gamma_k(X)$.

It follows that the poset inclusions

$$\mathrm{Br}_k(X) \subseteq \mathrm{L}_k(X) \subset \Gamma_k(X)$$

define strong deformation retractions, and that the respective contracting homotopies are compatible.

Recall from Lemma 3 that $\mathrm{Br}_0(X) = \mathrm{L}_0(X)$, so that the discussion simplifies for Vietoris-Rips complexes.

2 Stability

The general setup for stability of degree Rips complexes is the following: we suppose given finite metric spaces $X \subset Y$ such that the Hausdorff distance between the corresponding spaces X^k_{dis} and Y^k_{dis} of sets of $k+1$ distinct elements in X and Y respectively satisfies $d_H(X^k_{dis}, Y^k_{dis}) < r$, where r is a fixed non-zero positive real number.

Under these assumptions, the degree Rips stability theorem (Theorem 4 of [3]) says that there are homotopy commutative diagrams

$$\begin{array}{ccc} L_{s,k}(X) & \xrightarrow{\sigma} & L_{s+2r,k}(X) \\ {\scriptstyle i}\downarrow & \nearrow^{\theta} & \downarrow{\scriptstyle i} \\ L_{s,k}(Y) & \xrightarrow[\sigma]{} & L_{s+2r,k}(Y) \end{array} \tag{3}$$

Applying the path component functor π_0 gives commutative diagrams

$$\begin{array}{ccc} \pi_0 L_{s,k}(X) & \xrightarrow{\sigma} & \pi_0 L_{s+2r,k}(X) \\ {\scriptstyle i}\downarrow & \nearrow^{\theta} & \downarrow{\scriptstyle i} \\ \pi_0 L_{s,k}(Y) & \xrightarrow[\sigma]{} & \pi_0 L_{s+2r,k}(Y) \end{array} \tag{4}$$

and there is an induced commutative diagram of hierarchies

$$\begin{array}{ccc} \Gamma_k(X) & \xrightarrow{\sigma} & \Gamma_k(X) \\ {\scriptstyle i}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle i} \\ \Gamma_k(Y) & \xrightarrow[\sigma]{} & \Gamma_k(Y) \end{array} \tag{5}$$

Here,

$$\begin{aligned} &i((s,[x])) = (s,[i(x)]), \\ &\sigma((s,[x])) = (s+2r,[\sigma(x)]), \text{ and} \\ &\theta((s,[y]) = (s+2r,[\theta(y)]). \end{aligned}$$

Write $i_* : \mathrm{L}_k(X) \to \mathrm{L}_k(Y)$ for the composite poset morphism

$$\mathrm{L}_k(X) \subset \Gamma_k(X) \xrightarrow{i_*} \Gamma_k(Y) \xrightarrow{\max} \mathrm{L}_k(Y)$$

This map takes a layer point $(s,[x])$ to the maximal layer point below $(s,[i(x)])$.

Poset morphisms $\theta_* : \mathrm{L}_k(Y) \to \mathrm{L}_k(X)$ and $\sigma_* : \mathrm{L}_k(X) \to \mathrm{L}_k(X)$ are similarly defined, respectively, by the poset morphisms $\theta : \Gamma_k(Y) \to \Gamma_k(X)$ and the shift morphism $\sigma : \Gamma_k(X) \to \Gamma_k(X)$.

1) Consider the poset maps

$$\mathrm{L}_k(X) \xrightarrow{i_*} \mathrm{L}_k(Y) \xrightarrow{\theta_*} \mathrm{L}_k(X).$$

If $(s,[x])$ is a layer point for X, choose maximal layer points $(s_0,[x_0]) \leq (s,[i(x)]$, $(s_1,[x_1]) \leq (s_0+2r,[\theta(x_0)])$ and $(v,[y]) \leq (s+2r,[x])$ below the respective objects.

Then $\theta_* i_*(s,[x]) = (s_1,[x_1])$, and there is a natural relation

$$\theta_* i_*(s,[x]) = (s_1,[x_1]) \leq (v,[y]) = \sigma_*(s,[x])$$

by a maximality argument. We therefore have a homotopy of poset maps

$$\theta_* i_* \leq \sigma_* : \mathrm{L}_k(X) \to \mathrm{L}_k(X). \tag{6}$$

2) Similarly, if $(t,[y])$ is a layer point of Y, then

$$i_*\theta_*(t,[y]) \leq \sigma_*(t,[y]),$$

giving a homotopy

$$i_*\theta_* \leq \sigma_* : \mathrm{L}_k(Y) \to \mathrm{L}_k(Y). \tag{7}$$

There are relations

$$(s,[x]) \leq \sigma_*(s,[x]) \leq (s+2r,[x]) \tag{8}$$

for branch points $(s,[x])$. It follows that the poset map $\sigma_* : \mathrm{L}_k(X) \to \mathrm{L}_k(X)$ is homotopic to the identity on $\mathrm{L}_k(X)$.

