# Cohomological invariants associated to symmetric bilinear forms

J.F. Jardine[1]

Mathematics Department
University of Western Ontario
London, Ontario N6A 5B7
Canada

## Introduction

This paper is an outgrowth of the notes for a lecture given at the Mathematics Department of Queen's University in December of 1988. I would like to take this opportunity to thank the members of that department for their hospitality during my visit. I would also like to thank Bruno Kahn for reading early versions of this manuscript and making several helpful remarks.

The basic idea of the lecture was to summarize what was known (at least to me) at the time about characteristic classes in the *mod* 2 Galois cohomology of fields $K$ of characteristic not equal to 2 which arise either from symmetric bilinear forms or representations of Galois groups in their orthogonal groups. Some new results have appeared during the writing of this paper; they are discussed below.

The theory, at least through the lens of the current discussion, has a long, though somewhat desultory pedigree. The most prominent classical cohomological invariants are the Delzant Stiefel-Whitney classes [1] of a symmetric bilinear form. More recently, Fröhlich [2] introduced the spinor class $Sp_2(\rho)$ associated to an orthogonal representation $\rho : G \to O_n(K)$ of a Galois group $G$ of a finite Galois extension of $K$. $Sp_2(\rho)$ is defined to be the image in the Galois cohomology group $H^2_{et}(K, \mathbb{Z}/2)$ under cup product of the element in $H^1(G, H^1_{et}(K, \mathbb{Z}/2))$ defined by the composite

$$G \xrightarrow{\rho} O_n(K) \xrightarrow{\delta} H^1_{et}(K, \mathbb{Z}/2),$$

where $\delta$ is the classical spinor norm (see also the Appendix of this paper). But it has also been clear for a long time that a representation $\rho$ has characteristic classes associated to it which essentially come from the cohomology of the topological space $BO$, and therefore also have some right to be called its Stiefel-Whitney classes.

There has only recently been what one would call a burst of activity in this area; it was essentially motivated by a letter from Serre to Martinet (1982, see also [21]) in which he gave a formula for the "Hasse-Witt invariant" associated to the trace form of a finite separable field extension of $K$. It turns out, by results of

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Fröhlich [2], that this Hasse-Witt invariant is the second Delzant Stiefel-Whitney class of a "twisted" form $(\rho, \beta)$ which is canonically associated to some orthogonal representation of a Galois group $\rho : G \to O_\beta(K)$. Furthermore, the Hasse-Witt invariant does not coincide with what one would think of as the Stiefel-Whitney class of the representation: the difference is essentially given by the spinor class.

Some authors (myself included) now refer to the Delzant Stiefel-Whitney classes as Hasse-Witt classes. This part of the theory culminates with a formula relating the second Hasse-Witt class of the twisted form associated to a representation of a Galois group in the automorphism group of an arbitrary form with the Stiefel-Whitney classes of the representation, the Hasse-Witt classes of the underlying form, and the spinor class of the representation. Various people have worked on this result; I call it the Fröhlich-Kahn-Snaith formula [2], [10], [22], and give a different proof below.

My own involvement in this area started with the point of view that characteristic classes should be defined by homotopy theory in some sense. Perhaps the original motivation was naive, but there is a deep connection between this theory and traditional algebraic topological methods via the homotopy theory of simplicial sheaves. The isomorphism classes of non-degenerate symmetric bilinear forms of rank $n$ are classified by the non-abelian cohomology object $H^1_{et}(K, O_n)$, computed on the big étale site $(Sch|_K)_{et}$ of $K$, where $O_n$ is now the sheaf of groups represented by the corresponding $K$-group-scheme. In turn, $H^1_{et}(K, O_n)$ may be identified with the set of morphisms $[*, BO_n]$ in the homotopy category of simplicial sheaves on $(Sch|_K)_{et}$, where $BO_n$ is the classifying simplicial object of the sheaf of groups $O_n$ and $*$ is the terminal simplicial sheaf. The étale, or Galois, cohomology of the field $K$ is representable in this theory in the sense that one can identify $H^r_{et}(K, \mathbb{Z}/2)$ with the set of homotopy classes $[*, K(\mathbb{Z}/2, r)]$ from $*$ to a constant Eilenberg-Mac Lane object. Thus, elements $z$ of the cohomology group $H^r_{et}(BO_n, \mathbb{Z}/2) = [BO_n, K(\mathbb{Z}/2, r)]$ give rise to characteristic classes associated to forms $\beta$ of rank $n$: one simply evaluates $\beta^*(z)$ in $H^r_{et}(K, \mathbb{Z}/2)$. The central result from this point of view is that the ring $H^*_{et}(BO_n, \mathbb{Z}/2)$ is a polynomial ring over the *mod* 2 Galois cohomology ring of the field $K$, with generators $HW_i$, $i = 1, ..., n$. Furthermore, $\beta^*(HW_i)$ is the $i^{th}$ Hasse-Witt, or Delzant Stiefel-Whitney, class $HW_i(\beta)$ of the form $\beta$. These results are proven in [7]; the calculation of $H^*_{et}(BO_n, \mathbb{Z}/2)$ involves a comparison of Galois cohomological descent spectral sequences.

My real purpose in writing the notes for the Queen's lecture was to give a connected account of the existing theory from the simplicial point of view. The first section of this paper essentially consists of the original lecture notes, but one will find in it a new approach to the second Hasse-Witt class $HW_2(\beta)$ of a form $\beta$ which characterizes it as an explicit homotopy obstruction $[w_2^\pi(\beta)]$ to lifting a cocycle with values in the orthogonal group to the group $Pin_n$ along the canonical sheaf epimorphism $\pi : Pin_n \to O_n$ in the classical central extension $\mathbb{Z}/2 \to Pin_n \to O_n$. This presumes, of course, that one has a central extension of sheaves on the big étale site to play with, so the material in Fröhlich's appendix [2] is not enough, even though it is a useful guide. This problem is dealt with in the Appendix of this paper; I construct the central extension of sheaves of groups

$$\mathbb{Z}/2 \to Pin_\beta \xrightarrow{\pi_\beta} O_\beta$$

associated to the orthogonal group-scheme of any non-degenerate symmetric bilinear form $\beta$ over $K$.

The existence of the central extension for arbitrary forms $\beta$ means that one has an obstruction class $[w_2^{\pi_\beta}(\omega)]$ defined on $O_\beta$-cocycles by analogy with the obstruction-theoretic definition of $HW_2$. This obstruction class is used to prove the Fröhlich-Kahn-Snaith formula; this is Theorem 2.4 of this paper. The central point of that proof is the formula of Proposition 2.2 for the obstruction associated to a direct sum of cocycles.

But it is also possible to show that the simplicial sheaf $BO_\beta$ on $(Sch|_K)_{et}$ is weakly equivalent to $BO_n$ by comparing both to some intermediate object. It follows in particular that $H^*_{et}(BO_\beta, \mathbb{Z}/2)$ is a polynomial ring in generators $HW_i^\beta$, $i = 1, ..., n$ over the *mod* 2 Galois cohomology of the field $K$. If $\beta$ is a diagonal form (and this can always be presumed), the constant group $\Gamma^*\mathbb{Z}/2^{\times n}$ is canonically included in $BO_\beta$, and $HW_i$ maps to the $i^{th}$ elementary symmetric polynomial in certain linear translates of the generators of the cohomology ring $H^*_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)$. This result is Theorem 3.1 of the third section.

The derivation of the weak equivalence between $BO_\beta$ and $BO_n$ is the interesting part of Theorem 3.1 in that it was something of a surprise, since the two objects obviously have different sheaves of fundamental groups, namely $O_\beta$ and $O_n$ respectively. The reason that one gets away with this is that this weak equivalence has the form

$$BO_\beta \xleftarrow{\simeq} X \xrightarrow{\simeq} BO_n,$$

where $X$ is a simplicial sheaf with an empty set of global sections. If there's no global choice of base point in $X$, then no comparison of sheaves of fundamental groups is possible if these sheaves are to be pointed by a global section. In fact, the object $X$ is non-empty only over a Galois extension $L$ over which $\beta$ trivializes.

Theorem 3.1 gives "characteristic" classes $HW_i^\beta(\omega) = \omega^*(HW_i^\beta)$ for any cocycle $\omega$ representing an element of $[*, BO_\beta]$. Furthermore, one finds that $HW_i^\beta(\omega)$ coincides with the ordinary class $HW_i(\omega, \beta)$ arising from the twisted form $(\omega, \beta)$ associated to $\omega$. Along the way to proving the Fröhlich-Kahn-Snaith result, one derives (Theorem 2.5) a formula giving $HW_2(\omega, \beta)$ in terms of the obstruction class $[w_2^{\pi_\beta}(\omega)]$, plus some terms which are non-zero in general, so that $HW_2^\beta(\omega)$ can differ from $[w_2^{\pi_\beta}(\omega)]$; the relationship between the two is given by the formula (3.7). This formula (3.7) has a pleasant form: I think of it as the "real" Fröhlich-Kahn-Snaith formula.

I hesitate slightly in declaring that the $HW_i^\beta(\omega)$ are characteristic classes only because $HW_i^\beta(\omega) = HW_i(\beta)$ if $\omega$ is trivial, so that non-trivial classes can arise from trivial forms. Some consequences of this phenomenon are discussed at the end of the third section of this paper.

An explanation of the higher Hasse-Witt classes $HW_i(\rho, \beta)$ of the twisted form $(\rho, \beta)$ has been a primary goal of this theory of characteristic classes, since it became known that the spinor class $Sp_2^\beta(\rho)$ intervened in degree 2 in the Fröhlich-Kahn-Snaith formula. It is possible to use the action of the Steenrod algebra on all of the cohomology groups in sight in conjuction with a Wu formula to bootstrap the formula for $HW_2(\rho, \beta)$ to a formula for $HW_3(\rho, \beta)$ (and hence for $HW_3^\beta(\rho)$); this was done in [9] and it appears as Theorem 2.6 of this paper. This technique fails in all degrees which are higher powers of the prime 2. The fundamental open problem is really (and has been thought of as such for a while) to decide whether or not there are non-trivial higher spinor classes which are unrelated to $Sp_2^\beta(\rho)$. One may as well start with the degree 4 case, since it's still open.

Let me close this section by remarking that it is already possible to go beyond the material presented here (this will be the subject of a future paper). The calculation of the étale cohomology of $BO_n$ given in Theorem 1.4 below can be generalized to arbitrary base schemes $X$ defined over $\mathbb{Z}[1/2]$. More explicitly, if $O_{n,X}$ is the corresponding group-scheme defined over $X$, then $H^*_{et}(BO_{n,X}, \mathbb{Z}/2)$, as an algebra over $H^*_{et}(X, \mathbb{Z}/2)$, is a polynomial ring in classes $HW_i$, $i = 1, ..., n$, where $\deg(HW_i) = i$. The proof is the same sort of comparison of descent spectral sequences as was used to prove Theorem 1.4 in [8]. This gives a theory of Hasse-Witt classes for non-degenerate symmetric bilinear forms of rank $n$ on all such $X$, since isomorphism classes of these forms may be identified with morphisms $[*, BO_{n,X}]$ in the homotopy category of simplicial sheaves on the big étale site for $X$ (but see also [12]). Alternatively, Kahn has pointed out that if $Y$ is a scheme which is defined over a field $K$ of characteristic prime to 2, then one achieves a theory of Hasse-Witt classes for forms on $Y$ directly from Theorem 1.4 by defining the classes $HW_i$ to be the images of the corresponding classes from $H^*_{et}(BO_n, \mathbb{Z}/2)$ under the obvious canonical map. This amounts to the same theory, since the classes $HW_i$ in $H^*_{et}(BO_{n,X}, \mathbb{Z}/2)$ are stable under base change. Finally, there is another connection with classical geometry in the sense that standard techniques can be used to show that if $X$ is a smooth variety over $\mathbb{C}$, and $E$ is a non-degenerate symmetric bilinear form of rank $n$ over $X$, then $E$ gives rise to a quadratic form $E(\mathbb{C})$ over the analytic space $X(\mathbb{C})$ whose classical Stiefel-Whitney classes coincide with the Hasse-Witt classes of $E$ under the isomorphism $H^*_{et}(X, \mathbb{Z}/2) \cong H^*(X(\mathbb{C}), \mathbb{Z}/2)$.

**1. Basic theory.**

Let $K$ be a field such that $char(K) \neq 2$, and let $V = K^n$ be an $n$-dimensional $K$-vector space. Let $\beta : V \times V \to K$ be an non-degenerate symmetric bilinear form (of rank $n$), with matrix $B$ relative to some choice of basis for $V$. $B$ is symmetric and invertible. It can also be diagonalized. In effect, there is an element $v$ of $V$ such that $v^2 = \beta(v, v) \neq 0$; otherwise for any two elements $v$ and $w$ of $V$ one finds $0 = (v + w)(v + w) = 2vw$, and so $vw = 0$, contradicting the non-degeneracy of $\beta$. Now $V = \langle v \rangle \oplus \langle v \rangle^\perp$; there is a short exact sequence of vector space maps

$$0 \to \langle v \rangle^\perp \to V \to K \to 0,$$

(where the map $V \to K$ is defined by $w \mapsto \beta(w, v)$), and so for dimensional reasons, the map $\langle v \rangle^\perp \oplus \langle v \rangle \to V$ is an isomorphism. Furthermore, for example, if an element $w$ of $\langle v \rangle^\perp$ annihilates all of $\langle v \rangle^\perp$ under the restriction of $\beta$, it annihilates all of $V$, so that the restriction of $\beta$ to each summand is non-degenerate. It follows that there is a matrix $A \in Gl_n(K)$ such that $B$ has the form

$$B = A^t \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} A,$$

where $a_i \in K$, $i = 1, ..., n$. We shall say that

$$A : B \to \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

is an *isomorphism* (or just a *map*) of forms.

The classes $[a_i] \in K^*/(K^*)^2$ are obstructions to finding an isomorphism of forms

$$B \xrightarrow{A} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

or alternatively, to writing $B = A^t A$ for some $A \in Gl_n(K)$. These vanish over some finite Galois extension $L/K$ with $G = Gal(L/K)$.

