Complexity reduction for path categories

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Introduction

A finite cubical complex K is a subobject $K \subseteq \square^n$ of a standard n-cell in the category of cubical sets.

The object $\square^n$ is represented by the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1, 2, \dots, n\}$. This poset is an object of the box category $\square$ that defines cubical sets (see, for example, [2]). The complex K is defined by a list of non-degenerate cells $\sigma : \square^k \subseteq \square^n$. These cells can be identified with poset inclusions $[A, B] \subseteq \mathcal{P}(\underline{n})$ of intervals, where

$$[A, B] = \{F \mid A \subseteq F \subseteq B\}$$

where $A \subseteq B$ are subsets of $\underline{n}$.

As such, K is a list of intervals $[A, B] \subseteq \mathcal{P}(\underline{n})$ which is closed under taking subintervals.

Finite cubical complexes are the higher dimensional automata of geometric concurrency theory. In that setting, the vertices of a cubical complex K model the states of a concurrent system, and its k-cells represent (where possible) the simultaneous action of k processors. The cells of the ambient n-cell which are not in K represent constraints on the system.

The main object of study associated to K in this form of concurrency theory is its collections of execution paths. These paths are the morphisms of the *path category* $P(K)$.

The path category functor is now well known — it is also called the fundamental category and denoted by $\tau_1(K)$ in the higher categories literature [5].

The emphasis in concurrency theory is different, and is completely concerned with giving exact specifications of path categories $P(K)$ in the geometric setting described above. Techniques leading to explicit, algorithmic calculations of path categories form the subject of this paper.

The *triangulation* $|K|$ of the finite cubical complex K is a finite simplicial complex that is defined by "putting in the missing edges". More explicitly,

$$|\square^n| = (\Delta^1)^{\times n} = B\mathcal{P}(\underline{n}),$$

is the nerve of the poset $\mathcal{P}(\underline{n})$, and $|K|$ is constructed by gluing together such objects along the incidence relations for the cells of K.

The path category functor $X \mapsto P(X)$ for simplicial sets X is most succinctly defined to be the left adjoint of the nerve functor. The path category construction for cubical sets is a specialization of this functor, and we can write

$$P(K) := P(|K|)$$

for cubical complexes K.

In practice, the objects of $P(K)$ are the vertices of K, and the morphisms are equivalence classes of paths in 1-cells, modulo commutativity conditions that are defined by 2-cells.

Similarly, the path category $P(L)$ of a finite simplicial complex L has the vertices of L as objects, and has morphisms given by equivalence classes of paths in 1-simplices, modulo commutativity conditions that are defined by 2-simplices.

There is an algorithm for computing $P(L)$ for finite simplicial complexes $L \subseteq \Delta^n$ that arises from a 2-category $P_2(L)$ that is defined by the simplices of L, and for which $P(L)$ is the path component category of $P_2(L)$ in the sense that there is a bijection

$$P(L)(x,y) \cong \pi_0(P_2(L)(x,y))$$

for all vertices x, y. The 2-category $P_2(L)$ is defined in [3].

The algorithm can be summarized as follows:

1) Restrict to the 2-skeleton $\mathrm{sk}_2(L)$ of L.

2) Find all paths (strings of non-degenerate 1-simplices)

$$\omega : v_0 \xrightarrow{\sigma_1} v_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_k} v_k$$

in L.

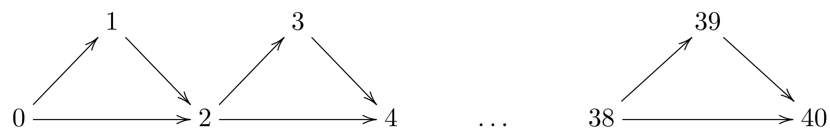
3) Find all morphisms in the category $P_2(L)(v,w)$ for all vertices $v < w$ in L (ordering in Δ^n).

4) Find the sets of path components for all categories $P_2(L)(v,w)$.

This algorithm is the *path category algorithm.* It has been coded in C and Haskell by M. Misamore — Misamore's code is published on github.com and hackage.haskell.org. The original test of concept was written by G. Denham in Macaulay 2.

Except for the first step, which is due to a basic result for path categories [3] that also appears in Lemma 2 below, the algorithm is brute force. It works well for toy examples, but it is easy to generate simple examples which output very large lists of morphism sets.

