# Galois descent criteria

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September 15, 2017

**Abstract**

This paper gives an introduction to homotopy theoretic descent, and its applications in algebraic $K$-theory computations for fields. On the étale site of a field, a fibrant model of a space can be constructed from naive Galois cohomological objects given by homotopy fixed point constructions, but only up to pro-equivalence. The homotopy fixed point spaces define finite Galois descent for spaces, and a pro categorical construction is a necessary second step for passage from finite descent conditions to full homotopy theoretic descent in a Galois cohomological setting.

## Introduction

Descent theory is a large subject, which appears in many forms in areas of geometry, number theory and topology.

Initially, descent was a set of methods for constructing global features of a "space" from a set of local data that satisfies patching conditions, or for defining a variety over a base field from a variety over a finite separable extension that comes equipped with a some type of cocycle. The latter field of definition problem appears in early work of Weil [20]; it was later subsumed by a general approach of Grothendieck to what we now call "faithfully flat descent" that appeared in FGA [7].

The early descriptions of patching conditions were later generalized to isomorphisms of structures on patches which are defined up to coherent isomorphism, in the formulation of the notion of effective descent that one finds in the theory of stacks and, more generally, higher stacks [16].

Cohomological descent is a spectral sequence technique for computing the cohomology of a "space" $S$ from the cohomology of the members of a covering. The theory is discussed in detail in SGA4, [1, Exp. Vbis], while the original spectral sequence for an ordinary covering was introduced by Godement [4].

The construction of the descent spectral sequence for a covering $U \to S$ (sheaf epimorphism) starts with a Čech resolution $\check{C}(U) \to S$ for the covering

*This research was supported by NSERC.

and an injective resolution $A \to I^\bullet$ of a coefficient abelian sheaf $A$. One forms the third quadrant bicomplex

$$\hom(\check{C}(U), I^\bullet)$$

and the resulting spectral sequence has the form

$$E_2^{p,q} = H^p(H^q(\check{C}(U), A)) \Rightarrow H^{p+q}(S, A). \tag{1}$$

It is a more recent observation that the spectral sequence (1) converges to $H^{p+q}(S, A)$, since the resolution $\check{C}(U) \to S$ is a stalkwise weak equivalence of simplicial sheaves. The Čech resolution $\check{C}(U)$ is a simplicial object which is made up of the components of the covering $U$ and their iterated intersections.

The variation of the descent spectral sequence that is discussed in SGA4 is constructed by replacing the Čech resolution by a hypercover $V \to S$. In modern terms, a hypercover is a local trivial fibration. Initially, a hypercover $V$ was a simplicial scheme over $S$ which satisfied a set of local epimorphism conditions defined by its coskeleta [2].

The key observation for these constructions is that, if $X \to Y$ is a stalkwise equivalence of simplicial sheaves (or presheaves), then the induced map of bicomplexes

$$\hom(Y, I^\bullet) \to \hom(X, I^\bullet) \tag{2}$$

induces a cohomology isomorphism of total complexes. Thus, one has a definition

$$H^n(X, A) := H^n(\operatorname{Tot}\hom(X, I^\bullet))$$

of the cohomology of a simplicial presheaf $X$ with coefficients in an abelian sheaf $A$ that is independent of the stalkwise homotopy type of $X$, along with a spectral sequence that computes it.

Descent theory became a homotopy theoretic pursuit with the introduction of local homotopy theories for simplicial presheaves and sheaves, and presheaves of spectra. These homotopy theories evolved from ideas of Grothendieck; their formalization essentially began with Illusie's thesis [8].

The local homotopy theories are Quillen model structures: a local weak equivalence of simplicial presheaves or sheaves is a map which induces weak equivalences at all stalks, and a cofibration is a monomorphism. The local homotopy theory of presheaves of spectra is constructed from the homotopy theory of simplicial presheaves by using methods of Bousfield and Friedlander. The fibrations for these theories are now commonly called injective fibrations.

In the setup for the cohomological descent spectral sequence (1), the injective resolution $I^\bullet$ "satisfies descent", in that it behaves like an injective fibrant object, with the result that a local weak equivalence $X \to Y$ induces a quasi-isomorphism (2). Homotopical descent theory is the study of simplicial objects and spectrum objects that are nearly fibrant.

One says that a simplicial presheaf $X$ *satisfies descent* (or homotopy theoretic descent) if any local weak equivalence $X \to Z$ with $Z$ injective fibrant is a sectionwise weak equivalence, in the sense that the maps $X(U) \to Z(U)$ are weak equivalences of simplicial sets for all objects $U$ in the underlying site.

This form of descent is a statement about the sectionwise behaviour of simplicial *presheaves*, or *presheaves* of spectra, and is oriented towards computing homotopy groups in sections. The role of sheaves is incidental, except in the analysis of local behaviour.

There are many examples:

1) Every local weak equivalence $Z \to Z'$ of injective fibrant objects is a sectionwise equivalence by formal nonsense (discussed below), so that all injective fibrant objects satisfy descent.

2) One can show [13, Sec. 9.2] that a sheaf of groupoids $G$ is a stack (ie. satisfies the effective descent condition) if and only if its nerve $BG$ satisfies homotopy theoretic descent.

The advantage of having in hand an object $X$ which satisfies descent is that there are machines (eg. Postnikov tower, or Godement resolution) that can be used to produce a spectral sequence

$$E_2^{s,t} = H^t(S, \tilde{\pi}_s X) \text{ “} \Rightarrow \text{” } \pi_{t-s}(X(S)) \tag{3}$$

which computes the homotopy groups of the space $X(S)$ in global sections from sheaf cohomology for $S$ with coefficients in the homotopy group sheaves of $X$. This is the homotopy theoretic descent spectral sequence.

The spectral sequence (3) is a Bousfield-Kan spectral sequence for a tower of fibrations, so convergence can be a problem, and there may also be a problem with knowing what it converges to. Both issues are circumvented in practice by insisting on a universal bound on cohomological dimension — see Section 4.

The availability of a calculational device such as (3) for objects $X$ which satisfy descent means that the hunt is on for such objects, for various topologies and in different contexts.

The algebraic $K$-theory presheaf of spectra $K$, for example, satisfies descent for the Nisnevich topology on the category $Sm|_S$ of smooth $S$-schemes, where $S$ is a regular Noetherian scheme of finite dimension. This follows from the existence of localization sequences in $K$-theory for such schemes, so that the $K$-theory presheaf satisfies a “*cd*-excision” property.

There is a general result of Morel and Voevodsky [14], [13, Th. 5.39], which says that any simplicial presheaf on $Sm|_S$ that satisfies the *cd*-excision property satisfies Nisnevich descent. The proof of the Morel-Voevodsky theorem is based on an earlier theorem of Brown and Gersten, which gives a descent criterion for simplicial presheaves on the standard site of open subsets of a Noetherian topological space. The descent criterion of Brown-Gersten amounts to homotopy cartesian patching for pairs of open subsets.

The arguments for the Morel-Voevodsky and Brown-Gersten descent theorems are geometric and a subtle, and depend strongly on the ambient Grothendieck topologies. Descent theorems are interesting and important geometric results, and finding one of them is a big event.

Homotopy theoretic descent problems originated in algebraic $K$-theory, in the complex of problems related to the Lichtenbaum-Quillen conjecture.

Suppose that $k$ is a field, that $\ell$ is a prime number which is distinct from the characteristic of $k$. The mod $\ell$ algebraic $K$-theory presheaf of spectra $K/\ell$ on smooth $k$-schemes is the cofibre of multiplication by $\ell$ on the algebraic $K$-theory presheaf $K$, and the stable homotopy groups $\pi_p K/\ell(k)$ are the mod $\ell$ $K$-groups $K_p(k, \mathbb{Z}/\ell)$ of the field $k$. The presheaf of spectra $K/\ell$ has a fibrant model $j : K/\ell \to LK/\ell$ for the étale topology on $k$, and the stable homotopy groups $\pi_p LK/\ell(k)$ are the étale $K$-groups $K_p^{et}(k, \mathbb{Z}/\ell)$ of $k$. The map $j$ induces a comparison

$$K_p(k, \mathbb{Z}/\ell) \to K_p^{et}(k, \mathbb{Z}/\ell), \tag{4}$$

and the Lichtenbaum-Quillen conjecture asserts that this map is an isomorphism in an infinite range of degrees above the Galois cohomological dimension of $k$.

The point of this conjecture is that algebraic $K$-theory with coefficients should be computable from étale (or Galois) cohomology. At the time that it was formulated, the conjecture was a striking leap of faith from calculations in low degrees. The precise form of the conjecture that incorporates the injective fibrant model $j : K/\ell \to LK/\ell$ followed much later.

Thomason's descent theorem for Bott periodic $K$-theory [18] was a first approximation to Lichtenbaum-Quillen. His theorem says that formally inverting the Bott element $\beta$ in $K_*(k, \mathbb{Z}/\ell)$ produces a presheaf of spectra $K/\ell(1/\beta)$ which satisfies homotopy theoretic descent for the étale topology on the field $k$. Formally inverting the Bott element has no effect on étale $K$-theory, so that the Bott periodic $K$-theory spectrum object $K/\ell(1/\beta)$ is sectionwise stably equivalent to the étale $K$-theory presheaf.

The Lichtenbaum-Quillen conjecture was proved much later — it is a consequence of the Bloch-Kato conjecture [17], while Voevodsky's proof of Bloch-Kato appears in [19].

Voevodsky's work on Bloch-Kato depended on the introduction and use of motivic techniques, and was a radical departure from the methods that were used in attempts to calculate the $K$-theory of fields up to the mid 1990s.