Choose such an $L$ and an $A \in Gl_n(L)$ such that $B = A^t A$, and consider the resulting automorphisms $g(A) \cdot A^{-1}$

$$1_n \xleftarrow{g(A)} B \xrightarrow{A} 1_n$$

(note that $g(A)^t g(A) = g(B) = B = A^t A$) of the trivial form $1_n$ on $L^n$. In other words, $g(A) \cdot A^{-1}$ is a member of the orthogonal group $O_n(L)$ consisting of matrices $C \in Gl_n(L)$ such that $C^t C = I_n$. The function $f_\beta : G \to O_n(L)$ defined by $f_\beta(g) = g(A) \cdot A^{-1}$ satisfies

$$\begin{cases} f_\beta(e) = 1 \quad \text{and} \\ f_\beta(g \cdot h) = gh(A) \cdot A^{-1} = gh(A) \cdot g(A)^{-1} \cdot g(A) \cdot A^{-1} = g(f_\beta(h)) \cdot f_\beta(g). \end{cases}$$

$f_\beta : G \to O_n(L)$ is a (contravariant) cocycle defined on $G$ which takes values in the group $O_n(L)$. Let $Z^1(G, O_n(L))$ be the set of all such cocycles.

The cocycles $f,\ f' : G \to O_n(L)$ are said to be *cohomologous* if there is an orthogonal matrix $E \in O_n(L)$ such that $g(E) \cdot f(g) = f'(g) \cdot E$ for all $g$ in $G$. For example, if $B = D^t B' D$ over $K$ and $B' = (A')^t A'$ over $L$, then there is a commutative diagram

$$\begin{array}{ccccc} 1_n & \xleftarrow{g(A)} & B & \xrightarrow{A} & 1_n \\ {\scriptstyle g(A'DA^{-1}) = g(A')Dg(A)^{-1}}\Big\downarrow & & D\Big\downarrow & & \Big\downarrow{\scriptstyle A'DA^{-1}} \\ 1_n & \xleftarrow[g(A')]{} & B & \xrightarrow[A']{} & 1_n \end{array}$$

of morphisms of forms over $L$, so that the cocycles $g(A) \cdot A^{-1}$ and $g(A') \cdot (A')^{-1}$ are cohomologous. Define (just like anybody would) $H^1(G, O_n(L)) = Z^1(G, O_n(L))/_\sim$, where $\sim$ denotes the cohomology relation.

The upshot of what we have done so far is that there is a well-defined function:

$$\left\{\begin{array}{l} \text{isomorphism classes of symmetric} \\ \text{non-degenerate bilinear forms} \\ \text{over } K \text{, trivial over } L \end{array}\right\} \xrightarrow{\varphi} H^1(G, O_n(L)).$$

Proposition 1.1. *The function $\varphi$ is a bijection.*

Proof (cf. [19] or [20]): Any cocycle $c : G \to O_n(L)$ determines a cocycle with coefficients in $Gl_n(L)$, by composition with the obvious inclusion. This composed cocycle must be homotopically trivial, by a souped-up version of "Hilbert's Theorem 90": $H^1(G, Gl_n(L))$ is trivial. Thus, $c$ has the form $c(g) = g(A) \cdot A^{-1}$ for some $A \in Gl_n(L)$. But $g(A)^t g(A) = A^t c(g)^t c(g) A = A^t A$ for all $g \in G$, and so $B := A^t A \in Gl_n(K)$ represents a preimage of the class $[c] \in H^1(G, O_n(L))$, and $\varphi$ is surjective. The injectivity of $\varphi$ is the observation that a collection of commutative diagrams of the form

$$\begin{array}{ccccc} 1_n & \xleftarrow{g(A')} & B' & \xrightarrow{A'} & 1_n \\ {\scriptstyle g(D)}\Big\downarrow & & & & \Big\downarrow{\scriptstyle D} \\ 1_n & \xleftarrow[g(A)]{} & B & \xrightarrow[A]{} & 1_n \end{array}$$

given by a homotopy of cocycles over $L$ determines a $K$-linear morphism of forms, namely $A^{-1}DA'$. ∎

Corollary 1.2.

(1) *There is a bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes of}\\ \textit{non-degenerate symmetric bilinear forms}\\ \textit{of rank } n \textit{ over } K\end{array}\right\} \leftrightarrow H^1_{Gal}(\Omega_K, O_n),$$

*where $\Omega_K$ is the absolute Galois group of $K$.*

(2) *There is a bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes of}\\ \textit{non-degenerate symmetric bilinear forms}\\ \textit{of rank } n \textit{ over } K\end{array}\right\} \leftrightarrow [*, BO_n],$$

*where the thing on the right denotes morphisms from the terminal (simplicial) object $*$ to the classifying simplicial object $BO_n$ in the category of simplicial sheaves on the big étale site $(Sch|_K)_{et}$* [5], [7].

The group-scheme $O_n$ represents a sheaf of groups on this site (by faithfully flat descent), and $BO_n$ is the simplicial sheaf whose $n$-simplices consist of the sheaf $O_n \times \cdots \times O_n$ ($n$-fold product), with faces and degeneracies defined in the usual way:

$$\begin{aligned}
d_0(g_n, ..., g_1) &= (g_n, ..., g_2)\\
d_i(g_n, ..., g_1) &= (g_n, ..., \overset{i}{g_{i+1}g_i}, ..., g_1) \qquad \text{for } 1 \le i \le n-1\\
d_n(g_n, ..., g_1) &= (g_{n-1}, ..., g_1)\\
s_i(g_n, ..., g_1) &= (g_n, ..., g_{i+1}, e, g_i, ..., g_1) \qquad \text{for } 0 \le i \le n.
\end{aligned}$$

Note that I've notationally suppressed the dependence on $K$; it's very important, however, to remember that everything in sight is defined over this field.

"Proof": (1) is just the definition of the non-abelian Galois $H^1$ invariant associated to $O_n$. The best way to see (2) is to observe that $H^1(G, O_n(L))$ may be identified via abstract nonsense with a set of simplicial homotopy classes of maps $\pi(EG \times_G Sp(L), BO(n))$. $EG \times_G Sp(L)$ is just fancy language for the *Čech* hypercover associated to the étale covering $Sp(L) \to Sp(K)$, namely

$$Sp(L) \leftleftarrows Sp(L) \times Sp(L) \leftleftarrows Sp(L) \times Sp(L) \times Sp(L) \ldots$$

The Borel construction notation is justified by the fact that $L \otimes_K L \cong \prod_{g \in G} L$, and so the $\check{C}ech$ hypercover is the simplicial gadget associated to the cosimplicial ring

$$\begin{array}{ccccc} L & \rightleftarrows & L \otimes_K L & \overrightarrow{\overleftarrow{\rightarrow}} & L \otimes_K L \otimes_K L \ldots. \\ & & \| & & \| \\ & & \prod_{g \in G} L & & \prod_{(g_2, g_1)} L \end{array}$$

It follows that this hypercover is the nerve of the translation category given by the action of $G$ on $Sp(L)$. Finally, one has to know that $\varinjlim \pi(EG \times_G Sp(L), BO_n)$ is one of the forms of $[*, BO_n]$ (whatever it is), but that's a bit technical.

REMARK 1.3. Note that we now think of $G$ as the group of automorphisms of $Sp(L)$ over $Sp(K)$, which group is contravariantly isomorphic to the original Galois group. This accounts for the contravariance in the original definition of $G$-cocyle: one wants the covariant definition in the simplicial setting.

There is a corresponding identification $H^1(\mathcal{C}, A) \cong [*, BA]$ of A-torsors with homotopy classes of maps for any sheaf of groups A on any Grothendieck site $\mathcal{C}$ [8].

There are various *characteristic classes* that one can extract from a non-degenerate symmetric bilinear form $\beta$ by these methods, since we have managed to represent the isomorphism class $[\beta]$ as a homotopical object. The basic reason for this is that there are isomorphisms

$$H^i_{Gal}(\Omega_k, A) \cong H^i_{et}(K, A) \cong [*, K(A, i)]$$

for any sheaf of groups A on the étale site of $K$. People are generally used to the fact that Galois cohomology coincides with étale cohomology for fields (the first isomorphism), but the identification with homotopy classes of maps is relatively new. $K(A, i)$ is the simplicial sheaf associated to a sheaf of abelian groups $A$ on $(Sch|_K)_{et}$ by naturally extending the Dold-Puppe theory to a functorial setting. In some sense, it's a very familiar topological object: it is a simplicial sheaf with one non-trivial sheaf of (globally pointed — see §3) homotopy groups, namely $A$, in degree $i$. Now $[BO_n, K(A, i)]$ is a perfectly good definition for $H^i_{et}(BO_n, A)$ (there are others, giving the same group), and so any form $[\beta] : * \to BO_n$ induces a map $\beta^* : H^i(BO_n, \mathbb{Z}/2) \to H^i(K, \mathbb{Z}/2)$, for example. The game, just as in the classical toplogical theory of characteristic classes, is therefore to find a canonical list of classes in $H^*(BO_n, \mathbb{Z}/2)$, whose pullbacks along $[\beta]$ will be thought of as characteristic classes of $\beta$ in the *mod* 2 Galois cohomology of $K$. Here's one basic source for these classes:

Theorem 1.4. *Let $K$ be a field of characteristic not equal to 2, and let $A$ denote the mod 2 Galois cohomology ring $H^*_{et}(K, \mathbb{Z}/2)$ of $K$. Then there is an isomorphism of graded algebras of the form*

$$H^*_{et}(BO_n, \mathbb{Z}/2) \cong A[HW_1, ..., HW_n],$$

*where the polynomial generator $HW_i$ has degree $i$.*

The proof of this theorem [8] is a comparison of étale chomological descent spectral sequences arising from the inclusion of the diagonal subgroup in $O_n$. This inclusion induces an inclusion map of simplicial sheaves $i : \Gamma^* B\mathbb{Z}/2^{\times n} \hookrightarrow BO_n$ and determines a monomorphism $i^* : H^*_{et}(BO_n, \mathbb{Z}/2) \hookrightarrow H^*_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)$. Here, $\Gamma^* B\mathbb{Z}/2^{\times n}$ is the constant simplicial sheaf on $B\mathbb{Z}/2^{\times n}$. Its cohomology is easy to describe: $H^*_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) \cong A[x_1, ...x_n]$, $\deg(x_i) = 1$. $x_i$ is the homotopy class of the simplicial presheaf map $\Gamma^* B\mathbb{Z}/2^{\times n} \to \Gamma^* B\mathbb{Z}/2$ which is induced by the $i^{th}$ projection $\mathbb{Z}/2^{\times n} \to \mathbb{Z}/2$. It follows from the proof of the theorem that $i^*(HW_i) = \sigma_i(x_1, ..., x_n)$, the $i^{th}$ elementary symmetric polynomial in the $x_i's$.

Recall that, in the setup above, we diagonalized the matrix $B$ associated to the form $\beta$ to get a matrix

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}.$$

By examining cocycles, one can show that there is a commutative diagram of the form

$$\begin{array}{ccc} & & \Gamma^* B\mathbb{Z}/2^{\times n} \\ & \overset{[a_1] \times \cdots \times [a_n]}{\nearrow} & \big\downarrow i \\ * & \xrightarrow[{[\beta]}]{} & BO_n \end{array}$$

in the homotopy category of simplicial sheaves on $(Sch|_K)_{et}$. In particular, $x_i$ pulls back to $[a_i]$ in $H^1_{et}(K, \mathbb{Z}/2)$, and so $\beta^*(HW_i) = \sigma_i([a_1], ..., [a_n])$. In other words, $\beta^*(HW_i)$ is the $i^{th}$ Hasse-Witt class of $\beta$, as defined by Delzant [1]. I call the $HW_i's$ the *universal Hasse-Witt classes.*

The classes $HW_i$ are completely determined by the fact that they pull back to symmetric polynomials. This allows us to represent them geometrically in low degrees. In particular, $HW_1$ pulls back to the polynomial $x_1 + ... + x_n$, as does the class $BO_n \to B\mathbb{Z}/2$ induced by the determinant map $O_n \to \mathbb{Z}/2$. Thus, the class $HW_1$ is represented by the determinant map.

$HW_2$ is more interesting, because we are forced to depart slightly from intuition to see it. First of all, observe that any central extension of sheaves of groups

$$e \to \mathbb{Z}/2 \to H' \xrightarrow{\pi} H \to e$$

and any class $[\beta] \in [*, BH]$ together determine a well-defined obstruction class $[w_2^\pi(\beta)] \in H^2_{et}(K, \mathbb{Z}/2)$. In effect, choose a cocycle $\beta : EG \times_G Sp(L) \to BH$ representing $[\beta]$, and now choose a refinement $EG_1 \times_{G_1} Sp(L_1) \to EG \times_G Sp(L)$ such that a lifting of the form $\hat{\beta}$ exists, making the diagram

$$\begin{array}{ccc} \bigsqcup_{G_1} Sp(L_1) & \xrightarrow{\hat{\beta}} & H' \\ \downarrow & & \downarrow \pi \\ \bigsqcup_{G} Sp(L) & \xrightarrow[\beta]{} & H \end{array}$$

commute. $\hat{\beta}$ can be identified with a function $\hat{\beta} : G_1 \to H'(L_1)$ such that $\pi\hat{\beta}(g_1) = \beta(g_1)$ in $H(L_1)$. Then the obstruction to lifting the *category* $EG_1 \times_{G_1} Sp(L_1)$ to the *category* $H'$ is given by the class of the 2-cocycle

$$(g_2, g_1) \mapsto \hat{\beta}(g_2 g_1)\hat{\beta}(g_1)^{-1}((g_1)^*(\hat{\beta}(g_2)))^{-1} \in \mathbb{Z}/2.$$

This is the obstruction cocycle $w_2^\pi(\beta)$; its associated cohomology class $[w_2^\pi(\beta)]$ is an invariant of $[*, BH]$ which is independent of the lift $\hat{\beta}$ in the usual way. There are a few things to observe:

(1) If the central extension has the form

$$e \to \Gamma^*\mathbb{Z}/2 \to \Gamma^* H' \to \Gamma^* H \to e$$

(constant sheaves) for some ordinary short exact sequence of groups, then the obstruction cocycle has the form

$$EG \times_G Sp(L) \xrightarrow{\beta} B\Gamma^* H = \Gamma^* BH \xrightarrow{\Gamma^* w_2} \Gamma^* K(\mathbb{Z}/2, 2),$$

where $w_2 : BH \to K(\mathbb{Z}/2, 2)$ is the canonical map associated to the principal $B\mathbb{Z}/2$ fibration $BH' \to BH$ by the Barratt-Guggenheim-Moore theory of twisted cartesian products (see [13]) of simplicial sets. In some sense, the point of the construction above is that we are attempting to dodge the fact that this theory does not work for principal fibrations of simplicial sheaves.

