The Barratt-Priddy-Quillen Theorem

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September 15, 2009

Introduction

The purpose of this note is to present a modern proof of the well known Barratt-Priddy-Quillen Theorem, which theorem asserts that the space $QS^0 = \Omega^\infty\Sigma^\infty S^0$ is weakly equivalent to the space

$$\bigsqcup_{\mathbb{Z}} (B\Sigma_\infty)^+.$$

Here, $(B\Sigma_\infty)^+$ is the effect of applying Quillen's plus construction to the classifying space of the infinite symmetric group Σ_∞ and $QS^0 = \Omega^\infty\Sigma^\infty S^0$ is the space at level 0 of the stably fibrant model of the sphere spectrum.

The proof is combinatorial and natural. It is based on the construction of an endo-functor Γ^* on the category of pointed simplicial sets. This functor Γ^* comes equipped with a natural transformation $X \to \Gamma^*(X)$ which induces a stable equivalence for the standard Γ-space model for a suspension spectrum. Applying this functor to the Γ-space corresponding to the sphere spectrum gives a special Γ-space whose corresponding spectrum has space at level one which is weakly equivalent to the simplicial monoid

$$\bigsqcup_{n\geq 0} B\Sigma_n$$

Then a standard group completion argument implies the Barratt-Priddy Theorem.

The functor Γ^* itself is constructed in this paper from a rather simple homotopy colimit which is easily manipulated with standard categorical techniques. As such, this construction is the real point of departure from the papers of Barratt and Priddy [1],[2]: there is no Γ-space theory in those papers, but they did use some very strongly related ideas.

The Barratt-Priddy-Quillen Theorem appears as Theorem 11 at the end of this paper.

1 Proof of the theorem

Write **Mon** for the category of monomorphisms $\theta : \underline{m} \to \underline{n}$ of finite sets. This category includes the empty set $\emptyset$, which is initial.

Suppose that X is a pointed simplicial set. Then X defines a functor

$$P_X : \mathbf{Mon} \to s\mathbf{Set}$$

with $\underline{m} \mapsto X^m$, and which takes a monomorphism $\theta : \underline{m} \to \underline{n}$ to the function $\theta_* : X^m \to X^n$ defined by extending functions $x : \underline{m} \to X$ to functions $\theta^*(X) : \underline{n} \to X$ by sending elements of $\underline{n} - \underline{m}$ to the base point $* \in X$.

Write

$$\Gamma^+(X) = \underrightarrow{\text{holim}}\ P_X,$$

meaning the unpointed homotopy colimit, and let $\Gamma^*(X)$ be the pointed homotopy colimit. Then it's a standard observation that there is a cofibre sequence

$$B\mathbf{Mon} \to \Gamma^+(X) \to \Gamma^*(X).$$

The space $B\mathbf{Mon}$ is contractible, so that the natural map

$$\Gamma^+(X) \to \Gamma^*(X)$$

is a weak equivalence.

The space $\Gamma^*(X)$ is supposed to be a model for Barratt's space $\Gamma^+(X)$ [1], and here's a reality check:

Lemma 1. *There is a weak equivalence*

$$\bigsqcup_{n \geq 0} B\Sigma_n \simeq \Gamma^+(S^0).$$

Proof. Write $S^0 = \{0, 1\}$, pointed by 0. Then $\Gamma^+(S^0)$ is the nerve of the translation EP_{S^0} category whose objects are all pairs $(\underline{n}, x)$ with $x \in \{0, 1\}^{\times n}$ and with morphisms

$$\theta : (\underline{m}, y) \to (\underline{n}, x)$$

given by monomorphisms θ such that $\theta_*(y) = x$.

Given $(\underline{m}, y)$, write $\underline{k}_y$ for the subset of $\underline{m}$ on which the function y is non-zero. Then there is a unique ordered monomorphism $m_y : \underline{k}_y \to \underline{m}$ and a corresponding morphism $m_y : (\underline{k}_y, 1) \to (\underline{m}, y)$, where 1 is the function taking all elements to $1 \in S^0$. One can show that two objects $(\underline{m}, y)$ and $(\underline{n}, x)$ are in the same path component of EP_{S^0} if and only if $\underline{k}_x = \underline{k}_y$, meaning that x and y have the same number of non-zero entries.