Before Voevodsky, the general plan for showing that the étale descent spectral sequence converged to the algebraic $K$-theory of the base field followed the methods of Thomason, and in part amounted to attempts to mimic, for $K$-theory, the observation that the Galois cohomology of a field $k$ can be computed from Čech cohomology. At the time, the $E_2$-term of the étale descent spectral sequence for the $K$-theory of fields was known from Suslin's calculations of the $K$-theory of algebraically closed fields.

Explicitly, if $A$ is an abelian sheaf on the étale site of $k$, then there is an isomorphism

$$H^p_{Gal}(k, A) \cong \varinjlim_{L/k} H^p \hom(EG \times_G \mathrm{Sp}(L), A), \tag{5}$$

Here, $L$ varies through the finite Galois extensions of $k$, and we write $G = Gal(L/k)$ for the Galois group of such an extension $L$. The simplicial sheaf

$EG \times_G \mathrm{Sp}(L)$ is the Borel construction for the action of $G$ on the étale sheaf represented by the $k$-scheme $\mathrm{Sp}(L)$. The complex $\hom(EG \times_G \mathrm{Sp}(L), A)$ has $n$-cochains given by

$$\hom(EG \times_G \mathrm{Sp}(L), A)^n = \prod_{G^{\times n}} A(L),$$

and is the "homotopy" fixed points complex for the action of $G$ on the abelian group $A(L)$ of $L$-points of $A$.

It is a critical observation of Thomason that if $B$ is an abelian presheaf which is additive in the sense that it takes finite disjoint unions of schemes to products, then there is an isomorphism

$$H^p_{Gal}(k, \tilde{B}) \cong \varinjlim_{L/k} H^p \hom(EG \times_G \mathrm{Sp}(L), B), \tag{6}$$

which computes cohomology with coefficients in the associated sheaf $\tilde{B}$ from the presheaf-theoretic cochain complexes $\hom(EG \times_G \mathrm{Sp}(L), B)$.

The $K$-theory presheaf of spectra $K/\ell$ is additive, and it's still a leap, but one could hope that the analogous comparison map of spectra

$$K/\ell(k) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), K/\ell) \tag{7}$$

induces an isomorphism in stable homotopy groups in an appropriate range, and that the colimit on the right would be equivalent to the mod $\ell$ étale $K$-theory spectrum of the field $k$.

There are variations of this hope. The map (6) is a colimit of the comparison maps

$$K/\ell(k) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), K/\ell), \tag{8}$$

and one could ask that each such map induces an isomorphism in homotopy groups in an appropriate range.

The function complex spectrum $\mathbf{hom}(EG \times_G \mathrm{Sp}(L), K/\ell)$ is the homotopy fixed points spectrum for the action of $G$ on the spectrum $K/\ell(L)$, and the question of whether or not (8) is a weak equivalence is commonly called a homotopy fixed points problem. It is also a finite descent problem.

There were many attempts to solve homotopy fixed points problems for algebraic $K$-theory in the pre-motives era, with the general expectation that the question of identifying the colimit in (8) with the étale $K$-theory spectrum should take care of itself.

The identification problem, however, turned out to be hard. Attempts to address it at the time invariably ended in failure, and always involved the "canonical mistake", which is the false assumption that inverse limits to commute with filtered colimits.

It is a technical application of the methods of this paper that the identification of the colimit

$$\varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), K/\ell)$$

with the étale $K$-theory spectrum cannot work out, except in a suitable pro category.

This is expressed in more abstract terms as Theorem 24 below for a certain class of simplicial presheaves on the étale site for $k$. The mod $\ell$ $K$-theory presheaf of spectra $K/\ell$, or rather its component level spaces $(K/\ell)^n$ are examples of such objects.

The main body of this paper is set in the context of simplicial presheaves and sheaves on the site $G - \mathbf{Set}_{df}$ of discrete finite $G$-sets for a profinite group $G$ and their $G$-equivariant maps. The coverings for this site are the surjective maps.

If $k$ is a field, then the finite étale site is equivalent to the site $G - \mathbf{Set}_{df}$ for the absolute Galois group $G$ of $k$, via imbeddings of finite separable extensions of $k$ in its algebraic closure.

Until one reaches the specialized calculations of Section 4, everything that is said about simplicial presheaves and presheaves of spectra on the étale sites of fields is a consequence of general results about the corresponding objects associated to the sites $G - \mathbf{Set}_{df}$ for profinite groups $G$.

The local homotopy theory for general profinite groups was first explicitly described by Goerss [6], and has since become a central structural component of the chromatic picture of the stable homotopy groups of spheres.

This paper proceeds on a separate track, and reflects the focus on generalized Galois cohomology and descent questions which arose in algebraic $K$-theory, as partially described above. See also [10].

Some basic facts about the local homotopy theory for profinite groups are recalled in Section 1, and some basic facts about cosimplicial spaces are recalled in Section 2.

With this collection of ideas in hand, we arrive at the following:

**Theorem 1.** *Suppose that $f : X \to Y$ is a local weak equivalence between presheaves of Kan complexes on the site $G - \mathbf{Set}_{df}$ such that $X$ and $Y$ have only finitely many non-trivial presheaves of homotopy groups. Then the induced map*

$$f_* : \varinjlim_i \mathbf{hom}(EG_i \times_{G_i} G_i, X) \to \varinjlim_i \mathbf{hom}(EG_i \times_{G_i} G_i, Y)$$

*is a weak equivalence.*

This result appears as Theorem 9 below. It has the following special case:

**Corollary 2.** *Suppose that $f : X \to Y$ is a local weak equivalence between presheaves of Kan complexes on the finite étale site of a field $k$ such that $X$ and $Y$ have only finitely many non-trivial presheaves of homotopy groups. Then the induced map*

$$f_* : \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Y)$$

*is a weak equivalence.*

Theorem 1 is proved by inductively solving obstructions for cosimplicial spaces after refining along the filtered diagram associated to the profinite group $G$, by using methods that are displayed in Section 2. The assumption that the simplicial presheaf $X$ has only finitely many non-trivial presheaves of homotopy groups means that the obstructions can be solved in finitely many steps.

When specialized to the fields case, Theorem 1 implies that the colimit

$$\varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X)$$

is weakly equivalent to the simplicial set $Z(k)$ of global sections of a fibrant model $j : X \to Z$ on the finite étale site of a field $k$, provided that $X$ has only finitely many non-trivial presheaves of homotopy groups.

In particular, if $X$ satisfies this finiteness condition on its presheaves of homotopy groups and if $X$ also satisfies finite descent, then the map $X(k) \to Z(k)$ in global sections is a weak equivalence.

Generally, Theorem 1 means that you can use Galois cohomological methods to construct injective fibrant models for simplicial presheaves $X$ having finitely many non-trivial presheaves of homotopy groups. This construction specializes to (and incorporates) the identification (5) of Galois cohomology with Čech cohomology.

Going further involves use of the homotopy theory of pro objects, which is enabled by [11].

In general, a simplicial presheaf $Y$ is pro-equivalent to its derived Postnikov tower, via the canonical map $Y \to \mathbf{P}_*Y$. The Postnikov tower $\mathbf{P}_*Y$ has a fibrant model $\mathbf{P}_*(Y) \to L\mathbf{P}_*(Y)$ in the model category of pro objects and pro-equivalences of simplicial presheaves. One then has a string of pro-equivalences

$$Y \to \mathbf{P}_*Y \to L\mathbf{P}_*Y,$$

and it follows from Corollary 2 that the induced composite in global sections is weakly equivalent to the pro-map

$$\theta : Y(k) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), \mathbf{P}_*Y).$$

There are two questions:

1) Is the displayed map $\theta$ a pro-equivalence?

2) Is the map

$$LY(k) \to L\mathbf{P}_*Y(k) \xrightarrow{\simeq} \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), L\mathbf{P}_*Y)$$

a pro-equivalence?

If the answer to both questions is yes, then the map $Y(k) \to LY(k)$ is a pro-equivalence of spaces, and hence a weak equivalence by Corollary 13 of this paper. These questions are the Galois descent criteria for simplicial presheaves $Y$ on the étale site of a field.

The imposition of a global bound on cohomological dimension forces a positive answer to question 2) — see the proof of Theorem 24 below. In such cases, the simplicial presheaf $Y$ satisfies Galois descent if and only if the map $\theta$ is a pro-equivalence.

Observe finally that the necessity of answering question 2) disappears if we are willing to take our fibrant model of $Y$ in the pro category. This is consistent with the basic setup for étale homotopy theory and the extant definitions of continuous cohomology theory in number theory and geometry. Maybe one should have been working in the pro category all along.

# Contents



# 1 Profinite groups

We begin with a discussion of some generalities about profinite groups, in order to establish notation.

Suppose that the group-valued functor $G : I \to \mathbf{Grp}$ is a profinite group. This means that $I$ is left filtered (any two objects $i, i'$ have a common lower bound, and any two morphisms $i \rightrightarrows j$ have a weak equalizer), and that all of the constituent groups $G_i$, $i \in I$, are finite. We shall also assume that all of the transition homomorphisms $G_i \to G_j$ in the diagram are surjective.

**Example 3.** The standard example is the absolute Galois group $G_k$ of a field $k$. One takes all finite Galois extensions $L/k$ inside an algebraically closed field $\Omega$ containing $k$ in the sense that one has a fixed imbedding $i : k \to \Omega$, and the Galois extensions are specific field extensions $L = k(\alpha)$ of $k$ inside $\Omega$.