Example 1. The "necklace" $L \subseteq \Delta^{40}$ be the subcomplex



(1)

This is 20 copies of the complex $\partial\Delta^2$ glued together. It is visually obvious that there are 2^{20} morphisms in $P(L)(0, 40)$, and the text file list of morphisms of $P(L)$ consumes 2 GB of disk space.

In general, the size of the path category $P(L)$ can grow exponentially with L.

Extreme examples aside, various complexity reduction methods have been developed for the path category algorithm, and the purpose of this note is to give an account of these techniques.

The mathematical results of this paper are quite simple. Most of the statements amount to constructions of subcomplexes $K \subseteq L$ such that the induced functor $P(K) \to P(L)$ between path categories is fully faithful.

Explicitly, this means that if v, w are vertices of K, then the induced function $P(K)(v, w) \to P(L)(v, w)$ of morphism sets is a bijection. In this case, the morphism set $P(L)(v, w)$ can be computed in the smaller context given by K, which can be much simpler computationally.

Most of the time, K is a "full" subcomplex of L. Fullness is a general criterion for the induced functor $P(K) \to P(L)$ to be fully faithful. The concept (appearing in Section 1 of this paper) is used repeatedly, for the method of deletions of sources and sinks from a simplicial complex in Section 2, and for deriving Mismore's method of removing corners from a cubical complex in Section 3.

Section 4, on refinement of cubical complexes, is the opposite in some sense. The idea is that one can use the data that constructs a finite cubical complex K to construct a more complicated object K_α in a way that produces a fully faithful functor $P(K) \to P(K_\alpha)$. One expects that this idea will be useful for studies of successive approximations of cubical structures.

The last section, Section 5, gives a first, coarse method for parallelizing the path category algorithm for calculating $P(K)$ for a cubical complex K. All vertices of K have a size, or cardinality, that they inherit from the ambient cell $\square^n$. The resulting size functor can be used to isolate disjoint full subcomplexes, say A and B, for which $P(A)$ and $P(B)$ can be computed independently. All paths $u \to v$ of K which start in A and end in B cross a "frontier subcomplex" whose cells define a coequalizer picture (see (4)) that allows one to compute $P(K)(u, v)$ from the path categories $P(A)$ and $P(B)$.

The size functor is also used in Section 4, and it is very likely to have continuing utility. One can think of this functor as a ticking clock, but the relationship between that "clock" and the higher dimensional automaton concept can be a bit fraught.

1 Basic results

The first step of the path category algorithm involves a direct appeal to the following result:

Lemma 2. *The inclusion* $\mathrm{sk}_2(X) \subseteq X$ *of the* 2*-skeleton of a simplicial set* X *induces an isomorphism of categories*

$$P(\mathrm{sk}_2(X)) \xrightarrow{\cong} P(X).$$

This result follows from the fact that the nerve BC of a small category C is a 2-coskeleton [1, Lem. 3.5], which means that there is a bijection

$$\hom(X, BC) \cong \hom(\mathrm{sk}_2(X), BC).$$

Lemma 2 is a substantial complexity reduction step, in that it means that one can ignore much of the data for a finite simplicial complex L before computing $P(L)$.

We now discuss a concept and result that has appeared in connection with work on homotopy types of categories [4, Lem. 4].

Suppose that $L_0 \subseteq L$ is a subcomplex of a finite simplicial complex L. We say that L_0 is a *full subcomplex* of L if the following conditions hold:

1) L_0 is path-closed in L, in the sense that, if there is a path

$$v = v_0 \to v_1 \to \cdots \to v_n = v'$$

in L between vertices v, v' of L_0, then all $v_i \in L_0$,

2) if all the vertices of a simplex $\sigma \in L$ are in L_0 then the simplex σ is in L_0.

Lemma 3. *Suppose that* L_0 *is a full subcomplex of* L*. Then the functor* $P(L_0) \to P(L)$ *is fully faithful.*

Recall that a functor $F : C \to D$ is *fully faithful* if all induced functions

$$f : C(x, y) \to D(f(x), f(y))$$

of morphism sets are bijections.

The proof of Lemma 3 follows from the fact that the path category $P(L)$ is constructed by taking the category freely associated to the graph given by the 1-skeleton $\mathrm{sk}_1(L)$, modulo relations defined by 2-simplices of L [3]. The conditions imply that every path in L between vertices v, w of L_0 consists of simplices which are in L_0, and that all 2-simplices which define relations of paths in L between $v, w \in L_0$ are also in L_0.