The construction of the poset maps i_*, θ_* and σ_*, together with the relations (6) and (7), complete the construction/proof of the following result:

Theorem 8. *Suppose that $X \subset Y$ is an inclusion of finite metric spaces, and that $d_H(X^k_{dis}, Y^k_{dis}) < r$. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} L_k(X) & \xrightarrow{\sigma_*} & L_k(X) \\ {\scriptstyle i_*}\downarrow & \nearrow^{\theta_*} & \downarrow{\scriptstyle i_*} \\ L_k(Y) & \xrightarrow[\sigma_*]{} & L_k(Y) \end{array} \tag{9}$$

that relates the layer posets $\mathrm{L}_k(X)$ and $\mathrm{L}_k(Y)$ of the spaces X and Y, respectively.

Remark 9. The element $\sigma_*(s, [x]) = (t, [x])$ is close to $(s, [x])$ in the sense that there are relations

$$(s, [x]) \leq (t, [x]) \leq (s + 2r, [x])$$

so that $0 \leq t - s \leq 2r$. Thus, the layer points $(s, [x])$ and $\theta_* i_*(s, [x])$ have a common upper bound, namely $\sigma_*(s, [x])$, which is close to $(s, [x])$.

If $(t, [y])$ is a layer point of $\Gamma_k(Y)$, the layer point $\sigma_*(t, [y]) \leq (t + 2r, [y])$ is similarly an upper bound for $(t, [y])$ and $i_* \theta_*(t, [y])$, and is close to $(t, [y])$.

The subobject of $\mathrm{L}_k(X)$ consisting of all layer points of the form $(s, [x])$ as s varies has an obvious notion of distance: the distance between points $(s, [x])$ and $(t, [x])$ is $|t - s|$.

Suppose that

$$0 < t_1 < \cdots < t_k$$

are the phase change numbers for the system $L_{s,k}(X)$.

The assumption that $d_H(X^k_{dis}, Y^k_{dis}) < r$ forces the function

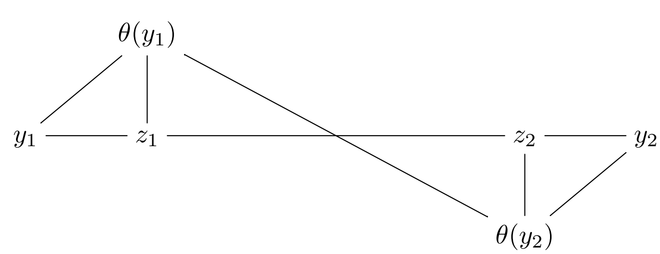
$$\pi_0 L_{s,k}(X) \to \pi_0 L_{s,k}(Y)$$

to be surjective if $s \geq r$.

Lemma 10. *Suppose, that $y_1, y_2 \in Y$ have elements $\theta(y_1), \theta(y_2) \in X$ such that $d(y_i, \theta(y_i)) < r$. Then $d(y_1, y_2)$ is in the interval $(t - 2r, t + 2r)$, where $t = d(\theta(y_1), \theta(y_2))$.*

Proof. We shall assume that $t - 2r > 0$.

Consider the picture



Suppose that v is the point of intersection of the lines (z_1, z_2) and $(\theta(y_1), \theta(y_2))$. Then

$$d(\theta(y_1), \theta(y_2)) \geq d(z_1, z_2) = d(z_1, v) + d(v, z_2) \geq d(y_1, y_2) - 2r.$$

The assertion that $d(\theta(y_1), \theta(y_2)) < d(y_1, y_2) + 2r$ is a simple application of the triangle inequality. □

Corollary 11. *All phase change numbers s for Y lie in intervals $(t-2r, t+2r)$ around phase change numbers t of X.*

There is a finite collection of numbers t such that $(t, [x])$ is a layer point for $\Gamma_k(X)$. Say that such numbers t are the layer parameters for X. Each layer parameter is a phase change number.

Observe that the inclusions $\sigma : L_{s,k}(X) \subseteq L_{t,k}(X)$ for $s \leq t$ induce inclusions $[x]_s \subset [x]_t$ for all vertices x of $L_{s,k}(X)$.

Recall from the proof of Lemma 5 that the maximal layer point below $(s, [x])$ can be constructed by finding the smallest phase change number t such that there is a relations $(t, [u]) \leq (s, [x])$ such that $[u]_t = [x]_s$ as subsets of X.

Lemma 12. *Suppose that $s < t$ and there are no layer points of the form $(u, [x])$ in $\Gamma_k(X)$, where $s < u \leq t$. Then the induced function*

$$\sigma_* : \pi_0 L_{s,k}(X) \to \pi_0 L_{t,k}(X)$$

is a bijection.