These are the objects of a right filtered category, for which the morphisms $L \to L'$ are extensions inside $\Omega$. The contravariant functor $G_k$ that associates the Galois group $G_k(L) = G(L/k)$ to each of these extensions is the absolute Galois group.

It is basic field theory that if $L \subset L'$ inside $\Omega$ which are finite Galois extensions, then every field automorphism $\alpha : L' \to L'$ that fixes $k$ also permutes the roots of which define $L$ over $k$, and hence restricts to an automorphism

$\alpha|_L : L \to L$. The assignment $\alpha \mapsto \alpha|_L$ determines a surjective group homomorphism $G(L'/k) \to G(L/k)$.

Let $G - \mathbf{Set}_{df}$ be the category of finite discrete $G$-sets, as in [10]. A discrete $G$-set is a set $F$ equipped with an action

$$G \times F \to G_i \times F \to F,$$

where $G = \varprojlim_i G_i$, and a morphism of discrete $G$-sets is a $G$-equivariant map.

Every finite discrete $G$-set $X$ has the form

$$X = G/H_1 \sqcup \dots G/H_n,$$

where the groups $H_i$ are stabilizers of elements of $x$. In this way, the category $G - \mathbf{Set}_{df}$ is a thickening of the orbit category $\mathcal{O}_G$ for the profinite group $G$, whose are quotients $G/H$, and with equivariant maps between them. The subgroups $H$ are special: they are preimages of normal subgroups of the $G_i$ under the maps $G \to G_i$.

**Example 4.** The finite étale site $fet|_k$ of $k$ is a category of schemes which has as objects all finite disjoint unions

$$\mathrm{Sp}(L_1) \sqcup \dots \sqcup \mathrm{Sp}(L_n)$$

of schemes defined by finite separable extensions $L_i/k$. The morphisms of $fet|_k$ are the scheme homomorphisms

$$\mathrm{Sp}(L_1) \sqcup \dots \sqcup \mathrm{Sp}(L_n) \to \mathrm{Sp}(N_1) \sqcup \dots \sqcup \mathrm{Sp}(N_m),$$

or equivalently $k$-algebra homomorphisms

$$\prod_j N_j \to \prod_i L_i.$$

A finite separable extension $N = k(\alpha)$ of $k$ is specified by the root $\alpha$ of some separable polynomial $f(x)$. The set of $k$-linear maps $N \to \Omega$ is specified by the roots of $f(x)$, and thus determines $f(x)$ and hence determines the field $L = k[x]/f(x)$.

One finds a finite Galois extension $L$ of $N$ by adjoining all roots of $f(x)$ to $N$. Then $L/N$ is Galois with Galois group $H = G(L/N)$ which is a normal subgroup of $G = G(L/k)$. The set of $k$-linear imbeddings $N \to \Omega$ can be identified with the set $G/H$.

It follows that there is a one-to-one correspondence

$$\{\text{finite separable } L/k\} \leftrightarrow \{G_k\text{-sets } G/H,\ G = G(L/k) \text{ finite},\ H \trianglelefteq G\}$$

This correspondence determines an isomorphism of categories

$$fet|_k \cong G_k - \mathbf{Set}_{df}.$$

If $k \subset L \subset N$ are finite separable extensions in $\Omega$, then the function

$$\hom_k(N, \Omega) \to \hom_k(L, \Omega)$$

is surjective, while the scheme homomorphism $\mathrm{Sp}(N) \to \mathrm{Sp}(L)$ is an étale cover.

For a general profinite group $G$, the category $G - \mathbf{Set}_{df}$ has a Grothendieck topology for which the covering families are the $G$-equivariant surjections $U \to V$.

A presheaf $F$ is a sheaf for this topology if and only if $F(\emptyset)$ is a point, and every surjection $\phi = (\phi_i) : \sqcup\, U_i \to V$ (covering family) induces a coequalizer

$$F(V) \to \prod_i F(U_i) \rightrightarrows \prod_{i \neq j} F(U_i \times_V U_j). \tag{9}$$

The resulting sheaf category

$$\mathcal{B}G := \mathbf{Shv}(G - \mathbf{Set}_{df})$$

is the often called the *classifying topos* for the profinite group $G$.

**Lemma 5.** *A presheaf $F$ on $G - \mathbf{Set}_{df}$ is a sheaf if and only if*

*1) $F$ takes disjoint unions to products, and*

*2) each canonical map $G_i \to G_i/H$ induces a bijection*

$$F(G_i/H) \xrightarrow{\cong} F(G_i)^H.$$

The assertion that a presheaf $F$ takes disjoint unions to products is often called the *additivity* condition for $F$.

*Proof.* If $F$ is a sheaf, then the covering given by the inclusions $U \to U \sqcup V$ and $V \to U \sqcup V$ defines an isomorphism

$$F(U \sqcup V) \xrightarrow{\cong} F(U) \times F(V),$$

since $U \times_{U \sqcup V} V = \emptyset$. Also, if $H \subset G_i$ is a subgroup then $G_i \times_{G_i/H} G_i \cong \sqcup_H G_i$, and the coequalizer

$$F(G_i/H) \to F(G_i) \rightrightarrows \prod_H F(G_i)$$

identifies $F(G_i/H)$ with the set of $H$-invariants $F(G_i)^H$.

Conversely, if the presheaf $F$ satisfies conditions 1) and 2) and $G_i/K \to G_i/H$ is an equivariant map, then $H$ is a subgroup of $K$ and $F(G)^K$ is the set of $K$-invariants of $F(G)^H$, so that the $G$-equivariant covering $G_i/K \to G_i/H$ defines an equalizer of the form (9). □

It follows from Lemma 5 that every discrete $G$-set $F$ represents a sheaf

$$F := \hom(\ , F)$$

on $G - \mathbf{Set}_{df}$.

Let

$$\pi : G - \mathbf{Set}_{df} \to \mathbf{Set}$$

be the functor which takes a finite discrete $G$-set to its underlying set. Every set $X$ represents a sheaf $\pi_* X$ on $G - \mathbf{Set}_{df}$ with

$$\pi_* X(U) = \hom(\pi(U), X).$$

The left adjoint $\pi^*$ of the corresponding functor $\pi_*$ has the form

$$\pi^* F = \varinjlim_i \; F(G_i),$$

by a cofinality argument.

A map $f : F \to G$ of presheaves is a *local epimorphism* if, given $y \in G(U)$ there is a covering $\phi : V \to U$ such that $\phi^*(y)$ is in the image of $f : F(V) \to G(V)$.

The presheaf map $f : F \to G$ is a *local monomorphism* if, given $x, y \in F(U)$ such that $f(x) = f(y)$ there is a covering $\phi : V \to U$ such that $\phi^*(x) = \phi^*(y)$ in $F(V)$.

It is a general fact that a morphism $f : F \to G$ of sheaves is an isomorphism if and only if it is both a local monomorphism and a local epimorphism.

Finally, one can show that a map $f : F \to G$ of sheaves on $G - \mathbf{Set}_{df}$ is a local epimorphism (respectively local monomorphism) if and only the induced function

$$\pi^*(f) : \pi^*(F) \to \pi^*(G)$$

is surjective (respectively injective). It follows that $f$ is an isomorphism if and only the function $\pi^*(f)$ is bijective.

We have a functor $\pi^*$ which is both exact (ie. preserves finite limits) and is faithful. This means that the corresponding geometric morphism

$$\pi = (\pi^*, \pi_* : \mathbf{Shv}(G - \mathbf{Set}_{df}) \to \mathbf{Set}$$

is a stalk (or Boolean localization) for the category of sheaves and presheaves on $G - \mathbf{Set}_{df}$, and gives a complete description of the local behaviour of sheaves and presheaves on this site.

We use these observations to start up a homotopy theoretic machine [13]. A map $f : X \to Y$ of simplicial presheaves (or simplicial sheaves) on $G - \mathbf{Set}_{df}$ is a *local weak equivalence* if and only if the induced map $\pi^* X \to \pi^* Y$ is a weak equivalence of simplicial sets.

The local weak equivalences are the weak equivalences of the *injective model structure* on the simplicial presheaf category for the site $G - \mathbf{Set}_{df}$. The *cofibrations* are the monomorphisms of simplicial presheaves (or simplicial sheaves). The fibrations for this structure, also called the *injective fibrations*, are the maps which have the right lifting property with respect to all cofibrations which are local weak equivalences.

There are two model structures here, for the category $s\,\mathrm{Pre}(G - \mathbf{Set}_{df})$ of simplicial presheaves and the category $s\mathbf{Shv}(G - \mathbf{Set}_{df})$ of simplicial sheaves, respectively. The forgetful and associated sheaf functors determine an adjoint pair of functors

$$L^2 : s\operatorname{Pre}(G - \mathbf{Set}_{df}) \leftrightarrows s\mathbf{Shv}(G - \mathbf{Set}_{df}) : u,$$

which is a Quillen equivalence, essentially since the canonical associated sheaf map $\eta : X \to L^2X = uL^2(X)$ is a local isomorphism, and hence a local weak equivalence.

The associated sheaf functor $L^2$ usually has a rather formal construction, but in this case there is a nice description:

$$L^2F(G_i/H) = \varinjlim_{G_j \to G_i} F(G_j)^{p^{-1}(H)},$$

where $p : G_j \to G_i$ varies over the transition maps of $G$ which take values in $G_i$.

Here's a trick: suppose that $f : E \to F$ is a function, and form the groupoid $E/f$ whose objects are the elements of $S$, and such that there is a unique morphism $x \to y$ if $f(x) = f(y)$. The corresponding nerve $B(E/f)$ has contractible path components, since each path component is the nerve of a trivial groupoid, and there is an isomorphism $\pi_0 B(E/f) \cong f(E)$. It follows that there are simplicial set maps

$$B(E/f) \xrightarrow{\simeq} f(E) \subset F,$$

where the sets $f(E)$ and $F$ are identified with discrete simplicial sets. In particular, if $f$ is surjective then the map $B(E/f) \to F$ is a weak equivalence.