(2) If there is a pullback diagram of group homomorphisms of the form

$$\begin{array}{ccc} H_1' & \longrightarrow & H' \\ \pi_1 \downarrow & & \downarrow \pi \\ H_1 & \xrightarrow[f]{} & H \end{array}$$

and $\beta : EG \times_G Sp(L) \to BH_1$ is a cocycle, then $[w_2^{\pi}(f\beta)] = [w_2^{\pi_1}(\beta)]$.

(3) There is a pullback diagram of sheaves

$$\begin{array}{ccc} \Gamma^* E & \longrightarrow & Pin_n \\ \downarrow & & \downarrow \pi \\ \Gamma^* \mathbb{Z}/2^{\times n} & \hookrightarrow & BO_n \end{array}$$

where $\pi$ is a central $\mathbb{Z}/2$-extension of $O_n$, and such that

$$e \to \mathbb{Z}/2 \to E \to \mathbb{Z}/2^{\times n} \to e$$

represents $\sigma_2(x_1, ..., x_n)$. This is "classical" theory, but see the Appendix.

It follows that $HW_2(\beta) = [w_2^{\pi}(\beta)]$ for any cocycle $\beta : EG \times_G Sp(L) \to BO_n$. In other words, $HW_2(\beta)$ can be calculated as an obstruction to lifting, for any form $\beta$.

Now let's suppose that $G = Gal(L/K)$, and that $\rho : G \to O_n(K)$ is an orthogonal representation of $G$. Then $\rho$ canonically determines a map $\rho : \Gamma^* BG \to BO_n$, essentially by an adjointness argument. There are two ways to go with $\rho$ :

(A) Base change to the algebraic closure $\overline{K}$ of $K$ to get a representation $G \to O_n(K) \subset O_n(\overline{K})$ and hence a map of simplicial sheaves $\rho : \Gamma^* BG \to BO_n$ over $\overline{K}$. This map induces a map

$$H^*_{et}(BO_{n,\overline{K}}, \mathbb{Z}/2) \to H^*_{et}(\Gamma^*_{\overline{K}} BG, \mathbb{Z}/2) \cong H^*(BG, \mathbb{Z}/2) \xrightarrow{can} H^*_{et}(K, \mathbb{Z}/2),$$

where $\Gamma^*_{\overline{K}}$ denotes the constant sheaf functor which takes values in sheaves on $(Sch|_{\overline{K}})_{et}$, and the map *can* is the obvious map from the discrete cohomology $H^*(BG, \mathbb{Z}/2)$ of the group $G$ to the Galois cohomology of $K$. The image of the generator $HW_i$ under this map is called the $i^{th}$ *Stiefel-Whitney class* $SW_i(\rho)$ of $\rho$.

(B) There is a canonical map $\gamma : EG \times_G Sp(L) \to \Gamma^* BG$ of simplicial sheaves over $K$, defined on the level of $n$-simplices by

$$\bigsqcup_{(g_n, \dots, g_1)} Sp(L) \to \bigsqcup_{(g_n, \dots, g_1)} *.$$

Alternatively, it's the map of sheaves of translation categories over $G$ induced by the $G$-equivariant map $Sp(L) \to *$. The composition

$$EG \times_G Sp(L) \xrightarrow{\gamma} \Gamma^* BG \xrightarrow{\rho} BO_n$$

is a cocycle with values in $O_n$, and hence represents a form, which we're just going to call $\rho$. This form has Hasse-Witt classes $HW_i(\rho)$.

REMARK 1.5. It's worth noting at this juncture that $H^*_{et}(\Gamma^* BG, \mathbb{Z}/2)$ is isomorphic to a tensor product of the ordinary *mod* 2 cohomology ring $H^*(BG, \mathbb{Z}/2)$ with the *mod* 2 Galois cohomology ring $A$ of $K$, essentially since $G$ is a finite group (see [8]). It is possible, therefore, to think of $\rho^*(HW_i)$ as a sum

$$\rho^*(HW_i) = \sum_{j=0}^{i} HW_{i,j}(\rho)$$

of classes in $H^i(\Gamma^* BG, \mathbb{Z}/2)$: $HW_{i,j}(\rho)$ is the summand of $\rho^*(HW_i)$ living in bidegree $(j, i-j)$ in $A \otimes_{\mathbb{Z}/2} H^*(BG, \mathbb{Z}/2)$.

A better way to look at this is to observe that, quite generally, for arbitrary simplicial sets $X$, there is an isomorphism of the form

$$H^i(\Gamma^* X, \mathbb{Z}/2) \cong \bigoplus_{j=0}^{i} hom(H_{i-j}(X, \mathbb{Z}/2), H^j_{et}(K, \mathbb{Z}/2)).$$

This is seen by a standard universal coefficients argument, applied to the bicomplex $hom(\mathbb{Z}/2(X), I^*(K))$ which computes $H^*(\Gamma^* X, \mathbb{Z}/2)$ ($I^*(K)$ is global sections of an injective resolution $\mathbb{Z}/2 \to I^*$ of the sheaf $\mathbb{Z}/2$ by 2-torsion injectives). This isomorphism is dual to a collection of pairings

$$H^i(\Gamma^* X, \mathbb{Z}/2) \otimes H_{i-j}(X, \mathbb{Z}/2) \to H^j_{et}(K, \mathbb{Z}/2),$$

which are defined by observing that a homology class of degree $i-j$ in $H_*(X, \mathbb{Z}/2)$ can be identified with a chain homotopy class of maps $\mathbb{Z}/2[i-j] \to \mathbb{Z}/2(X)$, which in turn induces a chain homotopy class of maps of total complexes associated to $hom(\mathbb{Z}/2(X), I^*(K)) \to hom(\mathbb{Z}/2[i-j], I^*(K))$.

Thus, if $H$ is arbitrary discrete group, then there is a direct sum decomposition

$$H^i(\Gamma^* BH, \mathbb{Z}/2) \cong \bigoplus_{j=0}^{i} hom(H_{i-j}(BH, \mathbb{Z}/2), H^j_{et}(K, \mathbb{Z}/2)),$$

(see also [4, p.273]). Note that $H^*(\Gamma^* BH, \mathbb{Z}/2)$ coincides with Grothendieck's mixed cohomology theory associated to the trivial action of $H$ on $Sp(K)$ [3], [4]. This formula is the foundation of a theory of mixed, or equivariant, characteristic classes. In particular, if $\rho : H \to O_n(K)$ is an orthogonal representation of $H$, then the correponding map of simplicial sheaves $\rho_* : \Gamma^* BH \to BO_n$ and the class $HW_i \in H^i_{et}(BO_n, \mathbb{Z}/2)$ together induce a system of classes

$$HW_{i,j}(\rho) : H_{i-j}(BH, \mathbb{Z}/2) \to H^j_{et}(K, \mathbb{Z}/2)$$

which coincide with the classes given above in the case that $H$ is a finite Galois group. Also, Kahn has observed that, by the same technique applied to the infinite orthogonal group, the classes $HW_i \in H^i_{et}(BO, \mathbb{Z}/2)$ and the canonical map $\epsilon : \Gamma^* BO(K) \to BO$ associated to the identity representation $O(K) \to O(K)$ give rise to classes

$$HW_{i,j}(\epsilon) : H_{i-j}(BO(K), \mathbb{Z}/2) \to H^j_{et}(K, \mathbb{Z}/2).$$

These classes may then be composed with a Hurewicz homomorphism

$${}_1L_{i-j}(K, \mathbb{Z}/2) \to H_{i-j}(BO(K), \mathbb{Z}/2)$$

defined on the *mod* 2 Karoubi ${}_1L$-theory of $K$ to give analogues

$${}_1w_{i,j} : {}_1L_{i-j}(K, \mathbb{Z}/2) \to H^j_{et}(K, \mathbb{Z}/2)$$

of the Soulé Chern classes (see [24] — the Soulé Chern classes are constructed by the same method).

In the context of (B) above, the canonical map $H^n(BG, \mathbb{Z}/2) \to H^n_{et}(K, \mathbb{Z}/2)$ is equal to the composition

$$\begin{aligned}
[BG, K(\mathbb{Z}/2, n)] \xrightarrow{\Gamma^*} [\Gamma^* BG, \Gamma^* K(\mathbb{Z}/2, n)] \xrightarrow{\gamma^*} & [EG \times_G Sp(L), \Gamma^* K(\mathbb{Z}/2, n)] \\
& \cong [*, \Gamma^* K(\mathbb{Z}/2, n)] \\
& = H^n_{et}(K, \mathbb{Z}/2).
\end{aligned}$$

It follows that $SW_1(\rho)$ is represented in $H^1(K, \mathbb{Z}/2)$ by the composite

$$EG \times_G Sp(L) \xrightarrow{\gamma} \Gamma^* BG \xrightarrow{\rho} BO_n \xrightarrow{det} B\mathbb{Z}/2,$$

and hence coincides with $HW_1(\rho)$.

The image of $HW_2$ under the composition

$$H^*_{et}(BO_{n,\overline{K}}, \mathbb{Z}/2) \to H^*_{et}(\Gamma^* BG, \mathbb{Z}/2) \cong H^*(BG, \mathbb{Z}/2)$$

is the obstruction to lifting $\rho : G \to O_n(\overline{K})$ to $Pin_n(\overline{K})$. In effect, for $m$ sufficiently large, $i^* : H^2(BO_m(\overline{K}), \mathbb{Z}/2) \to H^2(B\mathbb{Z}/2^{\times m}, \mathbb{Z}/2)$ is a monomorphism, by Vogtmann's stability results [25] and the fact that the *mod* 2 étale and discrete cohomology of $BO_{\overline{K}}$ coincide [11], [6]. The obstruction to lifting $O_m(\overline{K})$ to $Pin_m(\overline{K})$ restricts to $\sigma_2(x_1, ..., x_m)$ in $H^2(B\mathbb{Z}/2^{\times m}, \mathbb{Z}/2)$, and so this obstruction must be the image of $HW_2$ under the map

$$H^2_{et}(BO_{m,\overline{K}}, \mathbb{Z}/2) \xrightarrow{\epsilon^*} H^2(\Gamma^* BO_m(\overline{K}), \mathbb{Z}/2) \cong H^2(BO_m(\overline{K}), \mathbb{Z}/2).$$

But then, by drawing the appropriate commutative diagram, one finds that $HW_2$ hits the obstruction to lifting $O_n(\overline{K})$ to $Pin_n(\overline{K})$ for all $n$. The claim follows.

The obstruction to lifting

$$G \xrightarrow{\rho} O_n(K) \subset O_n(\overline{K})$$

can be calculated by finding a finite Galois extension $N/L$ such that $G \to O_n(K) \subset O_n(N)$ lifts elementwise to $\hat{\rho} : G \to Pin_n(N)$, and then by defining the obstruction cocycle $w_2(\rho)$ by $w_2(\rho)(g_2, g_1) = \hat{\rho}(g_2 g_1)\hat{\rho}(g_1)^{-1}\hat{\rho}(g_2)^{-1}$.

Now let's calculate the obstruction cocycle for lifting the form

$$EG \times_G Sp(L) \xrightarrow{\gamma} \Gamma^* BG \xrightarrow{\rho} BO_n$$

(defined over $K$, of course). We are entitled to use the lifting $\hat{\rho}$ of $\rho$ given in the previous paragraph, so that, explicitly

$$\begin{aligned} w_2^{\pi}(\rho\gamma)(g_2, g_1) &= \hat{\rho}(g_2 g_1)\hat{\rho}(g_1)^{-1}((g_1)^*(\hat{\rho}(g_2)))^{-1} \\ &= \hat{\rho}(g_2 g_1)\hat{\rho}(g_1)^{-1}\hat{\rho}(g_2)^{-1}\hat{\rho}(g_2)((g_1)^*(\hat{\rho}(g_2)))^{-1} \end{aligned}$$

The cocycle $(g_2, g_1) \mapsto \hat{\rho}(g_2)((g_1)^*(\hat{\rho}(g_2)))^{-1}$ is just the classical spinor class $Sp_2(\rho)$ of $\rho$ in a more modern form (see the Appendix), so we have given a conceptual proof of the following special case of a theorem of Fröhlich, Kahn and Snaith [2], [10], [22]:

THEOREM 1.6. *Suppose that char$(K) \neq 2$ and that $L/K$ is a finite Galois extension with Galois group $G$. Suppose that $\rho : G \to O_n(K)$ is an orthogonal representation of $G$. Then there is a formula $HW_2(\rho) = SW_2(\rho) + Sp_2(\rho)$, valid in $H^2_{et}(K, \mathbb{Z}/2)$.*

The full statement and proof of the Fröhlich-Kahn-Snaith formula appears in the next section.

REMARK 1.7. Theorem 1.6 implies that $SW_2(\rho)$ is not a cocycle homotopy invariant. Otherwise $\rho$ could be replaced in the formula above by a representation $\rho' : H \to \mathbb{Z}/2^{\times n}$. But $Sp_2(\rho') = 0$, since the pullback of $\pi : Pin_n \to O_n$ to $\mathbb{Z}/2^{\times n}$ is surjective in global sections: all reflections in anisotropic vectors of norm 1 lift to $Pin_n(K)$. This would mean that $Sp_2(\rho) = 0$ for all representations $\rho$.