Given $\theta : (\underline{m}, y) \to (\underline{n}, x)$ and $\underline{k}_y \neq \emptyset$, there is a commutative diagram

$$\begin{array}{ccc} (1, \underline{k}) & \xrightarrow{m_y} & (\underline{m}, y) \\ {\scriptstyle \sigma_\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ (1, \underline{k}) & \xrightarrow[m_x]{} & (\underline{n}, x) \end{array}$$

where $\underline{k} = \underline{k}_y = \underline{k}_x$ and σ_θ is a uniquely determined element of the symmetric group Σ_k. It follows that the component of $(\underline{m}, y)$ in EP_{S^0} is homotopy equivalent to Σ_k.

The component of $(\emptyset, 0)$ in EP_{S^0} is a copy of **Mon**, which is contractible. It is therefore equivalent to $B\Sigma_0 = *$. □

Write $\mathbf{Mon}_k$ for the full subcategory of **Mon** consisting of the sets $\underline{n}$ of cardinality less than or equal to k, and write $P_X^{(k)}$ for the left Kan extension to **Mon** of the restricted functor

$$\mathbf{Mon}_k \subset \mathbf{Mon} \xrightarrow{P_X} s\mathbf{Set}.$$

Let $\Gamma^+(X)^{(k)} = \underrightarrow{\mathrm{holim}}\ P_X^{(k)}$.

Example 2. The space $P_X^{(1)}(\underline{n}) = \bigvee_n X$ is the n-fold wedge of copies of X. $\mathbf{Mon}_1$ is the category $\emptyset \to \underline{1}$, and the restriction of P_X to $\mathbf{Mon}_1$ is the diagram $* \to X$, which is projective cofibrant. It follows that the Kan extension $P_X^{(1)}$ is also a projective cofibrant diagram, so that the map

$$\underrightarrow{\mathrm{holim}}\ P_X^{(1)} \to \underrightarrow{\lim}\ P_X^{(1)}$$

is a weak equivalence. The fold maps $\bigvee_n X \to X$ define an isomorphism $\underrightarrow{\lim}\ P_X^{(1)} \cong X$.

Example 3. More generally, the space $P_X^{(k)}(\underline{n})$ is isomorphic to the subset $A_X^{(k)}(\underline{n})$ of $X^{\times n}$ which consists of those n-tuples $(x_1, \dots, x_n) \in X^n$ such that $x_i \neq *$ in at most k places.

In effect, the map

$$P_X^{(k)}(\underline{n}) = \underrightarrow{\lim}_{\underline{m} \to \underline{n}, m \leq k} X^m \to X^n$$

factors through the inclusion $A_X^{(k)}(\underline{n}) \subset X^{\times n}$. Also, for subsets A, B of $\underline{n}$ the diagram

$$\begin{array}{ccc} X^{A \cap B} & \longrightarrow & X^B \\ \downarrow & & \downarrow \\ X^A & \longrightarrow & X^{\underline{n}} \end{array}$$

is a pullback, where the functions are defined by extension by base point. It follows that the function

$$P_X^{(k)}(\underline{n}) = \underrightarrow{\lim}_{\underline{m} \to \underline{n}, m \leq k} X^m \to A_X^{(k)}(\underline{n})$$

is injective.

Theorem 4. *1) There is a pointed weak equivalence*

$$E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k} \xrightarrow{\simeq} \Gamma^+(X)^{(k)} / \Gamma^+(X)^{(k-1)}.$$

2) The canonical maps $E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k} \to \Gamma^* X$ induce a weak equivalence

$$\bigvee_{k \geq 1} E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k} \xrightarrow{\simeq} \Gamma^* X.$$

Here, $E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k}$ is the pointed homotopy colimit for the Σ_k action on the k-fold smash $X^{\wedge k}$ which interchanges smash factors.

Proof. **NB**: Part 2) needs proof, but see [5].