This construction is functorial, and hence applies to presheaves and sheaves. In particular, suppose that $\phi : V \to U$ is a local epimorphism of presheaves. Then the simplicial presheaf map

$$\check{C}(V) := B(E/\phi) \to U$$

is a local weak equivalence of simplicial presheaves, because $B(E/\phi) \to \phi(V)$ is a sectionwise hence local weak equivalence and $\phi(V) \to U$ induces an isomorphism of associated sheaves.

As the notation suggests, $\check{C}(V)$ is the Čech resolution for the covering $\phi$. All Čech resolutions arise from this construction.

**Examples**: 1) Suppose that $G = \{G_i\}$ is a profinite group. The one-point set $*$ is a terminal object of the category $G = \mathbf{Set}_{df}$. The group $G_i$ defines a covering $G_i = \hom(\ ,G_i) \to *$ of the terminal object, while the group $G_i$ acts on the sheaf $G_i = \hom(\ ,G_i)$ by composition. There is a simplicial presheaf map

$$\eta : EG_i \times_{G_i} G_i \to \check{C}(G_i)$$

which takes a morphism $\phi \to g \cdot \phi$ to the pair $(\phi, g \cdot \phi)$. The map $\eta$ induces an isomorphism in sections corresponding to quotients $G_j/H$ for $j \geq i$, hence in stalks, and is therefore the associated sheaf map and a local weak equivalence.

2) Suppose that $L/k$ is a finite Galois extension with Galois group $G$, and let $\mathrm{Sp}(L) \to *$ be the corresponding sheaf epimorphism on the finite étale site for $k$. The Galois group $G$ acts on $\mathrm{Sp}(L)$, and there is a canonical map

$$\eta : EG \times_G \mathrm{Sp}(L) \to \check{C}(L).$$

For a finite separable extension $N/k$, the sections $\mathrm{Sp}(L)(N)$ are the $k$-linear field maps $L \to N$. Any two such maps determine a commutative diagram

$$\begin{array}{ccc} L & & \\ {\scriptstyle\sigma}\downarrow & \searrow & \\ & & N \\ & \nearrow & \\ L & & \end{array}$$

where $\sigma$ is a uniquely determined element of the Galois group $G$. It follows that $\eta$ is an isomorphism in sections corresponding to all such extensions $N$, and so $\eta$ is the associated sheaf map for the simplicial presheaf $EG \times_G \mathrm{Sp}(L)$, and is a local weak equivalence of simplicial presheaves for the étale topology.

**Remark 6.** Every category of simplicial presheaves has an auxiliary model structure which is defined by cofibrations as above and sectionwise weak equivalences. A map $X \to Y$ is a sectionwise weak equivalence if all induced maps $X(U) \to Y(U)$ in sections are weak equivalences of simplicial sets for all objects $U$ in the underlying site. This model structure is a special case of the injective model structure for simplicial presheaves on a site, for the so-called chaotic topology [13, Ex. 5.10].

The fibrations for this model structure will be called *injective fibrations of diagrams* in what follows. These are the maps which have the right lifting property with respect to all cofibrations $A \to B$ which are sectionwise weak equivalences.

Every injective fibration of simplicial presheaves is an injective fibration of diagrams, since every sectionwise weak equivalence is a local weak equivalence. The converse is not true.

In all that follows, an *injective fibrant model* of a simplicial presheaf $X$ is a local weak equivalence $j : X \to Z$ such that $Z$ is injective fibrant.

Every simplicial presheaf $X$ has an injective fibrant model: factorize the canonical map $X \to *$ to the terminal object as a trivial cofibration $j : X \to Z$, followed by an injective fibration $Z \to *$.

Here is an example: if $F$ is a presheaf, identified with a simplicial presheaf which is discrete in the simplicial direction, then the associated sheaf map $\eta : F \to \tilde{F}$ is an injective fibrant model.

Any two injective fibrant models of a fixed simplicial presheaf $X$ are equivalent in a very strong sense — they are homotopy equivalent.

In effect, every local weak equivalence $Z_1 \to Z_2$ of injective fibrant objects is a homotopy equivalence, for the cylinder object that is defined by the standard 1-simplex $\Delta^1$. It follows that all simplicial set maps $Z_1(U) \to Z_2(U)$ are homotopy equivalences.

In particular, every local weak equivalence of injective fibrant objects is a sectionwise equivalence.

The injective model structure on the simplicial presheaf category $s\,\mathrm{Pre}(G-\mathbf{Set}_{df})$ is a simplicial model structure, where the function complex $\mathbf{hom}(X,Y)$ has $n$-simplices given by the maps $X \times \Delta^n \to Y$.

All simplicial presheaves are cofibrant. It follows that, if $Z$ is an injective fibrant simplicial presheaf and the map $\theta : A \to B$ is a local weak equivalence, then the induced map

$$\theta^* : \mathbf{hom}(B,Z) \to \mathbf{hom}(A,Z)$$

is a weak equivalence of simplicial sets.

# 2 Cosimplicial spaces

We shall use the Bousfield-Kan model structure for cosimplicial spaces [3], [12]. The weak equivalences for this structure are defined sectionwise: a map $f : X \to Y$ is a weak equivalence of cosimplicial spaces if and only if all maps $X^n \to Y^n$ are weak equivalences of simplicial sets. The fibrations for the structure are those maps $p : X \to Y$ for which all induced maps

$$(p,s) : X^{n+1} \to Y^{n+1} \times_{M^nY} M^nX$$

are fibrations of simplicial sets. Recall that $M^nX$ is the subcomplex of $\prod_{j=0}^n X^n$ which consists of those elements $(x_0,\dots,x_n)$ such that $s^jx_i = s^ix_{j+1}$ for $i \leq j$, and the canonical map $s : X^{n+1} \to M^nX$ is defined by

$$s(x) = (s^0x, s^1x, \dots, s^nx).$$

The total complex $\mathrm{Tot}(X)$ for a fibrant cosimplicial space $X$ is defined by

$$\mathrm{Tot}(X) = \mathbf{hom}(\Delta, X),$$

where $\mathbf{hom}(\Delta,X)$ is the standard presheaf-theoretic function complex, and $\Delta$ is the cosimplicial space of standard simplices, given by the assignments $\mathbf{n} \mapsto \Delta^n$. The $p$-simplices of $\mathbf{hom}(\Delta,X)$ are the cosimplicial space maps $\Delta \times \Delta^p \to X$.

If $X$ and $U$ are simplicial presheaves, write $\mathrm{hom}(U_\bullet, X)$ for the cosimplicial space $\mathbf{n} \mapsto \mathrm{hom}(U_n, X)$. If $U$ is representable by a simplicial object $U$ in the underlying site, then $\mathrm{hom}(U_\bullet, X)$ can be identified up to isomorphism with the cosimplicial space $\mathbf{n} \mapsto X(U_n)$.

There are adjunction isomorphisms

$$\begin{aligned}\mathrm{hom}(\Delta \times \Delta^p, \mathrm{hom}(U_\bullet, X)) &\cong \mathrm{hom}(U, \mathbf{hom}(\Delta^p, X)) \\ &\cong \mathrm{hom}(U \times \Delta^p, X),\end{aligned}$$

which relate cosimplicial space maps to simplicial set maps. Letting $k$ vary gives a natural isomorphism

$$\mathrm{Tot}(\hom(U_\bullet, X)) = \mathbf{hom}(\Delta, \hom(U_\bullet, X)) \cong \mathbf{hom}(U, X)$$

of simplicial sets, for all simplicial presheaves $U$ and $X$.

**Lemma 7.** *Suppose that $U$ is a simplicial presheaf. Then the functor $X \mapsto \hom(U_\bullet, X)$ takes injective fibrations of diagrams to Bousfield-Kan fibrations of cosimplicial spaces.*

*Proof.* There is an isomorphism

$$M^n \hom(U_\bullet, X) \cong \hom(DU_{n+1}, X^{n+1}),$$

where $DU_{n+1} \subset U_{n+1}$ is the degenerate part of $U_{n+1}$ in the presheaf category. Suppose that $p : X \to Y$ is an injective fibration. Then $p$ has the right lifting property with respect to the trivial cofibrations

$$(U_{n+1} \times \Lambda^m_k) \cup (DU_{n+1} \times \Delta^m) \subset U_{n+1} \times \Delta^m,$$

so that the map

$$\hom(U_{n+1}, X) \to \hom(U_{n+1}, Y) \times_{\hom(DU_{n+1}, Y)} \hom(DU_{n+1}, X)$$

is a fibration. ☐

**Remark 8.** If $Y$ is a Bousfield-Kan fibrant cosimplicial space then there is a weak equivalence

$$\mathrm{Tot}\, Y = \mathbf{hom}(\Delta, Y) \simeq \underleftarrow{\mathrm{holim}}_n\ Y^n,$$

which is natural in $Y$.

This is most easily seen by using the injective model structure for cosimplicial spaces (ie. for cosimplicial diagrams) of Remark 6 (see also [12]).