Proof. We can assume that $L_{t,k}(X) \neq \emptyset$, for otherwise $L_{s,k}(X) = L_{t,k}(X) = \emptyset$.

Suppose that $(t, [x]) \in \Gamma_k(X)$ and that $(u, [y])$ is a maximal layer point with $(u, [y]) \leq (t, [x])$. Then $u \leq s$ and the relations $(u, [y]) \leq (s, [y]) \leq (t, [x])$ force $[y]_s = [x]_t$. In particular, the function σ_* is surjective.

If $[y_1], [y_s] \in \pi_0 L_{s,k}(X)$ have the same image $[x] \in \pi_0 L_{t,k}(X)$, then $[y_1]_s = [x]_t = [y_2]_s$ as subsets of X, so that $[y_1] = [y_2]$ in $\pi_0 L_{s,k}(X)$, and so σ_* is injective. □

Given a layer parameter t for X, write t_+ for the smallest layer parameter of X with $t < t_+$, and write t_- for the largest layer parameter of X with $t_- < t$.

Lemma 13. *Suppose that t is a layer parameter for X such that $r < t < t_+ - 2r$. Then the function $i : \pi_0 L_{t,k}(X) \to \pi_0 L_{t,k}(Y)$ is a bijection.*

Proof. The diagram

$$\begin{array}{ccc} \pi_0 L_{t,k}(X) & \xrightarrow{\cong} & \pi_0 L_{t+2r,k}(X) \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \theta} & \\ \pi_0 L_{t,k}(Y) & & \end{array}$$

commutes, and the displayed function is a bijection by Lemma 12, so the function i is injective. The surjectivity of i follows from the assumption $t > r$. □

Lemma 14. *Suppose that $(t,[x])$ is a layer point of $\Gamma_k(X)$ with $r < t < t_+ - 2r$, and suppose that $(s,[y])$ is a maximal layer point below $(t,[i(x)])$ in $\Gamma_k(Y)$. Then $t - 2r \leq s \leq t$.*

Proof. Suppose that $s < t - 2r$.

The map $i_* : \pi_0 L_{t,k}(X) \to \pi_0 L_{t,k}(Y)$ is a bijection by Lemma 13 and $i_*([x]) = i_*([\theta(y)]) = [i(x)]$ in $\pi_0 L_{t,k}(Y)$. It follows that there is a commutative diagram of functions

$$
\begin{array}{cccc}
 & [\theta(y)]_{s+2r} & \xrightarrow{\sigma} & [x]_t \\
 & \nearrow{\scriptstyle\theta} & & \downarrow{\scriptstyle i} \\
[y]_s & & \xrightarrow[\sigma]{\cong} & [i(x)]_t
\end{array}
$$

in which the map $i : [x]_t \to [i(x)]_t$ is a monomorphism since it is a subobject of a monomorphism of vertices.

The functions i and $\sigma \cdot \theta$ are bijections, and so $\sigma : [\theta(y)]_{s+2r} \to [x]_{t_j}$ is an epimorphism. This function σ is also a monomorphism, since it is a subobject of the monomorphism of vertices $L_{s+2r,k}(X)_0 \to L_{t,k}(X)_0$.

It follows that the function $\sigma : [\theta(y)]_{s+2r} \to [x]_t$ is a bijection, so that $(t,[x])$ is not a layer point. □

Corollary 15. *Suppose that $(t,[x])$ is a layer point for $\Gamma_k(X)$ such that $r < t < t_+ - 2r$. Then we have*

$$\theta_* i_*(t,[x]) = (t,[x]).$$

Proof. Suppose that $(s,[z])$ is a maximal layer point below $(t,[i(x)])$ in $\Gamma_k(Y)$. Then $t - 2r \leq s \leq t$ by Lemma 14, so that $t \leq s + 2r \leq t + 2r < t_+$.

The layer point $(t,[x])$ is a maximal layer point below $(t+2r,[x])$, since $t + 2r < t_+$, so that $[x]_t = [x]_{t+2r}$. The layer point $\theta_*(s,[z])$ is the maximal layer point below $(s+2r,[\theta(z)])$, and the relation

$$(s+2r,[\theta(z)]) \leq (t+2r,[x])$$

implies that $\theta(z) \in [x]_{t+2r} = [x]_{s+2r}$, so that $x \in [\theta(z)]_{s+2r}$. It follows that the maximal layer point below $(s+2r,[\theta(z)])$ must also be the maximal layer point below $(t+2r,[x])$, which is $(t,[x])$. □

Lemma 16. *Suppose that $(s,[y])$ is a layer point of $\Gamma_k(Y)$, and that $s < s_+ - 2r$. Suppose that $(t,[z])$ is a maximal layer point below $(s+2r,[\theta(y)])$. Then $s \leq t \leq s + 2r$.*

Proof. Suppose that $t < s$.