A non-trivial $Sp_2(\rho)$ over the rational numbers $\mathbb{Q}$ is given by taking $\rho$ to be any representation $G \to \mathbb{Z}/2$ which classifies the element $[7] \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$, and then by including $\mathbb{Z}/2$ into $O_n(\mathbb{Q})$ by mapping the non-trivial element of $\mathbb{Z}/2$ to the element $\sigma_w \in O_n(\mathbb{Q})$ defined by reflection in the hyperplane orthogonal to the vector $w = (2, 1, 0, ..., 0)$. But then the image of $\sigma_w$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ under the spinor norm $\delta : O_n(\mathbb{Q}) \to \mathbb{Q}^*/(\mathbb{Q}^*)^2$ is $\delta(\sigma_w) = [2^2 + 1^2] = [5]$, so that $Sp(\rho)$ can be identified with the cup product $[7] \cdot [5] \in H^2_{et}(K, \mathbb{Z}/2)$. By the Merkurjev-Suslin theorem, to show that $[7] \cdot [5] \neq 0$ in $H^2_{et}(\mathbb{Q}, \mathbb{Z}/2)$, it is enough to show that the symbol $\{7, 5\}$ is not a 2-divisible element of $K_2(\mathbb{Q})$. To see this, we compute the tame symbol $(5, 7)_5 = 1/7 \pmod 5$ in $\mathbb{F}_5^* \cong \mathbb{Z}/4\mathbb{Z}$ and observe that we get an element of order 4 (see [17, p.101]).

Going further requires knowledge of the fact that the Steenrod algebra acts on all the cohomology groups in sight, and respects all base change morphisms [9]. We also need to know that the usual list of properties for the Steenrod squares is satisfied. This implies, for example, that the universal classes $HW_i$ satisfy the Wu formulae, so that in particular $Sq^1 HW_2 = HW_1 HW_2 + HW_3$. It follows that the Wu formulae are satisfied by both the Stiefel-Whitney and the Hasse-Witt classes of the representation $\rho$.

Now apply $Sq^1$ to both sides of the equation $HW_2(\rho) = SW_2(\rho) + Sp(\rho)$. Another description of the spinor class (see the Appendix) is given by feeding the composite homomorphism

$$G \xrightarrow{\rho} O_n(K) \xrightarrow{\delta} K^*(K^*)^2$$

through the composition

$$\begin{aligned}
&hom(G, H^1_{et}(K, \mathbb{Z}/2)) \\
&\qquad \cong H^1(BG, \mathbb{Z}/2) \otimes H^1(K, \mathbb{Z}/2) \to H^1(K, \mathbb{Z}/2) \otimes H^1(K, \mathbb{Z}/2) \xrightarrow{\cup} H^2(K, \mathbb{Z}/2).
\end{aligned}$$

Here, $\delta$ is the boundary map appearing in the 6-term sequence associated to the short exact sequence of sheaves

$$e \to \mathbb{Z}/2 \to Pin_n \to O_n \to e.$$

It follows that $Sp(\rho)$ is decomposable (alternatively, one could invoke the Merkurjev-Suslin theorem [15], [14] which identifies the 2-torsion in the Brauer group of $K$ to show that all of $H^2(K, \mathbb{Z}/2)$ is decomposeable), so that $Sq^1(Sp(\rho)) = 0$. In effect, if $x$ and $y$ are two 1-dimensional Galois cohomology classes, then

$$Sq^1(xy) = x^2 y + xy^2 = \epsilon xy + x\epsilon y = 0,$$

where $\epsilon$ is the class $[-1] \in K^*/(K^*)^2$, by a cup-square formula $x^2 = \epsilon x$ for 1-dimensional classes $x$ in $H^*_{et}(K, \mathbb{Z}/2)$. Thus,

$$\begin{aligned}
HW_3 &= HW_1 HW_2 + SW_1 SW_2 + SW_3 \\
&= SW_1(SW_2 + Sp) + SW_1 SW_2 + SW_3 \\
&= SW_3 + SW_1 Sp,
\end{aligned}$$

and so

$$HW_3(\rho) = SW_3(\rho) + SW_1(\rho) Sp(\rho).$$

This is as far as we can get by this method, however, since there is no appropriate Wu formula in degree 4.

## 2. Fröhlich's twisted form.

Let $\beta$ be a non-degenerate symmetric bilinear form over $K$, which we shall also assume without loss of generality to be defined by a diagonal matrix

$$B = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}.$$

Let $O_\beta(K)$ denote its group of automorphisms over $K$: explicitly, $O_\beta(K)$ may be identified with the set of invertible $n \times n$ matrices $A$ such that $A^tBA = B$. As such, $O_\beta(K)$ is seen to be the group of $K$-rational points of a $K$-group-scheme $O_\beta$, and hence the group of global sections of a sheaf of groups $O_\beta$ represented on the big étale site $(Sch|_K)_{et}$ by this group-scheme.

Suppose that $G = Gal(L/K)$, and that $\rho : G \to O_\beta(K)$ is a representation of the group $G$ in $O_\beta(K)$. Choose the Galois extension $L$ of $K$ sufficiently large such that $\beta$ is trivial over $L$ in the sense that $B = C^tC$ for some invertible $n \times n$ matrix over $L$ (ie. $C$ is an isometry from $\beta$ to the trivial form over $L$). For what comes later, we shall also assume that $L$ is sufficiently large that the image of the composite

$$G \xrightarrow{\rho} O_\beta(K) \subset O_\beta(L)$$

is in the image of the homomorphism $\pi_\beta : Pin_\beta(L) \to O_\beta(L)$ (see Proposition A.2).

The composite $G \to O_\beta(K) \subset O_\beta(L)$ is the $G$-cocycle in $O_\beta$ which is canonically associated to $\rho$. It is easy to see that the function $g \mapsto g(C)\rho(g)C^{-1}$ defines a $G$-cocycle $(\rho, \beta)$ with values in $O_n(L)$. The cocycle $(\rho, \beta) : EG \times_G Sp(L) \to BO_n$ determines an isomorphism class of symmetric bilinear forms over $K$, which I call *Fröhlich's twisted form*. The Fröhlich-Kahn-Snaith formula (Theorem 2.4 below) is an explicit formula for $HW_2(\rho, \beta)$, given in terms of The Stiefel-Whitney classes of $\rho$, the Hasse-Witt classes of $\beta$, and the spinor norm of $\rho$ coming from the central extension

$$e \to \mathbb{Z}/2 \to Pin_\beta \xrightarrow{\pi_\beta} O_\beta \to e.$$

The basic aim of this section is to explain and prove this result.

Conjugating with the matrix $C$ determines a group isomorphism $C_* : O_\beta(L) \cong O_n(L)$. This conjugation isomorphism extends to an isomorphism of central extensions

$$\begin{array}{ccc} \mathbb{Z}/2 & = & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ Pin_\beta(L) & \xrightarrow{\cong} & Pin_n(L) \\ \downarrow & & \downarrow \\ O_\beta(L) & \xrightarrow[C_*]{\cong} & O_n(L) \end{array}$$

induced by the trivialization, and so one can define $SW_2(\rho)$ as the obstruction to lifting the composite

$$BG \xrightarrow{\rho_*} BO_\beta(K) \hookrightarrow BO_\beta(L)$$

to $BPin_\beta(L)$.

More generally, the Stiefel-Whitney classes $SW_i(\rho)$ are defined by analogy with the construction given above for $\beta$ trivial. The composite

$$G \xrightarrow{\rho} O_\beta(K) \subset O_\beta(L) \cong O_n(L) \subset O_n(\overline{K})$$

induces a map $\Gamma^* BG \to BO_n$ of simplicial sheaves over $\overline{K}$, and hence a composite

$$H^*_{et}(BO_{n,\overline{K}}, \mathbb{Z}/2) \to H^*_{et}(\Gamma^*_{\overline{K}} BG, \mathbb{Z}/2) \cong H^*(BG, \mathbb{Z}/2) \xrightarrow{can} H^*_K, \mathbb{Z}/2).$$

The image of $HW_i$ under this composite is $SW_i(\rho)$. It is easy to see that $SW_1(\rho)$ is the image in $H^1_{et}(K, \mathbb{Z}/2)$ under the canonical map of the element in $H^1(BG, \mathbb{Z}/2)$ which is determined by the composite

$$G \xrightarrow{\rho} O_\beta(K) \to \mathbb{Z}/2$$

of $\rho$ with the determinant homomorphism $det_\beta$. The point is that the determinant homomorphism commutes with the trivialization isomorphism $O_\beta(L) \cong O_n(L)$.

The determinant homomorphism $O_\beta(K) \to \mathbb{Z}/2$ is the evaluation at global sections of a morphism $det_\beta : O_\beta \to \Gamma^* \mathbb{Z}/2$ of sheaves of groups. This morphism, in turn, canonically determines a class in $H^1_{et}(BO_\beta, \mathbb{Z}/2)$ which I shall also denote by $det_\beta$. Recall that the class $det_n$ determined by $det_n : O_n \to \Gamma^* \mathbb{Z}/2$ coincides with the universal Hasse-Witt class $HW_1$.

The *spinor class* $Sp^\beta_2(\rho)$ is defined to be the image of the homomorphism

$$G \to O_\beta(K) \xrightarrow{\delta} H^1_{et}(K, \mathbb{Z}/2)$$

in $H^2_{et}(K, \mathbb{Z}/2)$. Then, calculating with explicit cocycles as in the proof of Theorem 1.6, we find a relation

$$[w_2^{\pi_\beta}(\rho)] = SW_2(\rho) + Sp^\beta_2(\rho),$$

where $[w_2^{\pi_\beta}(\rho)]$ is the obstruction to lifting the cocycle

$$EG \times_G Sp(L) \xrightarrow{\gamma} \Gamma^* BG \xrightarrow{\rho} BO_\beta$$

to $BPin_\beta$ along $\pi_\beta : BPin_\beta \to BO_\beta$.

We shall now compute $[w_2^{\pi_\beta}(\rho)]$ in some important special cases.

Suppose that $\beta = \langle a \rangle$ is the rank one form defined by an element $a$ of $K^*$, and that $\rho : G \to O_{\langle a \rangle}(K) = \mathbb{Z}/2$ is a representation of the Galois group $G$ for a Galois extension $L/K$ in which $a$ has a square root. Then $Pin_{\langle a \rangle}(L) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is a 2-torsion group, and so $SW_2(\rho) = 0$. Thus, the obstruction class $[w_2^{\pi\langle a \rangle}(\rho)]$ coincides with the spinor class $Sp_2^{\langle a \rangle}(\rho)$ corresponding to $\rho$ and the central extension

$$e \to \mathbb{Z}/2 \to Pin_{\langle a \rangle} \xrightarrow{\pi\langle a \rangle} \mathbb{Z}/2 \to e.$$

$Sp_2^{\langle a \rangle}(\rho)$ is the class in $H^2_{et}(K, \mathbb{Z}/2)$ determined by the composite

$$G \xrightarrow{\rho} \mathbb{Z}/2 \xrightarrow{[a]} K^*/(K^*)^2$$

(see the Appendix), which is the cup-product $[\rho] \cdot [a]$. One therefore has the formula

$$[w_2^{\pi\langle a \rangle}(\rho)] = [\rho] \cdot [a] \tag{2.1}$$

for a representation $\rho : G \to O_{\langle a \rangle}(K)$.

I shall call a map of simplicial sheaves of the form $EG \times_G Sp(L) \to BO_\beta$ a *$\beta$-cocycle.* A basic point in much of what follows is:

LEMMA 2.1. *Every $\beta$-cocycle $EG \times_G Sp(L) \to BO_\beta$ is a composite of the form*

$$EH \times_H Sp(N) \to B^*\mathbb{Z}/2^{\times n} \xrightarrow{i} BO_\beta$$

*up to refinement and homotopy.*

PROOF: Let $\omega : EG \times_G Sp(L) \to BO_\beta$ be a $\beta$-cocycle, and presume by refining $L$ if necessary that $\beta$ can be trivialized over $L$ via the matrix

$$C = \begin{bmatrix} \sqrt{a_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{a_n} \end{bmatrix}.$$

Then the assignment $g \mapsto g(C)\omega(g)C^{-1}$ defines a cocycle $(\omega, \beta) : EG \times_G Sp(L) \to BO_n$, which in turn represents a form over $K$. This form can be diagonalized over $K$, so that (up to possible refinement of $L$) there is an element $D$ of $O_n(L)$ such that the cocycle defined by $g \mapsto g(D)g(C)\omega(g)C^{-1}D^{-1}$ factors through the inclusion $\mathbb{Z}/2^{\times n} \subset O_n(L)$. But then $g(C)^{-1}g(D)g(C)\omega(g)C^{-1}D^{-1}C \in \mathbb{Z}/2^{\times n}$ for all $g \in G$, and so $\omega$ is homotopic to a $\mathbb{Z}/2^{\times n}$-cocycle, as claimed. ∎

Suppose that $\gamma_1 : EH \times_H Sp(N) \to BO_{\beta_1}$ and $\gamma_2 : EH \times_H Sp(N) \to BO_{\beta_2}$ are cocyles. Then the $(\beta_1 \oplus \beta_2)$-cocyle $\gamma_1 \oplus \gamma_2$ is defined to be the composite

$$\begin{array}{ccccc} EH \times_H Sp(N) & \xrightarrow{\Delta} & EH \times_H Sp(N) \times EH \times_H Sp(N) & \xrightarrow{\gamma_1 \times \gamma_2} & BO_{\beta_1} \times BO_{\beta_2} \\ & & & & \downarrow \oplus \\ & & & & BO_{\beta_1 \oplus \beta_2}, \end{array}$$

where $\Delta$ is the diagonal map and $\oplus$ is the map defined by direct sum on the matrix level.