Example 4. The inclusions $d^0 : \partial\Delta^2 \subseteq \Lambda^3_0$ and $d^3 : \partial\Delta^2 \subseteq \Lambda^3_3$ induced by the respective cofaces $\Delta^2 \subseteq \Delta^3$ both define full subcomplexes.

In the first case, an argument on orientation says that no path in Λ^0_0 that starts and ends in the set of vertices $\{1, 2, 3\}$ can pass through the vertex 0. The second case is similar.

Example 5. Suppose that $i \leq j$ in $\mathbf{n}$ and suppose that $L \subseteq \Delta^n$. $L[i,j]$ is the subcomplex of L such that $\sigma \in L[i,j]$ if and only if all vertices of σ are in the interval $[i,j]$ of vertices v such that $i \leq v \leq j$. Then $L[i,j]$ is a full subcomplex of L.

Example 6. Suppose that $v \leq w$ are vertices of $L \subseteq \Delta^n$. Let $L(v,w)$ be the subcomplex of L consisting of simplices whose vertices appear on a path from v to w. Then $L(v,w)$ is a full subcomplex of L, and of $L[v,w]$.

2 Sources and sinks

A vertex v is a *source* of L if there are no non-degenerate 1-simplices $u \to v$ in L. The vertex z is a *sink* of L if there are no non-degenerate 1-simplices $z \to w$ in L.

Every finite simplicial complex $L \subset \Delta^n$ has at least one source and one sink. These are the smallest and largest vertices of L, respectively, in the totally ordered set of vertices of the ambient simplex Δ^n.

Observe that 0 is a source of Λ^3_0 and 3 is a sink of Λ^3_3. The following result formalizes the assertions made in Example 4 above:

Lemma 7. *Suppose that S is a subset of the vertices of $L \subseteq \Delta^n$ which consists of sources and sinks. Let $L(-S)$ be the subcomplex of L which consists of simplices which do not have a vertex in S. Then $L(-S)$ is a full subcomplex of L.*

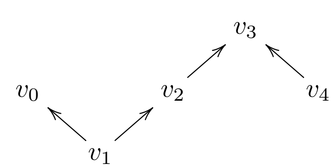
Proof. Suppose that $v < v'$ are vertices of $L(-S)$ and suppose that the string of 1-simplices

$$v = v_0 \to v_1 \to \cdots \to v_n = v'$$

is a path of L from v to w consisting of non-degenerate 1-simplices. Then no intermediate object v_i, $1 \leq i \leq n-1$ can be a source or a sink. It follows that all $v_i \in L(-S)$.

A simplex σ of L is in $L(-S)$ if and only if none of its vertices are in S, by definition. □

Example 8. Suppose that L is the complex



The set $S = \{v_1, v_3\}$ consists of sources and sinks, and $L(-S)$ is discrete on the vertices v_0, v_2, v_4. The isolated point v_2 is a source and a sink for $L(-S)$. Let $S' = \{v_2\}$. Then

$$P(L)(v_0,v_4) = P(L(-S))(v_0,v_4) = P(L(-S)(-S'))(v_0,v_4) = \emptyset.$$

Thus, removing sources and sinks can create new ones. The process of removing sources and sinks relative to a pair of vertices v, w of L must stop, since L is finite.

Lemma 9. *Suppose that $v < w$ in L and that S consists of sources and sinks of L which are distinct from v and w. Then*

$$L(-S)(v, w) = L(v, w).$$

Proof. Suppose that

$$\sigma : v = v_0 \to v_1 \to \cdots \to v_n = w$$

is a path from v to w in L. Then each intermediate vertex v_i is neither a source or a sink, and is therefore not in S, so that $v_i \in L(-S)$. The subcomplex $L(-S)$ is full so that the path σ is in $L(-S)$.

Thus, every vertex of $L(v, w)$ is a vertex of $L(-S)(v, w)$, so that the two complexes have the same set of vertices. These are full subcomplexes of L having the same sets of vertices, so that the inclusion

$$L(-S)(v, w) \subseteq L(v, w)$$

is an identity. □

Lemma 10. *Suppose that $v \leq w$ in L, where v is a source and w is a sink. Suppose given complexes*

$$L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_0 = L$$

where $v, w \in L_{i+1} = L_i(-S_i)$ and S_i is some set of sources and sinks in L_i. Suppose that L_n has a unique source v and a unique sink w. Then $L_n = L(v, w)$.