For $k \leq n$, there is a surjective map

$$\bigsqcup_{A \subset \underline{n},\ |A|=k} X^{\times k} \to P_X^{(k)}(\underline{n}),$$

where the induced map off the direct summmand corresponding to the k-element subset $A \subset \underline{n}$ is the map $\theta_{A*} : X^{\times k} \to X^{\times n}$ which is induced by the unique ordered function $\underline{k} \to \underline{n}$ which picks out the elements of A. If the n-tuple x is in the image of θ_{A*} and θ_{B*} in the sense that $x = \theta_{A*}(y) = \theta_{B*}(z)$ for distinct k-element subsets A and B, then $x \in P_X^{(k-1)}$, as are the elements y, z. It follows that there is an induced pointed isomorphism

$$\bigvee_{A \subset \underline{n},\ |A|=k} X^{\wedge k} \xrightarrow{\cong} P_X^{(k)}/P_X^{(k-1)}(\underline{n})$$

for $n \geq k$, while

$$P_X^{(k)}/P_X^{(k-1)}(\underline{n}) = *$$

for $n \leq k-1$.

The space $\Gamma^+(X)^{(k)}/\Gamma^+(X)^{(k-1)}$ is the pointed homotopy colimit of the functor

$$\underline{n} \mapsto P_X^{(k)}/P_X^{(k-1)}(\underline{n}).$$

There is a category $\mathbf{M}_k$ whose objects are the set of order preserving injections $A : \underline{k} \subset \underline{n}$, and whose morphisms are the commutative diagrams of injections

$$\begin{array}{ccc} \underline{k} & \xrightarrow{A} & \underline{m} \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \theta} \\ \underline{k} & \xrightarrow[B]{} & \underline{n} \end{array}$$

Note that $B = \theta(A)$ and $\sigma \in \Sigma_k$. It follows from the analysis above that $\Gamma^+(X)^{(k)}/\Gamma^+(X)^{(k-1)}$ is the pointed homotopy colimit $\underrightarrow{\text{holim}}_* X^\bullet$ of the functor

$$X^\bullet : \mathbf{M}_k \to s\mathbf{Set}_*$$

taking values in pointed simplicial sets, which is defined by sending $A : \underline{k} \subset \underline{n}$ to $X^{\wedge k}$, and which sends a morphism (θ, σ) to the induced isomorphism $\sigma_* : X^{\wedge k} \to X^{\wedge k}$.

There is a functor $f : \mathbf{M}_k \to \Sigma_k$ which sends (θ, σ) to σ, and the functor $X^\bullet$ is the composite

$$\mathbf{M}_k \xrightarrow{f} \Sigma_k \xrightarrow{X^{\wedge k}} s\mathbf{Set}_*.$$

There is also a functor $g : \Sigma_k \to \mathbf{M}_k$ which takes $*$ to the object $\underline{k} : \underline{k} \xrightarrow{1} \underline{k}$, and takes $\tau \in \Sigma_k$ to the morphism

$$\begin{array}{ccc} \underline{k} & \xrightarrow{1} & \underline{k} \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ \underline{k} & \xrightarrow[1]{} & \underline{k} \end{array}$$

Observe that $fg = 1$ and there is a natural transformation $gf \to 1$. It follows that the map (functor) f in the pullback diagram

$$\begin{array}{ccc} \underrightarrow{\mathrm{holim}}_{\mathbf{M}_k} X^\bullet & \xrightarrow{f_*} & \underrightarrow{\mathrm{holim}}_{\Sigma_k} X^{\wedge k} \\ \downarrow & & \downarrow \\ B\mathbf{M}_k & \xrightarrow[f]{} & B\Sigma_k \end{array}$$

induces a weak equivalence f_*, since the group Σ_k acts invertibly on $X^{\wedge k}$. It follows that the induced map of pointed homotopy colimits

$$\underrightarrow{\mathrm{holim}}_{*} X^\bullet \to E\Sigma_k \wedge_{\Sigma_k} X^{\wedge k}$$

is a weak equivalence. □

Remark 5. Generally, suppose that $X : I \to s\mathbf{Set}$ is a small diagram of simplicial sets which is a diagram of equivalences in the sense that all morphisms $\alpha : i \to j$ of the category I induce weak equivalences $X(i) \to X(j)$. Then all pullback diagrams

$$\begin{array}{ccc} Y \times_{BI} \underrightarrow{\mathrm{holim}}_{I} X & \longrightarrow & \underrightarrow{\mathrm{holim}}_{I} X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & BI \end{array}$$

are homotopy cartesian. This is a slight variation of Quillen's "Theorem B" — see [4, IV.5.7].