In effect, if $j : Y \to Z$ is an injective fibrant model for $Y$ in cosimplicial spaces, then $j$ is a weak equivalence of Bousfield-Kan fibrant objects, so the map $j_* : \mathrm{Tot}(Y) \to \mathrm{Tot}(Z)$ is a weak equivalence. It follows that there is a natural string of weak equivalences

$$\mathrm{Tot}(Y) \xrightarrow{\simeq} \mathrm{Tot}(Z) = \mathbf{hom}(\Delta, Z) \xleftarrow{\simeq} \mathbf{hom}(*, Z) = \underleftarrow{\lim}\ Z =: \underleftarrow{\mathrm{holim}}\ Y,$$

since the cosimplicial space $\Delta$ is cofibrant for the Bousfield-Kan structure.

It follows from Lemma 7 that if $U$ and $Z$ are simplicial presheaves such that $Z$ is injective fibrant, then there is a natural weak equivalence

$$\mathbf{hom}(U, Z) \simeq \underleftarrow{\mathrm{holim}}_n\ \hom(U_n, Z).$$

**Examples**: 1) Suppose that $L/k$ is a finite Galois extension with Galois group $G$, and let $Y$ be a presheaf of Kan complexes for the finite étale site over $k$. The function complex

$$\mathbf{hom}(EG \times_G \mathrm{Sp}(L), Y)$$

can be rewritten as a homotopy inverse limit

$$\underleftarrow{\text{holim}}_n\, Y(\sqcup_{G^{\times n}}\ \text{Sp}(L)) = \underleftarrow{\text{holim}}_n(\prod_{G^{\times n}} Y(L)) = \underleftarrow{\text{holim}}_G\, Y(L) = Y(L)^{hG},$$

which is the homotopy fixed points space for the action of $G$ on the space $Y(L)$ of $L$-sections of $Y$.

2) Similarly, if $G = \{G_i\}$ is a profinite group and $X$ is a presheaf of Kan complexes on $G - \mathbf{Set}_{df}$, then

$$\mathbf{hom}(EG_i \times_{G_i} G_i, X) \simeq \underleftarrow{\text{holim}}_{G_i}\, X(G_i) = X(G_i)^{hG_i}$$

is the homotopy fixed points space for the action of the group $G_i$ on the space $X(G_i)$.

We prove the following:

**Theorem 9.** *Suppose that $f : X \to Y$ is a local weak equivalence between presheaves of Kan complexes on the site $G - \mathbf{Set}_{df}$ such that $X$ and $Y$ have only finitely many non-trivial presheaves of homotopy groups. Then the induced map*

$$f_* : \varinjlim_i\ \mathbf{hom}(EG_i \times_{G_i} G_i, X) \to \varinjlim_i\ \mathbf{hom}(EG_i \times_{G_i} G_i, Y)$$

*is a weak equivalence.*

**Corollary 10.** *Suppose that $X$ is a presheaf of Kan complexes on $G - \mathbf{Set}_{df}$ which has only finitely many non-trivial homotopy groups, and let $j : X \to Z$ be an injective fibrant model. Then the induced map of simplicial sets weak equivalences*

$$\varinjlim_i\ \mathbf{hom}(EG_i \times_{G_i} G_i, X) \xrightarrow[j_*]{\simeq} \varinjlim_i\ \mathbf{hom}(EG_i \times_{G_i} G_i, Z)$$

*is a weak equivalence.*

*Proof.* If the presheaves of homotopy groups $\pi_i X$ are trivial for $i \geq N$, then the homotopy groups $\pi_i Z$ are trivial for $i \geq N$ — this is a special case of a very general fact [10, Prop. 6.11]. Thus, $j_*$ is a weak equivalence by Theorem 9. □

*Proof of Theorem 9.* We can suppose that $X$ and $Y$ are injective fibrant *as diagrams* on $G - \mathbf{Set}_{df}$ and that $f : X \to Y$ is an injective fibration of diagrams. By Lemma 7, all induced maps

$$f : \text{hom}((EG_i \times_{G_i} G_i)_\bullet, X) \to \text{hom}((EG_i \times_{G_i} G_i)_\bullet, Y)$$

are Bousfield-Kan fibrations of Bousfield-Kan fibrant cosimplicial spaces, and we want to show that the induced map

$$\varinjlim_i \text{Tot}\,\text{hom}((EG_i \times_{G_i} G_i)_\bullet, X) \to \varinjlim_i \text{Tot}\,\text{hom}((EG_i \times_{G_i} G_i)_\bullet, Y)$$

is a trivial fibration of simplicial sets.

The idea is to show that all lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \operatorname{Tot}\hom((EG_i \times_{G_i} G_i)_\bullet, X) \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & \operatorname{Tot}\hom((EG_i \times_{G_i} G_i)_\bullet, Y) \end{array}$$

can be solved in the filtered colimit. This is equivalent to the solution of all cosimplicial space lifting problems

$$\begin{array}{ccc} & & \hom((EG_i \times_{G_i} G_i)_\bullet, X^{\Delta^n}) \\ & \nearrow & \downarrow \\ \Delta & \longrightarrow & \hom((EG_i \times_{G_i} G_i)_\bullet, X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}) \end{array}$$

in the filtered colimit.

The induced map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

is an injective fibration of injective fibrant diagrams which is a local weak equivalence, between objects which have only finitely many non-trivial presheaves of homotopy groups. It therefore suffices to show that all lifting problems

$$\begin{array}{ccc} & & \hom((EG_i \times_{G_i} G_i)_\bullet, X) \\ & \nearrow & \downarrow f \\ \Delta & \xrightarrow[\alpha]{} & \hom((EG_i \times_{G_i} G_i)_\bullet, Y) \end{array} \tag{10}$$

can be solved in the filtered colimit, for maps $f : X \to Y$ which are locally trivial injective fibrations of diagrams between injective fibrant objects, which objects have only finitely many non-trivial presheaves of homotopy groups.

Suppose that $p : Z \to W$ is a locally trivial fibration of simplicial presheaves on $G - \mathbf{Set}_{df}$, and suppose given a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z((EG_i \times_{G_i} G_i)_n) \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & W((EG_i \times_{G_i} G_i)_n) \end{array} \tag{11}$$

There is a surjection

$$U \to (EG_i \times_{G_i} G_i)_n = \sqcup_{G_i^{\times n}} G_i$$

of finite discrete $G$-sets such that the lift exists in the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z((EG_i \times_{G_i} G_i)_n) \arrow[r] & Z(U) \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[urr] & W((EG_i \times_{G_i} G_i)_n) \arrow[r] & W(U) \end{tikzcd}$$

There is a transition morphism $\gamma : G_j \to G_i$ in the pro group $G$ and a discrete $G$-sets morphism $\sqcup_{G_j^{\times n}} G_j \to U$ such that the composite

$$\sqcup_{G_j^{\times n}} G_j \to U \to \sqcup_{G_i^{\times n}} G_i$$

is the $G$-sets homomorphism which is induced by $\gamma$. It follows that the lifting problem (11) has a solution

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z((EG_i \times_{G_i} G_i)_n) \arrow[r, "\gamma^*"] & Z((EG_j \times_{G_j} G_j)_n) \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[urr] & W((EG_i \times_{G_i} G_i)_n) \arrow[r, "\gamma^*"'] & W(EG_j \times_{G_j} G_j)_n) \end{tikzcd}$$

after refinement along the induced map $\gamma : EG_j \times_{G_j} G_j \to EG_i \times_{G_i} G_i$.

All induced maps

$$f : \hom((EG_i \times_{G_i} G_i)_\bullet, X) \to \hom((EG_i \times_{G_i} G_i)_\bullet, Y)$$

are Bousfield-Kan fibrations of cosimplicial spaces by Lemma 7, as is their filtered colimit

$$f_* : \varinjlim_i \hom((EG_i \times_{G_i} G_i)_\bullet, X) \to \varinjlim_i \hom((EG_i \times_{G_i} G_i)_\bullet, Y).$$

The map $f_*$ is a weak equivalence of cosimplicial spaces by the previous paragraph, and is therefore a trivial fibration.

In general, solving the lifting problem

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ \Delta \arrow[r, "\alpha"'] \arrow[ur, dotted] & W \end{tikzcd}$$

for a map of cosimplicial spaces $p : Z \to W$ amounts to inductively solving a sequence of lifting problems

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r] \arrow[d] & Z_{n+1} \arrow[d] \\ \Delta^{n+1} \arrow[r] \arrow[ur, dotted] & Y_{n+1} \times_{M^n Y} M^n Z \end{tikzcd} \tag{12}$$

It follows from the paragraphs above that, given a number $N \geq 0$, there is a structure map $\gamma : G_j \to G_i$ for the pro group $G$, such that the lifting problems (12) associated to lifting a specific map

$$\alpha : \Delta \to \hom((EG_i \times_{G_i} G_i)_\bullet, Y)$$

to the total space of the map

$$f_* : \hom((EG_i \times_{G_i} G_i)_\bullet, X) \to \hom((EG_i \times_{G_i} G_i)_\bullet, Y)$$

have a simultaneous solution in $\hom((EG_j \times_{G_j} G_j)_\bullet, X)$ for $n \leq N$.

If the presheaves of homotopy groups $\pi_k X$ and $\pi_k Y$ are trivial for $k \geq N$, then $\hom((EG_j \times_{G_j} G_j, X)_\bullet, X)$ and $\hom((EG_j \times_{G_j} G_j, X)_\bullet, Y)$ satisfy the conditions of Lemma 11 below. The obstructions to the lifting problem (12) for $f_*$ lie in $\pi_k X(\sqcup_{G_j^{\times(k+1)}} G_j)$ and in

$$\pi_{k+1}((\sqcup_{G_j^{\times(k+1)}} G_j) \times_{M^k \hom((EG_j \times_{G_j} G_j)_\bullet, Y)} M^k \hom((EG_j \times_{G_j} G_j)_\bullet, X),$$

which groups are 0 since $k \geq N$.