Proof. The connected component of v in L_n has a sink, which must be w. All other components would have sources and sinks, and must therefore be empty. It follows that L_n is connected.

If L_n has a vertex x other than v, w then there are non-degenerate 1-simplices

$$a_1 \to x \to b_1.$$

If a_1 is a source then $a_1 = v$. Otherwise, there is a 1-simplex $a_2 \to a_1$. This procedure must stop, to produce a path

$$v = a_r \to \cdots \to a_2 \to a_1 \to x.$$

Similarly, there is a path

$$x \to b_1 \to b_2 \to \cdots \to b_s = w.$$

If L_n has no vertices other than v, w, then L_n consists of the 1-simplex $v \to w$.

It follows that every vertex of L_n is on a path from v to w, so that $L_n(v, w) = L_n$. Then Lemma 9 implies that $L_n(v, w) = L(v, w)$, so that $L_n = L(v, w)$. □

Suppose that $v \leq w$ in L, and start with $L_0 = L[v,w]$. Let S_0 be the set of all sources and sinks of L_0, except for the elements v, w, and set $L_1 = L_0(-S_0)$. Repeat this procedure inductively to produce a descending chain of complexes

$$L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_0 = L[v,w],$$

with $S_n = \emptyset$. Then

$$L_n = L[v,w](v,w) = L(v,w),$$

by Lemma 10.

In other words, starting with the full subcomplex $L[v,w]$ we can successively delete sources and sinks to produce $L(v,w)$, which is the minimal full subcomplex of L that computes $P(L)(v,w)$.

3 Corners

Suppose that $i : K \subseteq \square^n$ is a finite cubical complex. The inclusion i induces a functor

$$i_* : P(K) \to P(\square^n) = \mathcal{P}(\underline{n}).$$

There is a poset map $t : \mathcal{P}(\underline{n}) \to \mathbb{N}$ that is defined by cardinality, in the sense that

$$F \mapsto t(F) = |F|$$

for all subsets F of $\underline{n}$. The composite functor

$$P(K) \xrightarrow{i_*} \mathcal{P}(\underline{n}) \xrightarrow{t} \mathbb{N}$$

will also be denoted by t.

One thinks of the functor t as a sort of time parameter for K. This functor also behaves like a total degree.

Suppose that x is a vertex of the finite cubical complex K. Say that x is a *corner* if it belongs to only one maximal cell of K.

The following result was proved by M. Misamore in [6]. The proof that is given here is quite different.

Lemma 11. *Suppose that x is a corner of K, and let K_x be the subcomplex of cells which do not have x as a vertex. Then the functor*

$$P(K_x) \to P(K)$$

is fully faithful.

Proof. Suppose that σ is the unique top cell containing x.

If x is either maximal or minimal in σ, then x is either a sink or a source, respectively, by the uniqueness of σ. In that case, the functor $P(K_x) \to P(K)$ is fully faithful, by Lemma 7.

Suppose that x is neither maximal nor minimal in σ, and suppose that P is a non-degenerate path in K which passes through x, as in

$$P: \ u = u_0 \to \cdots \to u_n = v,$$

where $u, v \in K_x$, and $u_i = x$. Then $i \neq 0, n$, and there is a unique i such that $u_i = x$. In effect, since P is non-degenerate, it induces a system of proper inequalities

$$|u| = |u_0| < |u_1| < \cdots < |x| < \cdots < |u_n| = |v|,$$

in which the number $|x|$ can only appear once.

Then u_{i-1} and u_{i+1} are in K_x, and both 1-simplices $u_{i-1} \to x$ and $x \to u_{i+1}$ are in σ since σ is the unique maximal cell that contains x.

Write $\sigma = [A, B]$.