The assignment $K \mapsto \Gamma^*(K)$ for finite pointed sets K defines a Γ-space, and the natural pointed maps $K \to \Gamma^*(K)$ form a morphism of Γ-spaces, which will be denoted by $\phi : \mathrm{Id} \to \Gamma^*(\mathrm{Id})$. Here, Id refers to the Γ-space defined by the identity functor $K \mapsto K$. Recall that the associated spectrum $\mathrm{Id}(S)$ is the sphere spectrum S^0.

More generally, for a pointed simplicial set X, the spectrum $\mathrm{Id} \wedge X(S)$ associated to the Γ-space $\mathrm{Id} \wedge X$ is the suspension spectrum $\Sigma^\infty X$. The Γ^* construction can be applied to any Γ-space Y, and there is a corresponding natural map of Γ-spaces

$$\phi_Y : Y \to \Gamma^*(Y).$$

Corollary 6. *The* Γ*-space map* $\phi_{\mathrm{Id} \wedge X}$ *induces a stable equivalence*

$$\Sigma^\infty X = \mathrm{Id} \wedge X(S) \to \Gamma^*(\mathrm{Id} \wedge X)(S)$$

of associated spectra.

Proof. In general, if Y is an k-connected pointed simplicial set, then the cofibre of the map $Y \to \Gamma^*(Y)$ is at least $(2k+1)$-connected on account of Theorem 4, and it follows (by relative Hurewicz) that the map $\pi_j Y \to \pi_j \Gamma^*(Y)$ is an isomorphism for $j \leq 2k$. The map $S^n \wedge X \to \Gamma^*(S^n \wedge X)$ therefore induces an isomorphism

$$\pi_i(S^n \wedge X) \xrightarrow{\cong} \pi_i(\Gamma^*(S^n \wedge X))$$

for $i \leq 2(n-1)$. □

The space $\Gamma^+(X)$ has the structure of a simplicial monoid. In effect, given morphisms $\theta : (\underline{n}, x) \to (\underline{m}, y)$ and $\theta' : (\underline{n}', x') \to (\underline{m}', y')$, in the translation category defining $\underrightarrow{\mathrm{holim}}\ P_X$, their sum is the map

$$\theta \oplus \theta := \theta \vee \theta' : (\underline{m} \vee \underline{m}', (x, x')) \to (\underline{n} \vee \underline{n}', (y, y'))$$

Here, for example, $(x, x') \in X^m \times X^{m'}$, and there is a canonical bijection $\underline{m} \vee \underline{m}' \cong \underline{m + m'}$ which identifies $\underline{m}$ with the first m elements of $\underline{m + m'}$, and identifies $\underline{m'}$ with the last m' elements, both identifications in order. This bijection also forces an identification

$$X^m \times X^{m'} = \hom(\underline{m}, X) \times \hom(\underline{m'}, X) \cong \hom(\underline{m + m'}, X) = X^{m+m'}.$$

The identity for the monoid structure is the object $(\emptyset, *)$, and the monoid is homotopy commutative since there are natural diagrams

$$\begin{array}{ccc}
(\underline{m} \vee \underline{m}', (x, x')) & \xrightarrow{\theta \vee \theta'} & (\underline{n} \vee \underline{n}', (y, y')) \\
{\scriptstyle \tau} \downarrow & & \downarrow {\scriptstyle \tau'} \\
(\underline{m}' \vee \underline{m}, (x', x)) & \xrightarrow[\theta' \vee \theta]{} & (\underline{n}' \vee \underline{n}, (y', y))
\end{array}$$

in the underlying translation category, where the maps τ and τ' are suitably defined shuffles.

We shall use the standard models for finite pointed sets: in particular $\mathbf{n}_+$ denotes the finite ordinal number $\mathbf{n} = \{0, 1, \ldots, n\}$, pointed by 0. Observe that the pointed set S^0 of Lemma 1 is the pointed ordinal number $\mathbf{1}_+$ in this notation.