It follows that, given a lifting problem (10), there is a structure homomorphism $\gamma : G_j \to G_i$ of the pro group $G$ such that the problem (10) is solved over $G_j$ in the sense that there is a commutative diagram

$$\begin{array}{ccccc} & & & & \hom((EG_j \times_{G_j} G_j)_\bullet, X) \\ & & & \nearrow & \downarrow f \\ \Delta & \xrightarrow{\alpha} & \hom((EG_i \times_{G_i} G_i)_\bullet, Y) & \xrightarrow{\gamma^*} & \hom((EG_j \times_{G_j} G_j)_\bullet, Y) \end{array}$$

□

**Lemma 11.** *Suppose that $p : X \to Y$ is a Bousfield-Kan fibration between Bousfield-Kan fibrant cosimplicial spaces. Suppose that the homotopy groups $\pi_k X^n$ and $\pi_k Y^n$ are trivial for $k \geq N$ and for all $n \geq 0$. Then the groups $\pi_k(M^n X \times_{M_n Y} Y^{n+1})$ are trivial for all $k \geq N$ and for all $n \geq 0$.*

*Proof.* The assertion that the homotopy groups $\pi_k X^n$ are trivial for $k \geq N$ means that $\pi_k(X^n, x) = 0$ for all choices of base point $x \in X^n$ and for all $k \geq N$.

Recall that $M^n X = M^n_n X$, where $M^n_p X$ is the iterated pullback of the maps $s^i : X^{n+1} \to X^n$ for $i \leq p$. Let $s : X^{n+1} \to M^n_p X$ be the map $(s^0, \dots, s^p)$. There are natural pullback diagrams

$$\begin{array}{ccc} M^n_p X & \longrightarrow & X^n \\ \downarrow & & \downarrow s \\ M^n_{p-1} X & \longrightarrow & M^{n-1}_{p-1} X \end{array}$$

The map $s : X^{n+1} \to M^n X = M^n_n X$ is a fibration, so that all of the maps $s : X^n \to M^n_p X$ are fibrations by an inductive argument.

Inductively, if all $\pi_k M^{n-1}_{p-1} X$ and $\pi_k M^n_{p-1} X$ are trivial for $k \geq N$, then $\pi_k M^n_p X$ is trivial for $k \geq N$. It follows that $\pi_k M^n X$ is trivial for $k \geq N$.

In the pullback diagram

$$\begin{array}{ccc} M^n X \times_{M^n Y} Y^{n+1} & \longrightarrow & Y^{n+1} \\ \downarrow & & \downarrow s \\ M^n X & \longrightarrow & M^n Y \end{array}$$

the map $s$ is a fibration and the groups $\pi_k X^{n+1}$, $\pi_k M^n X$ and $\pi_k M^n Y$ are trivial for $k \leq N$, and it follows that $\pi_k(M^n X \times_{M^n Y} Y^{n+1})$ is trivial for $k \geq N$. $\square$

# 3 Pro objects

Suppose that $X$ is a presheaf if Kan complexes on the site $G - \mathbf{Set}_{df}$ of discrete finite $G$-sets, where $G$ is a profinite group. Corollary 10 implies (see Corollary 15 below, or argue directly) that the space

$$\varinjlim_i \mathbf{hom}(EG_i \times_{G_i} G_i, X)$$

is weakly equivalent to the space $Z(*)$ of global sections of an injective fibrant model $Z$ of $X$, provided that the simplicial presheaf $X$ has only finitely many non-trivial presheaves of homotopy groups.

The main (and only) examples of such are the finite Postnikov sections $\mathbf{P}_n Y$ of a simplicial presheaf $Y$.

To be specific, every simplicial presheaf $Y$ has a sectionwise fibrant model $i : Y \to \mathrm{Ex}^\infty Y$, where the latter is constructed by applying Kan's $\mathrm{Ex}^\infty$-functor in all sections. Then

$$\mathbf{P}_n Y := P_n \,\mathrm{Ex}^\infty Y,$$

which is the result of applying the Moore Postnikov section functor $P_n$ [5, VI.3.4] sectionwise to the Kan complexes $\mathrm{Ex}^\infty Y(U)$, where $U$ varies through the underlying site.

To summarize, there are natural maps

$$Y(U) \xrightarrow{i} \mathrm{Ex}^\infty Y(U) \xrightarrow{p} P_n \,\mathrm{Ex}^\infty Y(U),$$

where $i$ is a weak equivalence.

The functor $P_n$ preserves weak equivalences of Kan complexes, so we can skip the derived step by assuming that $Y$ is a presheaf of Kan complexes, and write $\mathbf{P}_n Y = P_n Y$ for such objects.

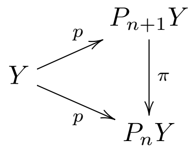
It is a basic property of Moore's construction [5, VI.3] that the map

$$p : Y(U) \to P_n Y(U)$$

is a Kan fibration which induces an isomorphism

$$\pi_k(Y(U), x) \xrightarrow{\cong} \pi_k(P_n Y(U)$$

for all vertices $x \in Y(U)$ and for $0 \leq k \leq n$. Furthermore $\pi_k(P_n Y(U), x) = 0$ for all vertices $x$ and $k \geq n$. The maps $p$ are arranged into a comparison diagram



in which all maps are sectionwise Kan fibrations. The tower $P_* Y$ of sectionwise fibrations is the Postnikov tower of the presheaf of Kan complexes $Y$.

A presheaf of Kan complexes $Y$ has only finitely many non-trivial presheaves of homotopy groups if and only if the map $p : Y \to P_n Y$ is a sectionwise weak equivalence for some $n$.

There are other, localization theoretic, constructions of the Postnikov tower which can be more useful, particularly for spectra — see [13].

Recall that a *pro-object* in a category $\mathcal{C}$ is a functor $I \to \mathcal{C}$, where $I$ is a small left filtered category.

**Examples**: 1) A profinite group is a pro-object $G : J \to \mathbf{Grp}$ in groups such that each $G_j = G(j)$ is a finite group. As noted in the first section, we also insist that all maps $i \to j$ in $J$ induce surjective group homomorphisms $G_i \to G_j$.

2) If $X$ is a simplicial presheaf, then associated the Postnikov tower $\mathbf{P}_* X$ is a pro-object in simplicial presheaves.

Every pro object $E : I \to \mathcal{C}$ in a category $\mathcal{C}$ represents a functor $h_E : \mathcal{C} \to \mathbf{Set}$, with

$$h_E(X) = \varinjlim_i \hom(E_i, X).$$

A *pro-map* $E \to F$ is a natural transformation $h_F \to h_E$.

Every object $Z$ in the category $C$ is a pro-object, defined on the one-point category. A Yoneda Lemma argument shows that a natural transformation $h_Z \to h_E$ can be identified with an element of the filtered colimit $\varinjlim_i \hom(E_i, Z)$, and we usually think of pro-maps $E \to Z$ in this way.

If $F : J \to \mathcal{C}$ is a pro-object and $i \in J$, then there is a pro-map $F \to F_i$ which is defined by the image of the identity on $F_i$ in the filtered colimit $\varinjlim_j \hom(F_j, F_i)$.

Any pro-map $\phi : E \to F$ can be composed with the canonical maps $F \to F_i$, and the map $\phi$ can be identified with an element of the set

$$\varprojlim_j \varinjlim_i \hom(E_i, F_j).$$

Every simplicial presheaf is a pro-object in simplicial presheaves, and the Postnikov tower construction

$$\begin{array}{ccc} & & \mathbf{P}_{n+1}X \\ & \nearrow & \downarrow \\ X & & \\ & \searrow & \\ & & \mathbf{P}_n X \end{array}$$

defines a natural pro-map $X \to \mathbf{P}_* X$.

There is a hierarchy of model structures for the category of pro-simplicial presheaves, which is developed in [11].

The "base" model structure is the Edwards-Hastings model structure, for which a cofibration $A \to B$ is map that is isomorphic in the pro category to a monomorphism in a category of diagrams. A weak equivalence for this structure, which is said to be an *Edwards-Hastings weak equivalence*, is a map $f : X \to Y$ of pro objects (that are defined on filtered categories $I$ and $J$, respectively) such that the induced map of filtered colimits

$$\varinjlim_{j \in J} \mathbf{hom}(Y_j, Z) \to \varinjlim_{i \in I} \mathbf{hom}(X_i, Z)$$

is a weak equivalence of simplicial sets for all injective fibrant objects $Z$.

Every pro-simplicial presheaf has a functorially defined Postnikov tower $\mathbf{P}_* X$, which is again a pro-object, albeit with a larger indexing category.

It is shown in [11] that the functor $X \mapsto \mathbf{P}_* X$ satisfies the criteria for Bousfield-Friedlander localization within the Edwards-Hastings model structure, and thus behaves somewhat like stabilization of spectra. In particular, one has a model structure for which the weak equivalences (the *pro equivalences*) are those maps $X \to Y$ which induce an Edwards-Hastings equivalence $\mathbf{P}_* X \to \mathbf{P}_* Y$. This is the *pro-equivalence structure* for pro-simplicial presheaves. It has the same cofibrations as the Edwards-Hastings structure.

The Edwards-Hastings structure and the pro-equivalence structure both specialize to model structures for pro objects in simplicial sets. The special case of the Edwards-Hastings structure for simplicial sets was first constructed by Isaksen in [9], where it is called the strict model structure.