PROPOSITION 2.2. *There is a formula*

$$[w_2^{\pi_{\beta_1 \oplus \beta_2}}(\gamma_1 \oplus \gamma_2)] = [w_2^{\pi_{\beta_1}}(\gamma_1)] + \gamma_1^*(det_{\beta_1})\gamma_2^*(det_{\beta_2}) + [w_2^{\pi_{\beta_2}}(\gamma_2)],$$

*valid in $H^2_{et}(K, \mathbb{Z}/2)$, where $det_{\beta_i}$ is the class in $H^1_{et}(BO_{\beta_i}, \mathbb{Z}/2)$ which is induced by the determinant homomorphism $O_{\beta_i} \to \Gamma^*\mathbb{Z}/2$ for $i = 1, 2$.*

PROOF: Presume, as always, that $\beta_1 = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$ and $\beta_2 = \langle b_1 \rangle \oplus \cdots \oplus \langle b_m \rangle$ are diagonal forms relative to specific orthogonal bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ respectively. All of the classes in the formula are homotopy invariant, so one can assume that $\gamma_1$ factors through the diagonal subgroup $\mathbb{Z}/2^{\times n}$ of $O_{\beta_1}$ and that $\gamma_2$ factors through the diagonal subgroup $\mathbb{Z}/2^{\times m}$ of $O_{\beta_2}$. Suppose also that the field extension $N$ is sufficiently large so that $N$ contains all of the square roots $\sqrt{a_i}$ and $\sqrt{b_j}$. Write $\overline{v}_i = v_i/\sqrt{a_i}$ and $\overline{w}_j = w_j/\sqrt{b_j}$ as elements of $Pin_{\beta_1 \oplus \beta_2}(L)$. Write $\pi = \pi_{\beta_1 \oplus \beta_2}$, and $\pi_i = \pi_{\beta_i}$ for $i = 1, 2$. Then $\pi(\overline{v}_i)$ is reflection in the hyperplane orthogonal to $v_i$, and $\pi(\overline{w}_j)$ is reflection in the hyperplane orthogonal to $w_j$.

Now write $\gamma_1(g) = (\gamma_1^1(g), \dots, \gamma_1^n(g)) \in \mathbb{Z}/2^{\times n}$ and $\gamma_2(g) = (\gamma_2^1(g), \dots, \gamma_2^m(g) \in \mathbb{Z}/2^{\times m}$ for each $g \in H$. Finally, think of each $\gamma_i^j(g)$ as being either 0 or 1, and write

$$\hat{\gamma}_i^j(g) = \begin{cases} \overline{v}_j^{\gamma_1^j(g)} & \text{if } i = 1, \\ \overline{w}_j^{\gamma_2^j(g)} & \text{if } i + 2. \end{cases}$$

Then the obstruction cocycle $w_2^\pi(\gamma_1 \oplus \gamma_2)$ is given by

$$\begin{aligned} w_2^\pi(\gamma_1 \oplus \gamma_2)(g_2, g_1) =& \hat{\gamma}_1^1(g_2 g_1) \cdot \ldots \cdot \hat{\gamma}_1^n(g_2 g_1) \hat{\gamma}_2^1(g_2 g_1) \cdot \ldots \cdot \hat{\gamma}_2^m(g_2 g_1) \\ & \cdot (\hat{\gamma}_1^1(g_1) \cdot \ldots \cdot \hat{\gamma}_1^n(g_1) \hat{\gamma}_2^1(g_1) \cdot \ldots \cdot \hat{\gamma}_2^m(g_1))^{-1} \\ & \cdot \overline{g}_1(\hat{\gamma}_1^1(g_2) \cdot \ldots \cdot \hat{\gamma}_1^n(g_2) \hat{\gamma}_2^1(g_2) \cdot \ldots \cdot \hat{\gamma}_2^m(g_2))^{-1} \end{aligned}$$

Observe that

$$\hat{\gamma}_1^j(g_1)^{-1} \cdot \overline{g}_1(\hat{\gamma}_2^i(g_2))^{-1} = (-1)^{\gamma_1^j(g_1)\gamma_2^i(g_2)} \overline{g}_1(\hat{\gamma}_2^i(g_2))^{-1} \cdot \hat{\gamma}_1^j(g_1)^{-1}$$

where the product $\gamma_1^j(g_1)\gamma_2^i(g_2)$ is taken in the field $\mathbb{F}_2$ of two elements. It follows that

$$w_2^{\pi}(\gamma_1 \oplus \gamma_2)(g_2, g_1) = w_2^{\pi_1}(\gamma_1)(g_2, g_1) \cdot w_2^{\pi_2}(\gamma_2)(g_2, g_1) \cdot (-1)^{\sum \gamma_1^j(g_1)\gamma_2^i(g_2)}$$

in $Pin_{\beta_1 \oplus \beta_2}(L)$, where the sum is indexed over all $i$ and $j$. Note that the 2-cocycle

$$(g_2, g_1) \mapsto (-1)^{\sum \gamma_1^j(g_1)\gamma_2^i(g_2)} \in \{-1, +1\}$$

represents the product

$$\gamma_1^*(det_{\beta_1})\gamma_2^*(det_{\beta_2}) = (\sum_j \gamma_1^j) \cdot (\sum_i \gamma_2^i). \quad \blacksquare$$

In view of formula (2.1), Proposition 2.2 leads to the following:

COROLLARY 2.3. *Suppose that $\rho = (\rho_1, \ldots, \rho_n) : G \to \mathbb{Z}/2^{\times n} \subset O_\beta(K)$ is a representation of $G = Gal(L/K)$, where $\beta = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$, and let $\rho$ also denote the associated $\beta$-cocycle*

$$EG \times_G Sp(L) \xrightarrow{\gamma} \Gamma^* BG \xrightarrow{\rho} BO_\beta.$$

*Then there is a relation*

$$[w_2^{\pi_\beta}(\rho)] = \sigma_2([\rho_1], \ldots, [\rho_n]) + \sum_{i=1}^{n} [\rho_i] \cdot [a_i].$$

Note that the sum $\sum_{i=1}^n [\rho_i] \cdot [a_i]$ is the spinor class associated to the central extension

$$e \to \mathbb{Z}/2 \to X_\beta \xrightarrow{\pi^*} \mathbb{Z}/2^{\times n} \to e$$

which is induced over $\mathbb{Z}/2^{\times n}$ by pulling back $\pi_\beta : Pin_\beta \to O_\beta$ along the canonical inclusion $\mathbb{Z}/2^{\times n} \subset O_\beta$.

Now recall that $HW_2(\rho, \beta)$ coincides with the obstruction $[w_2^{\pi}(\rho, \beta)]$ to lifting the cocycle $(\rho, \beta)$ to $BPin_n$ along the canonical map $\pi : BPin_n \to BO_n$. Recall that $(\rho, \beta)$ is given on the cocycle level by $g \mapsto g(C)\rho(g)C^{-1}$. $(\rho, \beta)$ can be stabilized to a $(\beta \oplus 1_n)$-cocycle $e \oplus (\rho, \beta)$, which can be represented by the block matrix

$$\begin{bmatrix} e & 0 \\ 0 & g(C)\rho(g)C^{-1} \end{bmatrix}.$$

Note that the obstruction $[w_2^{\pi_\beta}(e)]$ and the determinant class $e^*(det_\beta)$ are trivial, because $e$ is a trivial form, so that $[w_2^{\pi}(\rho, \beta)] = [w_2^{\pi_{\beta \oplus 1_n}}(e \oplus (\rho, \beta))]$ by Proposition

2.2. Recall that $C$ is an isomorphism of forms $\beta \to 1_n$ defined over $L$, with inverse $C^{-1} : 1_n \to \beta$; it follows that the block matrix

$$\begin{bmatrix} 0 & C^{-1} \\ C & 0 \end{bmatrix}$$

represents an element of $O_{\beta\oplus 1_n}(L)$. Then, by cocycle conjugating by this matrix, we find the relation

$$\begin{bmatrix} 0 & g(C)^{-1} \\ g(C) & 0 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & g(C)\rho(g)C^{-1} \end{bmatrix} \begin{bmatrix} 0 & C^{-1} \\ C & 0 \end{bmatrix} = \begin{bmatrix} \rho(g) & 0 \\ 0 & g(C)C^{-1} \end{bmatrix}$$

and so

$$[w_2^{\pi_{\beta\oplus 1_n}}(e \oplus (\rho, \beta))] = [w_2^{\pi_{\beta\oplus 1_n}}(\rho \oplus \beta)] = [w_2^{\pi_\beta}(\rho)] + \rho^*(det_\beta)\beta^*(det_n) + [w_2^{\pi}(\beta)]$$

where $\beta$ denotes the cocycle $g \mapsto g(C)C^{-1}$. We have seen above that $[w_2^{\pi_\beta}(\rho)] = SW_2(\rho) + Sp_2^{\beta}(\rho)$ and that $\rho^*(det_\beta) = SW_1(\rho)$. The results of the previous section imply that $\beta^*(det_n) = HW_1(\beta)$ and $[w_2^{\pi}(\beta)] = HW_2(\beta)$. The left hand side of the formula is $HW_2(\rho, \beta)$, so we have proved:

Theorem 2.4 (Fröhlich-Kahn-Snaith). *Suppose that $\rho : G \to O_\beta(K)$ is a representation of $G = Gal(L/K)$ in the orthogonal group $O_\beta(K)$ of the non-degenerate symmetric bilinear form $\beta$, and let $(\rho, \beta)$ denote its associated twisted form. Then there is relation*

$$HW_2(\rho, \beta) = SW_2(\rho) + SW_1(\rho)HW_1(\beta) + HW_2(\beta) + Sp_2^{\beta}(\rho),$$

*valid in $H^2_{et}(K, \mathbb{Z}/2)$.*

Recall that any $\beta$-cocycle $\omega : EG \times_G Sp(L) \to BO_\beta$ has a well-defined cocycle conjugate $(\omega, \beta) : EG \times_G Sp(L) \to BO_n$, given by $g \mapsto g(C)\omega(g)C^{-1}$. Furthermore, there is nothing special about the representation $\rho$ used in the proof of Theorem 2.4, other than the substitutions $[w_2^{\pi_\beta}(\rho)] = SW_2(\rho) + Sp_2^{\beta}(\rho)$ and $\rho^*(det_\beta) = SW_1(\rho)$. The techniques of the proof of Theorem 2.4 therefore carry over without change to give:

Theorem 2.5. *Suppose that $\omega : EG \times_G Sp(L) \to BO_\beta$ is a $\beta$-cocycle which is defined on a Galois extension $L/K$ which is big enough that the cocycle conjugate form $(\omega, \beta)$ is defined. Then there is a formula*

$$HW_2(\omega, \beta) = [w_2^{\pi_\beta}(\omega)] + \omega^*(det_\beta)HW_1(\beta) + HW_2(\beta).$$

I have refrained notationally from indicating any relationship between $\omega^*(det_\beta)$ and Hasse-Witt classes for reasons that will become apparent in the next section.

Observe that the cocycle $\omega$ in the statement of Theorem 2.5 can be assumed, by Lemma 2.1, to be a composite of the form

$$EG \times_G Sp(L) \xrightarrow{(\omega_1,\ldots,\omega_n)} B\Gamma^*\mathbb{Z}/2^{\times n} \subset BO_\beta,$$

up to homotopy and refinement, where the notation means that the composition of $(\omega_1, \ldots, \omega_n)$ with the morphism $B\Gamma^*\mathbb{Z}/2^{\times n} \to B\Gamma^*\mathbb{Z}/2$ induced by the $i^{th}$ projection homomorphism $pr_i : \mathbb{Z}/2^{\times n} \to \mathbb{Z}/2$ is $\omega_i$. For $\omega$ of this form, its cocycle conjugate $(\omega, \beta)$ has the form

$$EG \times_G Sp(L) \xrightarrow{(\zeta_1,\ldots,\zeta_n)} B\Gamma^*\mathbb{Z}/2^{\times n} \subset BO_n,$$

where each $\zeta_i$ is defined by $\zeta_i(g) = g(\sqrt{a_i})\omega_i(g)(\sqrt{a_i})^{-1}$. It follows that

$$HW_2(\omega, \beta) = \sigma_2([\omega_1] + [a_1], \ldots, [\omega_n] + [a_n])$$

in $H^2(K, \mathbb{Z}/2)$. Insofar as this can be done for any $\beta$-form $\omega$, Theorem 2.5 can be seen, via Proposition 2.2, as a consequence of the following formal identities of symmetric polynomials:

$$\sigma_2(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sigma_2(x_1, \ldots, x_n, y_1, \ldots, y_n) - \sum_{i=1}^{n} x_i y_i$$

$$= \sigma_2(x_1, \ldots, x_n) - \sum_{i=1}^{n} x_i y_i + \sigma_1(x_1, \ldots, x_n)\sigma_1(y_1, \ldots, y_n) + \sigma_2(y_1, \ldots, y_n).$$

As in the previous section, one can apply the Wu formula $Sq^1 w_2 = w_3 + w_2 w_1$ to both sides of the equation in Theorem 2.4, keeping in mind that $Sq^1$ is identically 0 on $H^2_{et}(K, \mathbb{Z}/2)$ since every class in $H^2(K, \mathbb{Z}/2)$ is decomposeable by the Merkurjev-Suslin theorem [15]. This means in particular that $HW_3(\rho, \beta) = HW_2(\rho, \beta) \cdot HW_1(\rho, \beta)$. Expanding this relation and rearranging terms gives

$$\begin{aligned} HW_3(\rho, \beta) &= SW_3(\rho) + SW_2(\rho)HW_1(\beta) + SW_1(\rho)HW_2(\beta) + HW_3(\beta) \\ &\quad + (SW_1(\rho) + HW_1(\beta))Sp_2^\beta(\rho) + SW_1(\rho)^2 HW_1(\beta) + SW_1(\rho)HW_1(\beta)^2. \end{aligned}$$

But

$$SW_1(\rho)^2 HW_1(\beta) + SW_1(\rho)HW_1(\beta)^2 = \epsilon SW_1(\rho)HW_1(\beta) + SW_1(\rho)\epsilon HW_1(\beta) = 0,$$

where $\epsilon = [-1]$ in $H^1_{et}(K, \mathbb{Z}/2)$. We have therefore proved:

Theorem 2.6. *Suppose that $\rho : G \to O_\beta(K)$ is a representation of the Galois group $G$ as in Theorem 2.4. Then there is a relation*

$$\begin{aligned} HW_3(\rho, \beta) &= SW_3(\rho) + SW_2(\rho)HW_1(\beta) + SW_1(\rho)HW_2(\beta) + HW_3(\beta) \\ &\quad + (SW_1(\rho) + HW_1(\beta))Sp_2^\beta(\rho), \end{aligned}$$

*valid in $H^3_{et}(K, \mathbb{Z}/2)$.*

This last formula was derived by a slightly different method in [9].