Write in_i for the pointed map function $\mathbf{1}_+ \to \mathbf{n}_+$ which picks out the number i for $1 \leq i \leq n$, and let ψ denote the composite

$$\Gamma^+(\mathbf{1}_+) \times \cdots \times \Gamma^+(\mathbf{1}_+) \xrightarrow{in_1 \times \cdots \times in_n} \Gamma^+(\mathbf{n}_+) \times \cdots \times \Gamma^+(\mathbf{n}_+) \xrightarrow{\oplus} \Gamma^+(\mathbf{n}_+).$$

Now here's a generalization of Lemma 1:

Lemma 7. *The map ψ is a weak equivalence.*

Proof. Two elements $(\underline{m}, x)$ and $(\underline{r}, y)$ are in the same path component if and only if the functions $x : \underline{m} \to \mathbf{n}_+$ and $y : \underline{r} \to \mathbf{n}_+$ have the same number k_i of images of each "colour" i for $1 \leq i \leq n$. It follows, by the same argument as for Lemma 1, that the path component of $\Gamma^+(\mathbf{n}_+)$ corresponding to the numbers k_i, $1 \leq i \leq n$ has the homotopy type of the product

$$B\Sigma_{k_1} \times \cdots \times B\Sigma_{k_n}.$$

Lemma 1 implies that the path components of the space

$$\Gamma^+(\mathbf{1}_+) \times \cdots \times \Gamma^+(\mathbf{1}_+)$$

have the same description, and the map ψ preserves them. □

Corollary 8. *There is a levelwise weak equivalence of simplicial spaces*

$$B\Gamma^+(\mathbf{1}_+) \xrightarrow{\simeq} \Gamma^+(S^1).$$

Proof. The simplicial space $\Gamma^+(S^1)$ is defined by $\mathbf{n} \mapsto \Gamma^+(S^1_n)$, and there are commutative diagrams

$$\begin{array}{ccc} \Gamma^+(\mathbf{1}_+)^{\times n} & \xrightarrow[\simeq]{\psi} & \Gamma^+(S^1_n) \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} \\ \Gamma^+(\mathbf{1}_+)^{\times m} & \xrightarrow[\psi]{\simeq} & \Gamma^+(S^1_n) \end{array}$$

for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, and where the copy of θ^* on the left is the simplicial structure map for $B\Gamma^+(\mathbf{1}_+)$. □

Corollary 9. *The Γ-space $\Gamma^*(\mathrm{Id})$ is special, meaning that the pinch maps form a weak equivalence*

$$\Gamma^*(\mathbf{n}_+) \to \prod_{i=1}^{n} \Gamma^*(\mathbf{1}_+) \tag{1}$$

for all $n \geq 1$.

Proof. The pinch maps induce a map $\Gamma^+(\mathbf{n}_+) \to \prod_{i=1}^n \Gamma^+(\mathbf{1}_+)$ which is weakly equivalent to the map in (1), and the composite

$$\prod_{i=1}^{n} \Gamma^+(\mathbf{1}_+) \xrightarrow{\psi} \Gamma^+(\mathbf{n}_+) \to \prod_{i=1}^{n} \Gamma^+(\mathbf{1}_+)$$

is homotopic to the identity. □

A standard group completion argument, arising from the action of the monoid $\bigsqcup_{n\geq 0} B\Sigma_n$ on the space $\bigsqcup_{\mathbb{Z}} B\Sigma_\infty$ then implies the following:

Lemma 10. *There is a weak equivalence*

$$\bigsqcup_{\mathbb{Z}} (B\Sigma_\infty)_+ \xrightarrow{\simeq} \Omega B(\bigsqcup_{n\geq 0} B\Sigma_n).$$

Here, $(B\Sigma_\infty)_+$ is the result of applying the plus construction to the space $B\Sigma_\infty$; it can also be characterized as an H-space having the homology of $B\Sigma_\infty$.

The spectrum $Y(S)$ associated to a special Γ-spaces Y is an Ω-spectrum above level 1 in the sense that the maps $Y(S)^n \to \Omega Y(S)^{n+1}$ are weak equivalences for $n \geq 1$ [3, Thm. 4.4]. It follows that the spectrum defined by the spaces

$$\Omega Y(S)^1, Y(S)^1, Y(S)^2, \ldots$$

is an Ω-spectrum having the same stable homotopy type as $Y(S)$.

In view of Corollary 6, Corollary 9 and Lemma 10, we therefore have the following:

Theorem 11 (Barratt-Priddy-Quillen)**.** *There is a weak equivalence*

$$QS^0 \simeq \bigsqcup_{\mathbb{Z}} (B\Sigma_\infty)^+.$$

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