We shall need the following:

**Lemma 12.** *Suppose that the map $f : Z \to W$ of simplicial presheaves is a pro equivalence. Then it is a local weak equivalence.*

**Corollary 13.** *Suppose that the map $f : Z \to W$ of simplicial sets is a pro equivalence. Then $f$ is a weak equivalence.*

*Proof of Lemma 12.* The natural map $Z \to \mathbf{P}_* Z$ induces an Edwards-Hastings weak equivalence

$$\mathbf{P}_n Z \to \mathbf{P}_n \mathbf{P}_* Z$$

for all $n \geq 0$. The induced map

$$\mathbf{P}_n\mathbf{P}_*Z \to \mathbf{P}_n\mathbf{P}_*W$$

is an Edwards-Hastings weak equivalence, since the Postnikov section functors preserve Edwards-Hastings equivalences (Lemma 25 of [11]). It follows that all simplicial presheaf maps

$$\mathbf{P}_nZ \to \mathbf{P}_nW$$

are Edwards-Hastings weak equivalences, and hence local weak equivalences of simplicial presheaves. This is true for all $n$, so the map $f : Z \to W$ is a local weak equivalence. □

# 4 Galois descent

Suppose again that $G = \{G_i\}$ is a profinite group, and let one of the groups $G_i$ represent a sheaf on the category $G - \mathbf{Set}_{df}$ of discrete finite modules.

Recall that the group $G_i$ acts on the sheaf $G_i$ which is represented by the $G$-set $G_i$, and the canonical map of simplicial sheaves $EG_i \times_{G_i} G_i \to *$ is a local weak equivalence, where $*$ is the terminal simplicial sheaf.

It follows that, if $Z$ is injective fibrant, then the induced map

$$\mathbf{hom}(*, Z) \to \mathbf{hom}(EG_i \times_{G_i} G_i, Z)$$

between function complexes is a weak equivalence of simplicial sets. The space $\mathbf{hom}(*, Z) = Z(*)$, and so we have a weak equivalence

$$Z(*) \xrightarrow{\simeq} \mathbf{hom}(EG_i \times_{G_i} G_i, Z)$$

between global sections of $Z$ and the homotopy fixed points for the action of $G_i$ on the simplicial set $Z(G_i)$. This is the *finite descent property* for injective fibrant simplicial presheaves $Z$.

More generally, if $X$ is a presheaf of Kan complexes on $G - \mathbf{Set}_{df}$, we say that $X$ *satisfies finite descent* if the induced map

$$X(*) \to \mathbf{hom}(EG_i \times_{G_i} G_i, X)$$

is a weak equivalence for each of the groups $G_i$ making up the profinite group $G$. We have just seen that all injective fibrant simplicial presheaves satisfy finite descent.

Recall (from Section 1) that, if $f : Z \to W$ is a local weak equivalence between injective fibrant objects, then $f$ is a sectionwise equivalence.

It follows that any two injective fibrant models $j : X \to Z$ and $j' : X \to Z'$ of a fixed simplicial presheaf $X$ are sectionwise equivalent.

To see this, we can assume that $j$ is a trivial cofibration, and then construct an extension $\theta$ so that the diagram

$$\begin{array}{ccc} & & Z \\ & \overset{j}{\nearrow} & \\ X & & \Big\downarrow \theta \\ & \underset{j'}{\searrow} & \\ & & Z' \end{array}$$

commutes. Then $\theta$ is a local weak equivalence between injective fibrant objects, and must therefore be a sectionwise equivalence.

One says that a simplicial presheaf $X$ *satisfies descent* if some (hence any) injective fibrant model $j : X \to Z$ is a sectionwise equivalence.

The general relationship between descent and finite descent is the following:

**Lemma 14.** *Suppose that the presheaf of Kan complexes $X$ on $G - \mathbf{Set}_{df}$ satisfies descent. Then it satisfies finite descent.*

*Proof.* Take an injective fibrant model $j : X \to Z$, and form the diagram

$$\begin{array}{ccc} X(*) & \xrightarrow[\simeq]{j} & Z(*) \\ \Big\downarrow & & \Big\downarrow \simeq \\ \mathbf{hom}(EG_i \times_{G_i} G_i, X) & \xrightarrow[j_*]{} & \mathbf{hom}(EG_i \times_{G_i} G_i, X) \end{array}$$

The map $j_*$ coincides with the map

$$\underleftarrow{\mathrm{holim}}_{G_i}\ X(G_i) \to \underleftarrow{\mathrm{holim}}_{G_i}\ Z(G_i)$$

of homotopy fixed point spaces which is defined by the $G_i$-equivariant weak equivalence $X(G_i) \to Z(G_i)$, and is therefore a weak equivalence. It follows that the map

$$X(*) \to \mathbf{hom}(EG_i \times_{G_i} G_i, X)$$

is a weak equivalence. □

It is unknown whether or not there is a converse to Lemma 14. The best statement of this kind that we have, for now, is the following consequence of Corollary 10:

**Corollary 15.** *Suppose that $X$ is a presheaf of Kan complexes on $G - \mathbf{Set}_{df}$ which has only finitely many non-trivial presheaves of homotopy groups, and suppose that $X$ satisfies finite descent. Suppose that $j : X \to Z$ is an injective fibrant model. Then the map $j : X(*) \to Z(*)$ in global sections is a weak equivalence.*

*Proof.* Form the diagram

$$\begin{array}{ccc} X(*) & \xrightarrow{\quad j \quad} & Z(*) \\ \downarrow & & \downarrow \\ \varinjlim_i \ \mathbf{hom}(EG_i \times_{G_i} G_i, X) & \xrightarrow[j_*]{\simeq} & \varinjlim_i \ \mathbf{hom}(EG_i \times_{G_i} G_i, Z) \end{array} \tag{13}$$

The map $j_*$ is a weak equivalence by Corollary 10. All maps

$$X(*) \to \mathbf{hom}(EG_i \times_{G_i} G_i, X) \text{ and } Z(*) \to \mathbf{hom}(EG_i \times_{G_i} G_i, Z)$$

are weak equivalences since $X$ and $Z$ satisfy finite descent, so the vertical maps in the diagram are weak equivalences. It follows that $j : X(*) \to Z(*)$ is a weak equivalence. □

The proof of Corollary 15 also implies the following:

**Corollary 16.** *Suppose that $X$ is a presheaf of Kan complexes on $G - \mathbf{Set}_{df}$ which has only finitely many non-trivial presheaves of homotopy groups. Suppose that $j : X \to Z$ is an injective fibrant model. Then the map $j : X(*) \to Z(*)$ is weakly equivalent to the map*

$$X(*) \to \varinjlim_i \ \mathbf{hom}(EG_i \times_{G_i} G_i, X).$$

We say, in general, that a simplicial presheaf $X$ on an arbitrary site *satisfies descent* if some (hence any) injective fibrant model $j : X \to Z$ is a sectionwise equivalence.

We can translate the finite descent concept to étale sites for fields: a presheaf of Kan complexes $X$ on the finite étale site $fet|_k$ of a field $k$ *satisfies finite descent* if, for any finite Galois extension $L/k$ with Galois group $G$, the local weak equivalence $EG \times_G \mathrm{Sp}(L) \to *$ induces a weak equivalence

$$X(k) \to \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X) = \varprojlim{}_G \mathrm{holim} \ X(L). \tag{14}$$

**Remark 17.** We have already seen the arguments for the following statements:

1) Every injective fibrant simplicial presheaf $Z$ on $fet|_k$ satisfies descent and satisfies finite descent.
2) If a presheaf of Kan complexes $X$ on $fet|_k$ satisfies descent, then it satisfies finite descent.

Theorem 9 and its corollaries also translate directly.

**Theorem 18.** *Suppose that $f : X \to Y$ is a local weak equivalence between presheaves of Kan complexes on the site $fet|_k$ such that $X$ and $Y$ have only finitely many non-trivial presheaves of homotopy groups. Then the induced map*

$$f_* : \varinjlim_{L/k} \ \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X) \to \varinjlim_{L/k} \ \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Y)$$

*is a weak equivalence.*

The colimits in the statement of Theorem 18 are indexed over finite Galois extensions $L/k$ in the algebraic closure $\overline{k}$, with Galois groups $G = G(L/k)$. Similar indexing will be used for all statements that follow.

**Corollary 19.** *Suppose that $X$ is a presheaf of Kan complexes on $fet|_k$ which has only finitely many non-trivial homotopy groups, and let $j : X \to Z$ be an injective fibrant model. Then the map $j$ induces a weak equivalence*

$$j_* : \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z).$$

**Corollary 20.** *Suppose that $X$ is a presheaf of Kan complexes on $fet|_k$ which has only finitely many non-trivial presheaves of homotopy groups, and suppose that $X$ satisfies finite descent. Suppose that $j : X \to Z$ is an injective fibrant model. Then the map $j : X(k) \to Z(k)$ in global sections is a weak equivalence.*

**Corollary 21.** *Suppose that $X$ is a presheaf of Kan complexes on $fet|_k$ which has only finitely many non-trivial presheaves of homotopy groups. Suppose that $j : X \to Z$ is an injective fibrant model. Then the map $j : X(k) \to Z(k)$ is weakly equivalent to the map*

$$X(k) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X).$$

Suppose now that $X$ is a presheaf of Kan complexes on the finite étale site $fet|_k$ of a field $k$. Let $j : X \to LX$ be a functorial choice of injective fibrant model for $X$. Let $\mathbf{P}_n X$ be the $n^{th}$ Postnikov section of $X$, with canonical maps $p : X \to P_n X$.

An example to keep in mind for $X$ is the mod $\ell$ $K$-theory presheaf $(\mathbf{K}/\ell)^n$, which is the "space" at level $n$ for the mod $\ell$ $K$-theory presheaf of spectra $\mathbf{K}/\ell$, where $\ell$ is a prime which is distinct from the characteristic of $k$.