## 3. The cohomology of $BO_\beta$.

Once again, assume that $\beta$ is the non-degenerate symmetric bilinear form defined over $K$ by the diagonal matrix

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

with $a_i \in K^*$, and recall that the automorphism group $O_\beta$ of $\beta$ has a subgroup $\mathbb{Z}/2^{\times n}$ included as diagonal matrices. The main result of this section is:

THEOREM 3.1. *Suppose that $K$ is a field such that $char(K) \neq 2$, and let $\beta$ be a non-degenerate symmetric bilinear form of rank $n$ over $K$ as above. Let $A$ denote the Galois cohomology ring $H^*_{et}(K, \mathbb{Z}/2)$ of $K$. Then there is ring isomorphism*

$$H^*_{et}(BO_\beta, \mathbb{Z}/2) \cong A[HW_1^\beta, \dots, HW_n^\beta],$$

$\deg(HW_i^\beta) = i$ *for* $i = 1, \dots, n$. *The canonical imbedding* $i : \Gamma^*\mathbb{Z}/2^{\times n} \to O_\beta$ *induces a map* $i^* : H^*_{et}(BO_\beta, \mathbb{Z}/2) \to H^*_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) \cong A[x_1, \dots, x_n]$ *(where* $\deg(x_i) = 1$*), which is a monomorphism of rings such that*

$$i^*(HW_i^\beta) = \sigma_i(x_1 + [a_1], \dots, x_n + [a_n]).$$

PROOF: Let $L$ be a Galois extension of $K$ with Galois group $G$ such that $\beta$ trivializes over $L$, with trivializing matrix

$$C = \begin{bmatrix} \sqrt{a_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{a_n} \end{bmatrix}$$

as in the last section. Note that the projection $pr : EG \times_G Sp(L) \times BO_\beta \to BO_\beta$ is a weak equivalence of simplicial sheaves over $K$. The basic idea of proof is to construct a map $\gamma : EG \times_G Sp(L) \times BO_\beta \to BO_n$ of simplicial sheaves over $K$, and then show that this map is a weak equivalence.

Recall that $EG \times_G Sp(L)$ is the nerve of the sheaf of translation categories given by the action of $G$ on the sheaf $Sp(L)$. Thus, any such simplicial map $\gamma$ is induced by some functor, and can be identified with a functor $EG \times_G Sp(L) \to \mathbf{hom}(O_\beta, O_n)$, where $\mathbf{hom}(O_\beta, O_n)$ is the sheaf of categories whose sheaf of objects is the sheaf of global homomorphisms from the sheaf of groups $O_\beta$ to the sheaf of groups $O_n$, and whose morphisms consist of the global sheaf of natural transformations. A functor $\gamma : EG \times_G Sp(L) \to \mathbf{hom}(O_\beta, O_n)$ therefore consists of:

(1) a homomorphism $\gamma_0 : O_\beta|_L \to O_n|_L$ defined over $L$ (the object morphism), and

(2) a collection $\gamma_g : O_\beta|_L \times 1 \to O_n|_L$, $g \in G$, of homotopies (global conjugations) $\gamma_g : \gamma_0 \to g^*\gamma_0$, which homotopies satisfy

   (a) the cocycle condition:

$$\begin{array}{ccc} \gamma_0 & \xrightarrow{\gamma_{g_1}} & g_1^*\gamma_0 \\ {\scriptstyle \gamma_{g_2 g_1}}\downarrow & & \downarrow{\scriptstyle g_1^*\gamma_{g_2}} \\ (g_2 g_1)^*\gamma_0 & = & g_1^* g_2^* \gamma_0 \end{array}$$

commutes for all $g_1, g_2 \in G$, and

   (b) the homotopy $\gamma_e$ is the identity.

Set $\gamma_0$ to be the (globally defined) conjugation map $X \mapsto CXC^{-1}$, so that $g^*\gamma_0$ is conjugation by $\overline{g}(C)$ where $\overline{g}$ is the image of $g : Sp(L) \to Sp(L)$ in the ordinary Galois group under the standard contravariant isomorphism. Now define the homotopy $\gamma_g$ to be conjugation by the global section $\overline{g}(C)C^{-1}$ of $O_n(L)$. One verifies easily that conditions (a) and (b) are satisfied, so that we have constructed a map of simplicial $K$-sheaves $\gamma : EG \times_G Sp(L) \times BO_\beta \to BO_n$.

I claim that this map is a weak equivalence. The two sheaves clearly have the same sheaf of path components, namely the terminal sheaf $*$, and every base point of $EG \times_G Sp(L) \times BO_\beta$ must be defined over $L$, so it suffices to show that the restriction of $\gamma$ to $L$ defines a weak equivalence of simplicial sheaves over $L$. But $EG \times_G Sp(L)|_L$ may be identified with the constant object $\Gamma^*(EG \times_G G)$, where the isomorphism $\Gamma^*(EG \times_G G) \to EG \times_G Sp(L)|_L$ is determined by the $G$-equivariant map $G \to Sp(L)(L)$ which takes $g$ to $g$. It follows that $\gamma|_L$ may be identified with the functor $\Gamma^*(EG \times_G G) \to \mathbf{hom}(O_\beta, O_n)|_L$ which is defined on objects $g \in G$ by

$$\gamma|_L(g) = g^*\gamma_0$$

and defined on morphisms $(g_2, g_1)$ by

$$\gamma|_L(g_2, g_1) = \text{the morphism} \quad g_1^*\gamma_0 \xrightarrow{g_1^*\gamma_{g_2}} g_1^* g_2^* \gamma_0 = (g_2 g_1)^*\gamma_0.$$

Then the conjugation maps $\gamma_g : \gamma_0 \to g^*\gamma_0$ define a natural transformation and hence a homotopy of maps of simplicial sheaves over $L$ from the composite

$$\Gamma^*(EG \times_G G) \times BO_\beta \xrightarrow{pr} BO_\beta \underset{\cong}{\xrightarrow{\gamma_0}} BO_n$$

to $\gamma|_L$. This composite is a weak equivalence of simplicial sheaves over $L$, so $\gamma|_L$ is a weak equivalence, as claimed.

Define $HW_i^\beta$ to be the image in $H^i_{et}(BO_\beta, \mathbb{Z}/2)$ under the resulting isomorphisms

$$H^*_{et}(BO_\beta, \mathbb{Z}/2) \xleftarrow[\cong]{pr^*} H^*_{et}(EG \times_G Sp(L) \times BO_\beta, \mathbb{Z}/2) \xrightarrow[\cong]{\gamma^*} H^*_{et}(BO_n, \mathbb{Z}/2).$$

Then $H^*_{et}(BO_\beta, \mathbb{Z}/2) \cong A[HW_1^\beta, \ldots, HW_n^\beta]$, with $\deg(HW_i^\beta) = i$ for $i = 1, \ldots, n$, as claimed.

To finish the proof of the theorem, observe that $\gamma$ restricts to a map $\gamma : EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2^{\times n} \to \Gamma^* B\mathbb{Z}/2^{\times n}$. In turn, $\gamma$ is a product of maps of the form $\gamma_i : EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2 \to \Gamma^* B\mathbb{Z}/2$, each of which is a cocycle of the form $\gamma_i(x, g) = x + g(\sqrt{a_i})(\sqrt{a_i})^{-1}$, when defined with respect to the additive structure of the group $\mathbb{Z}/2$. Alternatively, each $\gamma_i$ is the $\gamma$ associated to the form $\langle a_i \rangle$. It follows in particular that the map $\gamma : EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2^{\times n} - \to \Gamma^* B\mathbb{Z}/2^{\times n}$ is a weak equivalence. Observe that there is a commutative diagram of the form

$$(3.2) \qquad \begin{array}{ccccc} \Gamma^* B\mathbb{Z}/2^{\times n} & \xleftarrow{pr} & EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2^{\times n} & \xrightarrow{\gamma} & \Gamma^* B\mathbb{Z}/2^{\times n} \\ \downarrow & & \downarrow & & \downarrow \\ BO_\beta & \xleftarrow[pr]{} & EG \times_G Sp(L) \times BO_\beta & \xrightarrow[\gamma]{} & BO_n \end{array}$$

where the vertical maps are induced by the canonical inclusions and the horizontal maps are weak equivalences. There is also a commutative diagram of the form

$$(3.3) \qquad \begin{array}{ccccc} \Gamma^* B\mathbb{Z}/2^{\times n} & \xleftarrow{pr} & EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2^{\times n} & \xrightarrow{\gamma} & \Gamma^* B\mathbb{Z}/2^{\times n} \\ {\scriptstyle x_i}\downarrow & & \downarrow & & \downarrow{\scriptstyle x_i} \\ \Gamma^* B\mathbb{Z}/2 & \xleftarrow[pr]{} & EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2 & \xrightarrow[\gamma_i]{} & \Gamma^* B\mathbb{Z}/2 \end{array}$$

in which each of the vertical maps are induced by the projection $\mathbb{Z}/2^{\times n} \to \mathbb{Z}/2$ onto the $i^{th}$ factor. Recall that the $i^{th}$ projection induces the $i^{th}$ generator $x_i$ in $H^1_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)$, where $H^*_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) \cong A[x_1, \ldots, x_n]$ and $A = H^*_{et}(K, \mathbb{Z}/2)$. It follows from the commutativity of (3.3) and the description of $\gamma_i$ given above that $x_i \mapsto x_i + [a_i]$ in $H^1(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)$ under the composite

$$(3.4) \qquad \begin{array}{ccc} H^1_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) & \dashrightarrow & \\ \cong\downarrow\gamma^* & & \\ H^1_{et}(EG \times_G Sp(L) \times \Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) & \xleftarrow[pr^*]{\cong} & H^1_{et}(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) \end{array}$$

for $i = 1, \dots, n$. It follows that $\sigma_i(x_1, \dots, x_n) \mapsto \sigma_i(x_1 + [a_1], \dots, x_n + [a_n])$ under the composite (3.4) and so $HW_i^\beta \mapsto \sigma_i(x_1+[a_1], \dots, x_n+[a_n])$ under $i^*$ as claimed, by the commutativity of (3.2). ∎

REMARK 3.2. It is a somewhat delicate point that the morphism $\gamma$ of the proof of Theorem 3.1 is a weak equivalence of simplicial sheaves. On first blush, one might expect that $EG \times_G Sp(L) \times BO_\beta$ and $BO_n$ have different sheaves of fundamental groups, namely $O_\beta$ and $O_n$ respectively. But these sheaves of fundamental groups must be defined with respect to base points somewhere, and for the vertex set of the simplicial set of $U$-sections $(EG \times_G Sp(L) \times BO_\beta)(U)$ to be non-empty, $U$ must be an $L$-scheme, and so $O_\beta(U)$ and $O_n(U)$ coincide.

We've seen the classes $HW_i^\beta(\omega)$ before: it is a trivial consequence of Theorem 3.1 that $HW_i^\beta(\omega) = HW_i(\omega, \beta)$. Thus, for whatever reason (ultimately a symmetric polynomial relation), one finds a direct sum formula

$$HW_i^{\beta_1 \oplus \beta_2}(\omega_1 \oplus \omega_2) = HW_i^{\beta_1}(\omega_1) + HW_{i-1}^{\beta_1}(\omega_1) HW_1^{\beta_2}(\omega_2) + \cdots + HW_i^{\beta_2}(\omega_2), \tag{3.5}$$

as one would expect.

The catch with the $HW_i^\beta(\omega)$ is that they are a bit counter-intuitive. If $\omega$ is trivial, then it is easy to see that $HW_i^\beta(\omega) = HW_i(\beta)$, which can be non-zero in general. It then follows from the direct sum formula, for example, that

$$HW_i^{\beta_1 \oplus \beta_2}(\omega \oplus e) = HW_i^{\beta_1}(\omega) + HW_{i-1}^{\beta_1}(\omega) HW_1(\beta_2) + \cdots + HW_i(\beta_2). \tag{3.6}$$

Secondly, $HW_2^\beta(\omega)$ does not coincide with the obvious obstruction to lift. But Theorem 2.5 implies a formula

$$HW_2^\beta(\omega) = [w_2^{\pi_\beta}(\omega)] + \omega^*(det_\beta) HW_1(\beta) + HW_2(\beta),$$

and $\omega^*(det_\beta) = HW_1^\beta(\omega) + HW_1(\beta)$, so we have

$$HW_2^\beta(\omega) = [w_2^{\pi_\beta}(\omega)] + (HW_1^\beta(\omega) + HW_1(\beta)) HW_1(\beta) + HW_2(\beta). \tag{3.7}$$

## 4. Appendix.

### 4.1. $Pin_n(K)$ as a group.

The following description of $Pin_n(K)$ is based on the appendices of the Fröhlich paper [2].

$Cl(n)$ is the Clifford $K$-algebra corresponding to the trivial form $1_n$. It is the tensor algebra on $K^n$, modulo the relation $v^2 = \langle v, v\rangle$, where $\langle v, v\rangle = v^t v$. Let $e_1, \ldots, e_n$ denote the standard basis for $K^n$. Then $Cl(n)$ has a $K$-basis consisting of the products $e_{i_1} e_{i_2} \cdot \ldots \cdot e_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$. As a matter of notation, observe that the transformation $v \mapsto -v$ extends to an involution $I$ on $Cl(n)$; Note that $I(x) = \pm x$ for each homogeneous $x \in Cl(n)$.