As subsets of $B \subseteq \underline{n}$, $x = u_{i-1} \cup \{a\}$ and $u_{i+1} = x \cup \{b\}$, where a and b are distinct. The resulting 2-cell

$$\begin{array}{ccc} u_{i-1} & \longrightarrow & x \\ \downarrow & & \downarrow \\ u_{i-1} \cup \{b\} & \longrightarrow & u_{i+1} \end{array} \qquad (2)$$

in σ (hence in K) defines a morphism $u_{i-1} \to u_i$ in $P(K_x)$. Define $\psi(P)$ to be the composite of the morphisms

$$u = u_0 \to \cdots \to u_{i-1} \to u_{i+1} \to \cdots \to u_n = v$$

in $P(K_x)(u, v)$.

The 2-cell of the picture (2) is uniquely determined by the path P, as is its image $\psi(P)$.

If $Q : u \to v$ is a non-degenerate path which does not pass through x, let $\psi(Q)$ be the image of Q in $P(K_x)(u, v)$. We have therefore determined a function

$$\psi : \{\text{paths } u \to v\} \to P(K_x)(u, v).$$

If there is a 2-cell between paths $u \to v$ in K, then the corresponding images under ψ coincide. We therefore have an induced function

$$\psi_* : P(K)(u, v) \to P(K_x)(u, v).$$

The composite

$$P(K_x)(u, v) \to P(K)(u, v) \xrightarrow{\psi_*} P(K_x)(u, v)$$

is the identity by construction. The construction of $\psi(P)$ for paths P passing through x shows that the function

$$P(K_x)(u, v) \to P(K)(u, v)$$

is surjective, and is therefore a bijection. $\square$

Suppose that $x \subseteq \underline{n}$. Then x is an object of the poset $\mathcal{P}(\underline{n})$ and is a vertex of the simplicial set $B\mathcal{P}(\underline{n})$.

Let $\square^n_x$ be the cubical subcomplex of $\square^n$ consisting of cells which do not have x as a vertex.

Let D_x be the subcomplex of $B\mathcal{P}(\underline{n})$ consisting of those simplices which do not have x as a vertex. D_x is the nerve $B\mathcal{P}(\underline{n})_x$ of the full subcategory of $\mathcal{P}(\underline{n})$ with objects not equal to x. In particular, the functor $P(D_x) \to P(\square^n)$ is fully faithful.

The isomorphism $|\square^n| \cong B\mathcal{P}(\underline{n})$ restricts to a monomorphism of simplicial complexes

$$\gamma : |\square^n_x| \to D_x.$$

Observe that if x is neither the minimal element $\emptyset$ nor maximal element $\underline{n}$ of $\mathcal{P}(\underline{n})$, then $\emptyset \subseteq \underline{n}$ is a 1-simplex of D_x which cannot be in the image of the map γ.

If x is either the maximal or minimal element of $\mathcal{P}(\underline{n})$, then the map γ is an isomorphism. In effect, if $x = \underline{n}$, then a simplex $F_0 \subseteq \cdots \subseteq F_k$ is in D_x if and only if $F_k \neq \underline{n}$, and in this case it is in the image of the cell $|[\emptyset, F_k]|$. The case $x = \emptyset$ is argued similarly.

Corollary 12. *The functor $P(\square^n_x) \to P(\square^n)$ is fully faithful, and the induced functor*

$$\gamma_* : P(|\square^n_x|) \to P(D_x)$$

is an isomorphism of path categories.

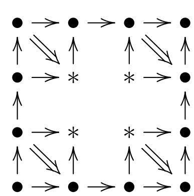
Proof. The functor

$$i_* : P(\square^n_x) \to P(\square^n)$$

is fully faithful by Lemma 11.

The functor γ_* is bijective on vertices, and is also fully faithful by the previous paragraph. It is therefore an isomorphism of categories as claimed. $\square$

Example 13. The Swiss flag (2-cells indicated by double arrows, centre region is empty)



has six corners, one sink, and one source, aside from the initial and terminal vertices. Remove the four "inner" corners to show that there are two morphisms from the initial vertex to the terminal vertex in the corresponding path category.

4 Refinement

Suppose that $\alpha : \mathcal{P}(\underline{m}) \to \mathcal{P}(\underline{n})$ is a poset monomorphism that preserves meets and joins.

Every interval $[A, B]$ in $\mathcal{P}(\underline{m})$ determines an interval $[\alpha(A), \alpha(B)]$ in $\mathcal{P}(\underline{n})$, and α restricts to a poset monomorphism $\alpha : [A, B] \to [\alpha(A), \alpha(B)]$. The assignment

$$[A, B] \mapsto [\alpha(A), \alpha(B)]$$

preserves inclusion relations between intervals, and preserves meets and joins of intervals.