As usual, we let $L/k$ be a finite Galois extension in $\overline{k}$, with Galois group $G$. Letting these extensions vary gives a commutative diagram

$$\begin{array}{ccc}
\varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), \mathbf{P}_n X) & \xrightarrow[\simeq]{j_*} & \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), L\mathbf{P}_n X) \\
\uparrow{\scriptstyle \alpha} & & \simeq \uparrow{\scriptstyle \alpha} \\
\mathbf{P}_n X(k) & \xrightarrow{j} & L\mathbf{P}_n X(k) \\
\uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p'} \\
X(k) & \xrightarrow[j]{} & LX(k)
\end{array} \tag{15}$$

in simplicial sets, where the indicated colimits are indexed on the finite Galois extensions $L/k$.

The indicated weak equivalence $\alpha$ is a filtered colimit of the weak equivalences

$$L\mathbf{P}_nX(k) \xrightarrow{\simeq} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), L\mathbf{P}_nX)$$

(see Remark 17). The map $j_*$ is a weak equivalence by Corollary 19.

The diagram (15) can be interpreted as a commutative diagram of pro objects in simplicial sets, which arises from a fibrant replacement $\mathbf{P}_nX \to L\mathbf{P}_nX$ of the Postnikov tower in the category of pro-simplicial presheaves [11]. This fibrant replacement is constructed by inductively finding local weak equivalences $j : \mathbf{P}_nX \to L\mathbf{P}_nX$ and injective fibrations $q : L\mathbf{P}_nX \to L\mathbf{P}_{n-1}X$ such that the diagrams

$$\begin{array}{ccc} \mathbf{P}_nX & \xrightarrow{j} & L\mathbf{P}_nX \\ \downarrow & & \downarrow q \\ \mathbf{P}_{n-1}X & \xrightarrow[j]{} & L\mathbf{P}_{n-1}X \end{array}$$

commute.

The map $p$ in (15) is a pro equivalence, by construction [11].

**Lemma 22.** *Suppose that $\ell$ is a prime with $\ell \neq char(k)$. Suppose that there is a uniform bound $N$ on the Galois cohomological dimension of $k$ with respect to $\ell$-torsion sheaves. Suppose that each of the presheaves $\pi_kX$ are $\ell^m$-torsion for some $m$. Then the map*

$$p' : LX(k) \to L\mathbf{P}_*X(k)$$

*is a pro equivalence.*

**Remark 23.** The uniform bound assumption implies that if $L/k$ is any finite separable extension and $x \in X(L)$ is a vertex, then $H^p_{et}(L, \tilde{\pi}_k(X|_L, x)) = 0$ for $p > N$.

Here, $X|_L$ is the restriction of the simplicial presheaf $X$ to the finite étale site of $L$. In effect, $\tilde{\pi}_k(X|_L, x)$ is an $\ell^m$-torsion sheaf, and the cohomological dimension of $L$ with respect to $\ell^m$-torsion sheaves is bounded above by that of $k$, by a Shapiro's Lemma argument [15, Sec 3.3].

The $K$-theory presheaves $(K/\ell)^n$ appearing in the mod $\ell$ $K$-theory presheaf of spectra have $\ell^2$-torsion presheaves of homotopy groups are standard (and motivating) examples.

*Proof of Lemma 22.* All presheaves $\mathbf{P}_nX = P_nX$ have the same presheaf of vertices, namely $X_0$, and there is a pullback diagram of simplicial presheaves

$$\begin{array}{ccc} K(\pi_nX, n) & \longrightarrow & P_nX \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & P_{n-1}X \end{array}$$

which defines the object $K(\pi_n X, n)$. In sections, the fibre of the map

$$K(\pi_n X, n)(U) \to X_0(U)$$

over the vertex $x \in X_0(U)$ is the space $K(\pi_n(X(U), x), n)$.

Form the diagram

$$\begin{array}{ccc} K(\pi_n X, n) & \xrightarrow[\simeq]{j} & LK(\pi_n X, n) \\ \downarrow & & \downarrow q \\ X_0 & \xrightarrow[j]{\simeq} & \tilde{X}_0 \end{array}$$

where the maps labelled by $j$ are injective fibrant models and $q$ is an injective fibration.

Suppose that $y \in LK(\pi_n X, n)(k)_0$. There is a finite separable extension $L/k$ such that $q(y) \in \tilde{X}_0(L)$ is in the image of the map $j : X_0(L) \to \tilde{X}_0(L)$, meaning that $q(y|_L) = j(z)$ for some $z \in X_0(L)$.

Form the pullback diagram

$$\begin{array}{ccc} q^{-1}(q(y)) & \longrightarrow & LK(\pi_n X, n) \\ \downarrow & & \downarrow q \\ * & \xrightarrow[q(y)]{} & \tilde{X}_0 \end{array}$$

Then

$$\pi_k(q^{-1}(q(y))(k), y) = \pi_k(LK(\pi_n X, n)(k), y).$$

The simplicial presheaf $q^{-1}(q(y))$ is injective fibrant, and has one non-trivial sheaf of homotopy groups, say $A$, in degree $n$. The sheaf $A$ is $\ell^m$-torsion, since its restriction to $fet|_L$ is the sheaf associated to the presheaf $\pi_n(X|_L, z)$, which is $\ell^m$-torsion.

It follows that

$$\pi_k LK(\pi_n X, n)(k), y) = \pi_k(q^{-1}(q(y))(k), y) \cong \begin{cases} H_{et}^{n-k}(k, A) & \text{if } k \leq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the homotopy groups $\pi_k LK(\pi_n X, n)(k), y)$ vanish for $k < n - N$.

It follows that the map

$$\varprojlim_m LP_m X(k) \to LP_n X(k)$$

induces a weak equivalence

$$P_r(\varprojlim_m LP_m X(k)) \to P_r(LP_n X(k))$$

for $n$ sufficiently large, and this is true for each $r$.

A similar statement holds for all finite separable extensions $L/k$, by the same argument: the map

$$\varprojlim_m LP_m X(L) \to LP_n X(L)$$

induces a weak equivalence

$$P_r(\varprojlim_m LP_m X(L)) \to P_r(LP_n X(L))$$

for $n$ sufficiently large, and this is true for each $r$.

There are several consequences:

1) The map $X \to \varprojlim_n L\mathbf{P}_n X$ is a local weak equivalence, since the map

$$\varprojlim_m LP_m X \to LP_n X$$

induces a sectionwise weak equivalence

$$P_r(\varprojlim_m LP_m X) \to P_r(LP_n X)$$

for $n$ sufficiently large, on account of the uniform bound.

2) The map $X \to \varprojlim_n L\mathbf{P}_n X$ is an injective fibrant model for $X$.

3) The map $X \to \varprojlim_n L\mathbf{P}_n X$ is a pro equivalence, because the map

$$Y := \varprojlim_n L\mathbf{P}_n X \to L\mathbf{P}_* X$$

is a pro-equivalence.

It follows that the induced function

$$\varinjlim [\mathbf{P}_k L\mathbf{P}_n X, Z] \to \varinjlim [\mathbf{P}_k Y, Z]$$

is a bijection. □

The existence of a global bound in Galois cohomological dimension of Lemma 22 is commonly met in practice, such as for the mod $\ell$ $K$-theory presheaves $(\mathbf{K}/\ell)^n$, when defined over fields $k$ that arise from finite dimensional objects in number theory and algebraic geometry — see [18].

Thus, in the presence of the global bound on cohomological dimension assumption of Lemma 22, we see that, *with the exception of* the maps $j : X(k) \to LX(k)$ and

$$\alpha : \mathbf{P}_n X(k) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), \mathbf{P}_n X),$$

the maps in the diagram (15) are pro equivalences.

The simplicial set map $X(k) \to G(X)(k)$ is a weak equivalence if and only if it is a pro equivalence, by Lemma 12. We have the following consequence:

**Theorem 24.** *Suppose that $X$ is a presheaf of Kan complexes on the finite étale site $fet|_k$ of a field $k$ such that the presheaves $\pi_k X$ are $\ell^n$-torsion for some $n$ and some prime $\ell$ which is distinct from the characteristic of $k$. Let $j : X \to LX$ be an injective fibrant model of $X$. Suppose that there is a uniform bound on the Galois cohomological dimension of $k$ with respect to $\ell$-torsion sheaves associated to each of the presheaves $\pi_k X$.*

*Then the induced map $j : X(k) \to LX(k)$ in global sections is a weak equivalence if and only if the map of towers*

$$\alpha : \mathbf{P}_n X(k) \to \varinjlim_{L/k} \mathbf{hom}(EG \times_G \mathrm{Sp}(L), \mathbf{P}_n X)$$

*is a pro equivalence in simplicial sets.*

**Remark 25.** The statement of Theorem 24 is only an illustration. One can refine the extension $\overline{k}/k$ into a sequence of Galois subextensions

$$k = L_0 \subset L_1 \subset \cdots \subset L_N = k_{sep}$$

such that each of the Galois extensions $L_{i+1}/L_1$ has Galois cohomological dimension 1 with respect to $\ell$-torsion sheaves — see Section 7.7 of [10]. Then there is a statement analogous to Theorem 24 for the finite Galois subextensions $L/L_i$ of $L_{i+1}/L_i$.

Historically, the use of this decomposition was meant to break up the problem of proving the Lichtenbaum-Quillen conjecture into proving descent statements in relative Galois cohomological dimension 1. This attack on the conjecture was never successfully realized.

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