The *Clifford group* $C^*(n)$ is the group of homogeneous invertible elements of $Cl(n)$ consisting of those elements $x$ such that $xvx^{-1} \in V := K^n$ for all $v \in V$. If a vector $w$ of $V$ is *anisotropic* in the sense that $\langle w, w\rangle \neq 0$, then $w$ is an element of $C^*(n)$, for $w^{-1} = w/\langle w, w\rangle$, and one can show that $v \mapsto -wvw^{-1}$ is reflection in the hyperplane orthogonal to $w$, just by examining the effect of this transformation on the subspace $\langle w\rangle$ and its orthogonal complement. In particular, $w$ is in $C^*(n)$. Note that, for $x \in C^*(n)$, the transformation $v \mapsto xvx^{-1}$ is orthogonal, since $xvx^{-1}xvx^{-1} = v^2$ for all $v \in V$, so that $\langle xvx^{-1}, xvx^{-1}\rangle = \langle v, v\rangle$. Now $I(x) = \pm x$, so that the map $x \mapsto (v \mapsto I(x)vx^{-1})$ defines a group homomorphism $r : C^*(n) \to O_n(K)$ such that $r(v)$ is the reflection in the hyperplane orthogonal to $v$ for each anisotropic $v \in V$.

One checks that no non-zero homogeneous element $x$ of odd degree in $Cl(n)$ anticommutes with all $v \in V$ in the sense that $vxv^{-1} = -x$. In effect,

$$e_j e_{i_1} \cdots e_{i_r} e_j = \begin{cases} (-1)^r e_{i_1} \ldots e_{i_r} & \text{if } j \notin \{i_1, \ldots, i_r\} \\ (-1)^{r-1} e_{i_1} \ldots e_{i_r} & \text{if } j \in \{i_1, \ldots, i_r\}. \end{cases}$$

Thus, conjugation by $e_j$ preserves all monomial summands in all degrees, and for $r$ odd and $\alpha \in K$, $e_j \alpha e_{i_1} \cdot \ldots \cdot e_{i_r} e_j = -\alpha e_{i_1} \cdots e_{i_r}$ if and only if $j \notin \{i_1, \ldots, i_r\}$. The claim follows.

Similarly, no non-zero homogeneous element of strictly positive even degree commutes with all $v \in V$. It follows that there is a short exact sequence of the form

$$e \to K^* \to C^*(n) \xrightarrow{r} O_n(K) \to e,$$

where the inclusion of $K^*$ in $C^*(n)$ is induced by the inclusion of the one dimensional summand corresponding to the identity element of $C^*(n)$.

There is an "involutory antiautomorphism" $x \mapsto x^t$ on $Cl(n)$, given on basis elements by $(e_{i_1} \cdot \ldots \cdot e_{i_k})^t = e_{i_k} \cdot \ldots \cdot e_{i_1}$ (this antiautomorphism is induced by a tensor algebra antiautomorphism $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1$). One finds that $v^t = v$ for each vector $v$. Thus, for $x \in C^*(n)$ and $v \in V$, $xvx^{-1} \in V$, and so $xvx^{-1} = (xvx^{-1})^t = (x^{-1})^t vx^t$, whence $x^txv(x^tx)^{-1} = v$. It follows that $x^tx \in K$ for all $x \in C^*(n)$, and the map $x \mapsto x^tx$ defines a group homomorphism $N : C^*(n) \to K^*$.

Note that the restriction of $N$ to $K^*$ takes values in $(K^*)^2$. Also, define $Pin_n(K)$ to be the kernel of $N$. Thus, there is a diagram of the form

$$\begin{CD}
e @>>> \mathbb{Z}/2 @>>> Pin_n(K) @>>> \overline{O}_n(K) @>>> e \\
@. @VVV @VVV @VVV @. \\
e @>>> K^* @>>> C^*(n) @>r>> O_n(K) @>>> e \\
@. @V{sq}VV @V{N}VV @VV{\delta}V @. \\
e @>>> (K^*)^2 @>>> K^* @>>> K^*/(K^*)^2 @>>> e
\end{CD} \tag{A.8}$$

Here, $Pin_n(K) \twoheadrightarrow \overline{O}_n(K) \hookrightarrow O_n(K)$ is an epi-monic factorization of the restriction of $r$ to $Pin_n(K)$, and $sq$ is the squaring map. We shall see that the induced map $\delta$ is the classical spinor norm.

**4.2.** $Pin_n$ **as a sheaf.**

The Clifford algebra $Cl(n)$ is a finite dimensional $K$-vector space, say of dimension $N$. It therefore determines a sheaf of $K$-algebras on the big étale site $(Sch|_K)_{et}$, which sheaf we shall also denote by $Cl(n)$. In effect, the product on $Cl_n$ may be used to define a ring structure on the scheme (or sheaf) $\mathbb{A}^N$, and so on. The Clifford algebra product $Cl(n) \times Cl(n) \to Cl(n)$ determines a map of vector spaces (and hence of affine schemes) $m_* : Cl(n) \to End_K(Cl(n)) \cong M_N$, where $M_N$ is the affine scheme of $N \times N$ matrices over $K$. It is easy to see that there is a pullback diagram of sheaves

$$\begin{CD}
Cl(n)^* @>{m_*}>> Gl_N \\
@VVV @VVV \\
Cl(n) @>>{m_*}> M_N
\end{CD}$$

where $Cl(n)^*$ is the group of units in $Cl(n)$. It follows that $Cl(n)^*$ is an open subscheme of $\mathbb{A}^N$, and hence has the structure of an affine algebraic group-scheme over $K$.

Let $Gl_{N,V}$ denote the parabolic subgroup of $Gl_N$ which consists of those matrices (or linear automorphisms of $Cl(n)$) that preserve $V$, where $V$ is the subspace of $Cl(n)$ given above. $Gl_{N,V}$ is a closed subgroup-scheme of $Gl_N$, and so a closed subgroup-scheme $\overline{C}^*(n)$ of $Cl(n)^*$ is defined by the requirement that the following diagram is a pullback:

$$\begin{array}{ccc} \overline{C}^*(n) & \longrightarrow & Gl_{N,V} \\ \big\downarrow & & \big\downarrow \\ Cl(n)^* & \longrightarrow & Gl_N \end{array}$$

where the map $Cl(n)^* \to Gl_N$ is defined by $x \mapsto (v \mapsto xvx^{-1})$. In particular, $\overline{C}^*(n)$ is affine. The induced composition

$$\overline{C}^*(n) \to Gl_{N,V} \to Gl(V) = Gl_n$$

factors through the inclusion of $O_n$ in $Gl_n$; the point is that $\langle xvx^{-1}, xvx^{-1}\rangle = xvx^{-1}xvx^{-1} = v^2 = \langle v, v\rangle$ in each algebra of sections, just as above.

Let $C^p(n)$ denote the presheaf of subgroups of $\overline{C}^*(n)$ which is given by the homogeneous elements. $C^p(n)$ is a finite disjoint union of sheaves represented by closed affine subschemes of $\overline{C}^*(n)$. It follows that its associated sheaf, which we shall denote by $C^*(n)$, is represented by an affine subgroup-scheme of $\overline{C}^*(n)$ (albeit a disconnected one), which we shall also denote by $C^*(n)$. Let $r : C^p(n) \to O_n$ be the homomorphism of presheaves of groups which is given in sections by $x \mapsto (v \mapsto I(x)vx^{-1})$. Then one shows just as before, by examining centralizers (and "anticentralizers") of elements of $V(R)$ in each section that the sequence of presheaves

$$e \to \mathbb{G}_m \xrightarrow{i} C^p(n) \xrightarrow{r} O_n$$

is exact, where $\mathbb{G}_m$ is identified via the homomorphism $i$ with the subgroup of invertible elements of degree 0 in $Cl(n)$. The sequence of associated sheaves

$$e \to \mathbb{G}_m \xrightarrow{i} C^*(n) \xrightarrow{r} O_n$$

must therefore also be exact. Notice that $C^p(n)(L) = C^*(n)(L)$ for all fields $L/K$.

An argument involving centralizers of $V$ also shows that the norm map $x \mapsto x^t x$ induces a presheaf of group homomorphism $C^p(n) \to \mathbb{G}_m$, and hence a morphism of group-schemes $N : C^*(n) \to \mathbb{G}_m$. Observe that the composite

$$\mathbb{G}_m \xrightarrow{i} C^*(n) \xrightarrow{N} \mathbb{G}_m$$

is the squaring map.

$Pin_n$ is the subgroup-scheme of $C^*(n)$ which is defined by the requirement that the sequence

$$e \to Pin_n \to C^*(n) \xrightarrow{N} \mathbb{B}_m$$

should be exact in the category of sheaves of groups. Then one immediately finds that there is an exact sequence of sheaves of groups over $K$ of the form

$$e \to \mathbb{Z}/2 \to Pin_n \xrightarrow{\pi} O_n,$$

where $\pi : Pin_n \to O_n$ is defined to be the composition

$$Pin_n \to C^*(n) \xrightarrow{r} O_n.$$

$\pi$ is also a map of group-schemes over $K$, with group-scheme kernel given by the constant group $\mathbb{Z}/2$. It follows in particular that $\pi$ is unramified (see [16, p.21], for example). But now let $\overline{K}$ be the algebraic closure of $K$. Then $\pi$ base changes to the corresponding construction over $\overline{K}$, and, on $\overline{K}$-rational points, $\pi : Pin_n(\overline{K}) \to O_n(\overline{K})$ is surjective. Thus, the affine group-scheme homomorphism $Pin_{n,\overline{K}} \to O_{n,\overline{K}}$ is faithfully flat, so that the $K$-group-scheme map $\pi : Pin_n \to O_n$ is faithfully flat as well. $\pi$ is therefore a covering étale map, and is hence an epimorphism in the category of sheaves on the big étale site $(Sch|_K)_{et}$. We have proved

PROPOSITION A.1. *There is a short exact sequence*

$$e \to \mathbb{Z}/2 \to Pin_n \xrightarrow{\pi} O_n \to e$$

*of sheaves of groups on $(Sch|_K)_{et}$ whose global sections gives the exact sequence*

$$e \to \mathbb{Z}/2 \to Pin_n(K) \to O_n(K)$$

*which arises from the diagram (A.8).*

### 4.3. The sheaf of groups $Pin_\beta$.

Let $\beta : V \times V \to K$ be an arbitrary non-degenerate symmetric bilinear form of rank $n$ over $K$, and choose an orthogonal basis $\{e_1, \dots, e_n\}$ for $\beta$. The Clifford algebra $Cl(\beta)$ is, as before, the tensor algebra on $V = K^n$, modulo the relation $v^2 = \beta(v, v)$ for $v \in V$. It is still true that $v \cdot w = -w \cdot v$ in $Cl(\beta)$ for $v$ and $w$ in $V$ with $v$ orthogonal to $w$, so that $Cl(\beta)$ has a $K$-vector-space basis consisting of the identity element and monomials of the form $e_{i_1} \cdot \dots \cdot e_{i_r}$ with $1 \leq i_1 < \dots < i_r \leq n$. In particular, $Cl(\beta)$ is a finite dimensional $K$-vector space, say of dimension $N$, so that the product determines a sheaf of rings structure on the affine space $\mathbb{A}^N$ over $K$. As before, the multiplication on $Cl(\beta)$ determines a morphism of schemes and sheaves $Cl(\beta) \to M_N$ over $K$, so that the group of units $Cl(\beta)^*$ has the structure of an affine algebraic group-scheme over $K$. One can also form the Clifford group-scheme $C^*(\beta)$ as the subgroup-scheme of homogeneous elements of $Cl(\beta)^*$ which conjugate $V$ into itself, and then one finds the group-scheme homomorphism $r : C^*(\beta) \to O_\beta$ just as before, where $O_\beta$ is the group-scheme of automorphisms of $\beta$.

Suppose that $x \in C^*(\beta)(K)$ has even degree and commutes with all elements of $V$. Choose a finite Galois extension $L/K$ such that $\beta(e_i, e_i) = \alpha_i^2$ for some $\alpha_i \in L$, for all $i = 1, \ldots, n$. Then the base extension $x_L$ of $x$ in $C^*(\beta)(L)$ has even degree and commutes with all elements of $V(L)$. In particular, $x_L$ commutes with each of the $\alpha_i^{-1} e_i$. On the other hand,

$$(\alpha_j^{-1} e_j) e_{i_1} \cdot \ldots \cdot e_{i_r} (\alpha_j^{-1} e_j) = \begin{cases} (-1)^r e_{i_1} \cdot \ldots \cdot e_{i_r} & \text{if } j \notin \{i_1, \ldots, i_r\} \\ (-1)^{r-1} e_{i_1} \cdot \ldots \cdot e_{i_r} & \text{if } j \in \{i_1, \ldots, i_r\} \end{cases}$$

so that $x$ had better not involve $e_j$ in any of its non-trivial monomial summands. But this is true for all $j$, so no such $x$ can exist in positive degrees. Similarly, no element of $C^*(\beta)(K)$ of odd degree can anti-commute with all elements of $V$. This argument works more generally, since the projection $X \times_{Sp(K)} Sp(L) \to X$ is covering for any $X$ in $(Sch|_K)_{et}$, so that $C^*(\beta)(X)$ maps injectively into $C^*(\beta)(X \times_{Sp(K)} Sp(L))$. We have in particular a short exact sequence of sheaves of the form

$$e \to \mathbb{G}_m \to C^*(\beta) \xrightarrow{r} O_\beta.$$

The norm homomorphism $N : C^*(\beta) \to \mathbb{G}_m$ is defined by a similar argument. The group-scheme $Pin_\beta$ is defined by the requirement that the following sequence should be exact:

$$e \to Pin_\beta \to C^*(\beta) \xrightarrow{N} \mathbb{G}_m \to e.$$

Let $\pi : Pin_\beta \to O_\beta$ be defined to be the composite homomorphism

$$Pin_\beta \to C^*(\beta) \xrightarrow{N} O_\beta.$$

Then the group-scheme kernel of $\pi$ may be identified with the constant group $\mathbb{Z}/2$, just as before, so that $\pi$ is unramified. Furthermore, since $\beta$ trivializes over the algebraic closure $\overline{K}$, $\pi$ base changes to the map $Pin_{n,\overline{K}} \to O_{n,\overline{K}}$ up to isomorphism, so that $\pi$ is faithfully flat. We have shown:

PROPOSITION A.2. *Each non-degenerate symmetric bilinear form $\beta$ over $K$ determines a short exact sequence of sheaves*

$$e \to \mathbb{Z}/2 \to Pin_\beta \xrightarrow{\pi_\beta} O_\beta \to 0.$$

Note that the induced map of sections $r : C^*(\beta)(K) \to O_\beta(K)$ is surjective. In effect, $\pi$ is a sheaf epimorphism so $r$ is, and $H^1_{et}(K, \mathbb{G}_m) = 0$ (Hilbert Theorem 90);

use the six term exact sequence associated to the short exact sequence of sheaves of groups

$$e \to \mathbb{G}_m \to C^*(\beta) \xrightarrow{r} O_\beta \to e.$$

One therefore recreates the Fröhlich diagram that defines the spinor norm

$$\delta : O_\beta(K) \to K^*/(K^*)^2.$$

Note also that these groups and maps are invariant up to isomorphism of the choice of orthogonal basis of $\beta$.