The cubical subcomplex of $\square^n$ that is generated by the intervals $[\alpha(A), \alpha(B)]$ associated to the intervals $[A, B]$ of K is denoted by K_α, and there is a simplicial set map $\alpha_* : |K| \to |K_\alpha|$ that makes the diagram

$$\begin{array}{ccc} |K| & \xrightarrow{\alpha_*} & |K_\alpha| \\ \downarrow & & \downarrow \\ B\mathcal{P}(\underline{m}) & \xrightarrow[\alpha]{} & B\mathcal{P}(\underline{n}) \end{array}$$

commute. The simplicial set map α_* is induced by the restricted poset morphisms $\alpha : [A, B] \to [\alpha(A), \alpha(B)]$. These poset morphisms are not face inclusions in general.

Suppose that $K \subseteq \square^m$ and $L \subseteq \square^n$ are higher dimensional automata. We say that L is a *refinement* of K if there is a poset monomorphism $\alpha : \mathcal{P}(\underline{m}) \to \mathcal{P}(\underline{n})$ that preserves meets and joins, and an inclusion $i : K_\alpha \subseteq L$ of cubical subcomplexes of $\square^n$. In this case, there is a commutative diagram of simplicial set maps

$$\begin{array}{ccccc} |K| & \xrightarrow{\alpha_*} & |K_\alpha| & \xrightarrow{i_*} & |L| \\ \downarrow & & \downarrow & \swarrow & \\ B\mathcal{P}(\underline{m}) & \xrightarrow[\alpha]{} & B\mathcal{P}(\underline{n}) & & \end{array}$$

Lemma 14. *Suppose that $\alpha : \mathcal{P}(\underline{m}) \to \mathcal{P}(\underline{n})$ is a poset monomorphism which preserves meets and joins, and suppose that $K \subseteq \square^r$ is a cubical subcomplex.*

Then the induced functor $\alpha_* : P(K) \to P(K_\alpha)$ is fully faithful.

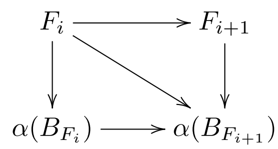
Proof. Suppose that F is a vertex of K_α. Then $F \subseteq [\alpha(A), \alpha(B)]$ for some interval $[A, B]$ of K, so there is a vertex B of K such that $F \subseteq \alpha(B)$. There is a minimal such B, call it B_F, since α preserves meets.

If $F = \alpha(C)$ for some C, then $B_F = C$, since α is a monomorphism. In effect, $\alpha(C) \leq \alpha(B_F) \leq \alpha(C)$, so $B_F = C$ in this case.

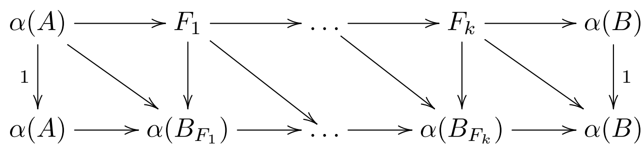
Suppose that

$$\omega : \alpha(A) \to F_1 \to \cdots \to F_k \to \alpha(B)$$

is a path in K_α. Then each $F_i \to F_{i+1}$ is in an interval $[\alpha(C_i), \alpha(D_i)]$, so that the diagram of inclusions



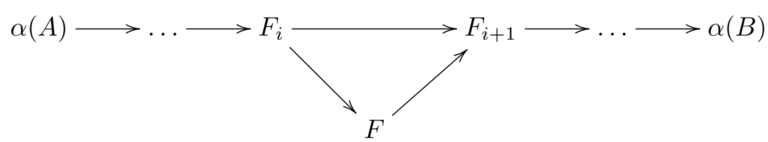
is in that same interval. The inclusion $\alpha(B_{F_i}) \to \alpha(B_{F_{i+1}})$ is the image of an inclusion $B_{F_i} \to B_{F_{i+1}}$ by the minimality of B_{F_i}. It follows that the diagram



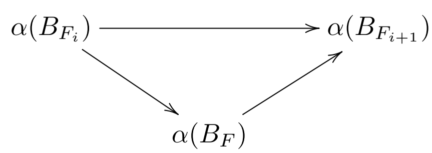
(3)

defines a homotopy in $|K_\alpha|$ from the path ω to the path along the bottom, which path is in the image of the function $P(K)(A,B) \to P(K_\alpha)(\alpha(A), \alpha(B))$, because all displayed simplices are in $|K_\alpha|$.