The map $\sigma : \pi_0 L_{s,k}(Y) \to \pi_0 L_{s+2r,k}(Y)$ is a bijection, since $\Gamma_k(Y)$ has no layer parameters in the interval $(s, s+2r]$, by assumption and Lemma 12. It follows that the map $\theta : \pi_0 L_{s,k}(Y) \to \pi_0 L_{s+2r}(X)$ is a monomorphism.

Then $\theta([y]) = \theta([i(z)]$ implies that $[y]_s = [i(z)]_s$, so the diagram

$$\begin{array}{ccccccc} [i(z)]_t & & \xrightarrow{\sigma} & & [y]_s & \xrightarrow{\sigma} & [y]_{s+2r} \\ {\scriptstyle i}\uparrow & \searrow^{\theta} & & & & \searrow^{\theta} & \uparrow{\scriptstyle i} \\ [z]_t & \xrightarrow[\sigma]{} & [z]_{t+2r} & & \xrightarrow[\sigma]{} & & [\theta(y)]_{s+2r} \end{array}$$

commutes.

The commutativity of the triangle on the right implies that $\theta : [y]_s \to [\theta(y)]_{s+2r}$ is a monomorphism.

The function $\sigma : [z]_t \to [\theta(y)]_{s+2r}$ a bijection, so $\theta : [y]_s \to [\theta(y)]_{s+2r}$ is a bijection.

The composite

$$[z]_t \xrightarrow{i} [i(z)]_t \xrightarrow{\sigma} [y]_s$$

a bijection, so $\sigma : [i(z)]_t \to [y]_s$ is a bijection, and it follows that $(s, [y])$ is not a layer point. □

The analysis of the morphism

$$\theta_* : V(Y) = \mathrm{L}_0(Y) \to \mathrm{L}_0(X) = V(X)$$

for Vietoris-Rips complexes is sharper, because all complexes $V_s(Y)$ share the same set of vertices, namely Y. In this case, we have a stronger version of Lemma 16, with a very different argument.

Lemma 17. *Suppose that $(s, [y])$ is a layer point of $\Gamma_0(Y)$, and that $(t, [z])$ is a maximal layer point of $\Gamma_0(X)$ below $(s + 2r, [\theta(y)])$. Then $s \leq t \leq s + 2r$.*

Proof. The sets $[z]_t$ and $[\theta(y)]_{s+2r}$ have the same cardinality, and so $\theta(y) \in [z]_t$.

Consider the collection of elements $[u] \in \pi_0 V_{t-2r}(Y)$ which map to $[z]_t = [\theta(y)]_t$ in $\pi_0 V_t(X)$. Then $\theta^{-1}([z]_t) = \sqcup\ [u]$ as a subset of the vertices Y of $V_{t-2r}(Y)$, and $y \in [u]$ for some $[u]$. All such components $[u]$ map to the same path component $[y]_t$ in $V_t(X)$.

In the diagram

$$\begin{CD} \theta^{-1}([z]_t) @>>> \theta^{-1}([\theta(y)]_{s+2r}) @>>> Y \\ @VVV @VVV @VV{\theta}V \\ [z]_t @>>{\cong}> [\theta(y)]_{s+2r} @>>> X \end{CD}$$

both squares are pullbacks, so the function

$$\theta^{-1}([z]_t) \to \theta^{-1}([\theta(y)]_{s+2r}$$

is a bijection.

Suppose that $t < s$. Then

$$\theta^{-1}([z]_t) = \sqcup\, [u] \subset [y]_t \subset [y]_s \subset \theta^{-1}([\theta(y)]_{s+2r})$$

while $\theta^{-1}([z]_t) = \theta^{-1}([\theta(y)]_{s+2r})$ as subsets of Y.

It follows that $[y]_t = [y]_s$, so that $(s, [y])$ is not a layer point. □

Lemma 17 and Lemma 14 together impose rather tight constraints on the layer points of $\Gamma_0(Y)$, in relation to those of $\Gamma_0(X)$. Recall that the comparison $\Gamma_0(X) \to \Gamma_0(Y)$ arises from applying path component functors to the comparison $V_*(X) \to V_*(Y)$. In this case, $d_H(X,Y) = r$ is the bound on Hausdorff distance which leads to the interleaving diagrams (3), (4) and (5).

To repeat the statement of Lemma 17, suppose that $(s, [y])$ is a layer point for $\Gamma_0(Y)$, and suppose that $(t, [x])$ is a maximal layer point below $(s+2r, [\theta(y)])$. Then $s \leq t \leq s + 2r$.

It follows, in particular, that all layer points of $\Gamma_0(Y)$ are in the intervals $[t - 2r, t]$ corresponding to layer points $(t, [x])$ of $\Gamma_0(X)$.

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