**4.4. The spinor class.**

Suppose that

$$e \to A \to P \to O \to e$$

is a central extension of sheaves of groups on $(Sch|_K)_{et}$ with constant kernel $A = \Gamma^* A$, and suppose that $\rho : G \to O(K)$ is a representation of $G$, where $G = Gal(L/K)$ for some finite Galois extension $L$ of $K$. Choose an extension $N/K$ with Galois group $H$ such that $L$ can be identified with a subfield of $N$, and such that the composite group homomorphism

$$G \xrightarrow{\rho} O(K) \subset O(N)$$

lifts to a function $\hat{\rho} : G \to P(N)$. Then, by definition of the boundary map $\delta : O(K) \to H^1_{et}(K, A)$, the composite map

$$G \xrightarrow{\rho} O(K) \xrightarrow{\delta} H^1_{et}(K, A)$$

is given by $g \mapsto [h \mapsto \hat{\rho}(g)h(\hat{\rho}(g))^{-1}]$, where the thing in the square brackets is the element of $H^1_{et}(K, A)$ which is represented by the cocyle $h \mapsto \hat{\rho}(g)h(\hat{\rho}(g))^{-1}$ defined on $H$ which takes values in $A$.

Consider the function $\rho_*(g, h) = \hat{\rho}(g)h(\hat{\rho}(g))^{-1}$, defined on $G \times H$ and taking values in A. Then $\rho_*$ is a homomorphism in both variables. In effect,

$$\begin{aligned}\hat{\rho}(g)h_2h_1(\hat{\rho}(g))^{-1} &= \hat{\rho}(g)h_2(\hat{\rho}(g))^{-1}h_2(\hat{\rho}(g))h_2h_1(\hat{\rho}(g))^{-1} \\ &= \hat{\rho}(g)h_2(\hat{\rho}(g))^{-1}\hat{\rho}(g)h_1(\hat{\rho}(g))^{-1}\end{aligned}$$

since $h_2$ fixes $\hat{\rho}(g)h_1(\hat{\rho}(g))^{-1} \in A$. Also, $\hat{\rho}(g_2g_1) = \hat{\rho}(g_2)\hat{\rho}(g_1)c(g_2, g_1)$ for some element $c(g_2, g_1) \in A$, so that

$$\begin{aligned}\hat{\rho}(g_2g_1)h(\hat{\rho}(g_2g_1))^{-1} &= \hat{\rho}(g_2)\hat{\rho}(g_1)c(g_2, g_1)h(\hat{\rho}(g_2)\hat{\rho}(g_1)c(g_2, g_1))^{-1} \\ &= \hat{\rho}(g_2)\hat{\rho}(g_1)c(g_2, g_1)h(c(g_2, g_1))^{-1}h(\hat{\rho}(g_1))^{-1}h(\hat{\rho}(g_2))^{-1}.\end{aligned}$$

But $c(g_2, g_1)h(c(g_2, g_1))^{-1} = e$ since $h$ fixes $c(g_2, g_1) \in A$, and $\hat{\rho}(g_1)h(\hat{\rho}(g_1)^{-1} \in A$, which is the centre of $P$, so that

$$\hat{\rho}(g_2g_1)h(\hat{\rho}(g_2g_1))^{-1} = \hat{\rho}(g_1)h(\hat{\rho}(g_1))^{-1}\hat{\rho}(g_2)h(\hat{\rho}(g_2))^{-1},$$

and the claim is proved.

It follows that there is a homomorphism $\tilde{\rho} : G \to \mathbf{hom}_{Gr}(H, A) = H^1(BH, A)$ defined by $g \mapsto [h \mapsto \hat{\rho}(g)h(\hat{\rho}(g))^{-1}]$, and that this homomorphism represents the map $\delta\rho : G \to H^1_{et}(K, A)$ in the sense that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}} & H^1(BH, A) \\ & \searrow^{\delta\rho} & \downarrow{\scriptstyle can} \\ & & H^1_{et}(K, A) \end{array}$$

commutes.

Now specialize to the case where $A$ is a $\mathbb{Z}/2$-vector space, and recall that $G$ is a finite group. For arbitrary $\mathbb{Z}/2$-vector spaces $B$ one finds a canonical isomorphism

$$hom_{Gr}(G, B) \cong hom_{Gr}(G, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} B.$$

Consider the canonical surjection

$$\pi : G \to G_{ab} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2$$

($k$ copies, say), and choose $g_i \in G$ such that

$$\pi(g_i) = e_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$$

in $(\mathbb{Z}/2)^k$. Let $pr_i$ denote the projection $(\mathbb{Z}/2)^k \to \mathbb{Z}/2$ onto the $i^{th}$ factor. Then the homomorphism $f : G \to B$ maps to the sum $\sum pr_i\pi \otimes f(g_i)$ in the tensor product $hom_{Gr}(G, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} B$. In particular, the map $\delta$ in $hom_{Gr}(G, H^1(BH, A))$ maps to $\sum pr_i\pi \otimes [h \mapsto \hat{\rho}(g_i)h(\hat{\rho}(g_i))^{-1}]$ in

$$hom_{Gr}(G, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^1(BH, A) = H^1(BG, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^1(BH, \mathbb{Z}/2).$$

Let $\gamma : H \to G$ denote the homomorphism of Galois groups which is induced by the inclusion $L \subset N$, and consider the maps

$$\begin{aligned} &hom(G, H^1(BH, A)) \cong hom(G, \mathbb{Z}/2) \otimes H^1(BH, A) \\ &= H^1(BG, \mathbb{Z}/2) \otimes H^1(BH, A) \xrightarrow{\gamma^* \otimes 1} H^1(BH, \mathbb{Z}/2) \otimes H^1(BH, A) \xrightarrow{\cup} H^2(BH, A) \end{aligned}$$

(all tensor products over $\mathbb{Z}/2$). By the above, the image of $\tilde{\rho}$ in the tensor product $H^1(BH,\mathbb{Z}/2)\otimes H^1(BH,\mathbb{Z}/2)$ is the sum $\sum pr_i\pi\gamma\otimes[h\mapsto\hat{\rho}(g_i)h(\hat{\rho}(g_i))^{-1}]$. It follows from the definition of cup product that the image of this element in $H^2(BH,A)$ is represented by the 2-cocycle $(h_2,h_1)\mapsto\hat{\rho}(\gamma h_2)h_1(\hat{\rho}(\gamma h_2))^{-1}$.

The corresponding image of $\tilde{\rho}$ in $H^2_{et}(K,A)$ is a generalized spinor class. The classical spinor norm

$$\theta : O_n(K)\to H^1_{et}(K,\mathbb{Z}/2)\cong K^*/(K^*)^2$$

is the map which is uniquely determined by sending each reflection $\tau_v$ in the hyperplane orthogonal to an anisotropic vector $v$ (since $O_n(K)$ is generated by such) to the element in $H^1_{et}(K,\mathbb{Z}/2)$ which is represented by $\langle v,v\rangle$ (see [18], for example). Specifically, one chooses a Galois extension $N/K$ with $H=Gal(N/K)$ such that $\langle v,v\rangle$ has a square root $\sqrt{\langle v,v\rangle}$ in $N$. Then $\theta(\tau_v)$ is the element in $H^1_{et}(K,\mathbb{Z}/2)$ which is represented by the $H$-cocyle $h\mapsto\sqrt{\langle v,v\rangle}\cdot h(\sqrt{\langle v,v\rangle})^{-1}$.

The spinor norm coincides with the boundary map $\delta : O_n(K)\to H^1_{et}(K,\mathbb{Z}/2)$ associated to the central extension $\mathbb{Z}/2\subset Pin_n\to O_n$ of sheaves of groups on $(Sch|_K)_{et}$. It suffices to check this on reflections $\tau_v$. But $\tau_v$ is the image of the element represented by $v$ in the Clifford group, under the map $r : C^*(n)(K)\to O_n(K)$, and so $\tau_v$ is in the image of $\sqrt{\langle v,v\rangle}^{-1}v\in Pin_n(N)$ in $O_n(N)$. Then

$$\sqrt{\langle v,v\rangle}^{-1}v\cdot h(\sqrt{\langle v,v\rangle}^{-1}v)^{-1}=h(\sqrt{\langle v,v\rangle})\sqrt{\langle v,v\rangle}^{-1}$$

in $Pin_n(K)$ since $h$ fixes $v$, and so the image of $\tau_v$ under $\delta$ coincides with $\theta(\tau_v)$.

Suppose that $G=Gal(L/K)$ once again. The spinor class $Sp_2(\rho)$ of a representation $\rho : G\to O_n(K)$ is defined to be the image in $H^2_{et}(K,\mathbb{Z}/2)$ of the group homomorphism $\delta\rho : G\to H^1_{et}(K,\mathbb{Z}/2)$ under the composition

$$\begin{aligned}&hom(G,H^1_{et}(K,\mathbb{Z}/2))\cong hom(G,\mathbb{Z}/2)\otimes H^1_{et}(K,\mathbb{Z}/2)=\\ &H^1(BG,\mathbb{Z}/2)\otimes H^1_{et}(K,\mathbb{Z}/2)\xrightarrow{can\otimes 1}H^1_{et}(K,\mathbb{Z}/2)\otimes H^1_{et}(K,\mathbb{Z}/2)\xrightarrow{\cup}H^2_{et}(K,\mathbb{Z}/2).\end{aligned}$$

From what we have shown above, $Sp_2(\rho)$ is the canonical image of a 2-cocycle of the form $(h_2,h_1)\mapsto\hat{\rho}(\gamma h_2)h_1(\hat{\rho}(\gamma h_2))^{-1}$ defined on some Galois group $H$.

There is a corresponding definition of the spinor class $Sp_2^{\beta}(\rho)$ for a representation $\rho : G\to O_{\beta}(K)$. $Sp_2^{\beta}(\rho)$ is the element of $H^2_{et}(K,\mathbb{Z}/2)$ which is associated to the composition

$$G\xrightarrow{\rho}O_{\beta}(K)\xrightarrow{\delta}H^1_{et}(K,\mathbb{Z}/2),$$

where $\delta$ is the boundary map coming from the short exact sequence of sheaves

$$e\to\mathbb{Z}/2\subset Pin_{\beta}\to O_{\beta}\to e.$$

# Bibliography


[1]. M.A. Delzant, *Définition des classes de Stiefel-Whitney d'un module quadratique sur un corps de charactéristique différent de 2*, C. R. Acad. Sci., Paris **255** (1962), 1366–1368.

[2]. A. Fröhlich, *Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants*, J. f. Reine ang math. **360** (1985), 85–123.

[3]. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9** (1957), 119–221.

[4]. A. Grothendieck, *Classes de Chern et représentations linéaires des groupes discrets*, Dix exposés sur la cohomologie des schémas, North Holland, Amsterdam, 1968, pp. 215–305.

[5]. J.F. Jardine, *Simplicial objects in a Grothendieck topos*, Contemp. Math. **55(I)** (1986), 193–239.

[6]. J.F. Jardine, *A rigidity theorem for $L$-theory*, Preprint (1984), ftp://jardine.math.uwo.ca.

[7]. J.F. Jardine, *Simplicial presheaves*, J. Pure Applied Algebra **47** (1987), 35–87.

[8]. J.F. Jardine, *Universal Hasse-Witt classes*, Contemp. Math. **83** (1989), 83–100.

[9]. J.F. Jardine, *Steenrod operations in the cohomology of simplicial presheaves*, NATO ASI Series C **279** (1989), 103–116.

[10]. B. Kahn, *Letter to A. Fröhlich, Dec. 29, 1984.*

[11]. M. Karoubi, *Relations between algebraic $K$-theory and hermitian $K$-theory*, J. Pure Applied Algebra **34** (1984), 259–263.

[12]. O. Laborde, *Classes de Stiefel-Whitney en cohomologie étale*, Mém. Soc. Math. France **48** (1976), 47–51.

[13]. J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton, 1967.

[14]. A.S. Merkurjev, *$K_2$ of fields and the Brauer group*, Contemp. Math. **55(II)** (1986), 529–546.

[15]. A.S. Merkurjev and A.A. Suslin, *$K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Math. of the USSR-Izvestiya **21** (1983), 307–340.

[16]. J.S. Milne, *Etale Cohomology*, Princeton University Press, Princeton, 1980.

[17]. J. Milnor, *Introduction to Algebraic $K$-theory*, Princeton University Press, Princeton, 1971.

[18]. O.T. O'Meara, *Introduction to Quadratic Forms*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.

[19]. J-P. Serre, *Cohomologie Galoisienne*, Springer Lecture Notes in Math. **5** (1973).

[20]. J-P. Serre, *Local Fields*, Springer-Verlag, New York, 1979.

[21]. J-P. Serre, *L'invariant de Witt de la forme $Tr(x^2)$*, Comm. Math. Helv. **59** (1984), 651–676.

[22]. V.P. Snaith, *Stiefel-Whitney classes of bilinear forms — a formula of Serre*, Can. Bull. Math. **28(2)** (1985), 218–222.

[23]. V.P. Snaith, *Topological Methods in Galois Representation Theory*, Wiley, New York, 1989.

[24]. C. Soulé, *$K$-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. 55 (1979), 251–295.

[25]. K. Vogtmann, *Homology stability for $O_{nn}$*, Comm. in Alg. **7(1)** (1979), 9–38.