Suppose given a commutative diagram



where ω is the path along the top, and the displayed triangle of inclusions defines a 2-simplex σ of $|K_\alpha|$. This 2-simplex is in some interval $[\alpha(C), \alpha(D)]$, and the corresponding diagram



is also in the interval $[\alpha(C), \alpha(D)]$, by minimality. This simplex is the image of a 2-simplex of $|K|$.

We have therefore defined a function

$$s : P(K_\alpha(\alpha(A), \alpha(B)) \to P(K)(A,B)$$

such that the composite $s \cdot \alpha_*$ is the identity on $P(K)(A,B)$. The construction of the function s and the existence of the homotopies (3) together imply that the function

$$\alpha_* : P(K)(A,B) \to P(K_\alpha)(\alpha(A), \alpha(B))$$

is surjective. It follows that the function α_* is a bijection, as required. □

5 Frontier subcomplex

Suppose that $i : K \subseteq \square^n$ is a finite cubical complex. Recall from Section 3 that the assignment $F \mapsto |F| =: t(F)$ that is defined by cardinality determines a poset morphism $\mathcal{P}(\underline{n}) \to \mathbb{N}$, and hence a composite functor

$$t : P(K) \xrightarrow{i_*} \mathcal{P}(\underline{n}) \xrightarrow{t} \mathbb{N}.$$

Observe that if $v \to w$ is a non-degenerate 1-simplex of $|K|$, then there is a strict containment relation $v \subset w$ as subsets of $\underline{n}$, so that $t(v) < t(w)$. A more precise version of this statement applies to all 1-cells $v \to w$ of K: $t(w) = t(v)+1$ for such a 1-cell.

The functor t defines full subcomplexes of the complex K and its triangulation $|K|$. In particular, if $r < s$ in $\mathbb{N}$, let $K(r,s)$ be the subcomplex of cells whose vertices F satisfy $r \leq t(F) \leq s$, and let $|K|(r,s)$ be the subcomplex of $|K|$ whose simplices have vertices F with $t(f)$ in the same range.

Then we have the following:

Lemma 15. *1) $|K|(r,s)$ is a full subcomplex of $|K|$.*

2) The canonical map

$$|K(r,s)| \to |K|(r,s)$$

is an isomorphism of simplicial complexes.

Proof. For statement 1), suppose that v, w are vertices of $|K|(r,x)$ and that

$$v = v_0 \to v_1 \to \cdots \to v_n = w$$

is a non-degenerate path from v to w in $|K|$. Then

$$r \leq t(v) = t(v_0) < t(v_1) < \cdots < t(v_n) = t(w) \leq s,$$

so that all vertices v_i are in $|K|(r,s)$. The higher simplex condition for fullness of $|K|(r,s)$ is automatic from the definition.

The canonical inclusion of statement 2) arises from the observation that $K(r,s)$ is a union of cells of K, and the induced inclusion $|K(r,s)| \subseteq |K|$ factors through $|K|(r,s)$.

To prove statement 2), it is enough to show that the inclusion

$$|K(r,s)| \to |K|(r,s)$$

is surjective on non-degenerate simplices. If σ is a simplex of $|K|(r,s)$, it is in the image of the map $|\square^k| \to |K|$ which is induced by a non-degenerate cell of K. The simplex σ has the form

$$F_0 \leq F_1 \leq \cdots \leq F_p$$

with

$$r \leq |F_0| \leq |F_1| \leq \cdots \leq |F_p| \leq s,$$

and it follows that the interval $[F_0, F_p]$ defines a cell of $K(r,s)$. The simplex σ is therefore in $|K(r,s)|$. $\square$

Suppose that $K \subseteq \square^n$ is a finite cubical complex such that $\mathrm{sk}_2(K) = K$, and pick M such that $0 < M < n$. Let $A = K(0, M)$ and $B = K(M+1, n)$. Then $|A|$ and $|B|$ are full subcomplexes of $|K|$ by Lemma 15.

Every path

$$v_0 \to v_1 \to \cdots \to v_k$$

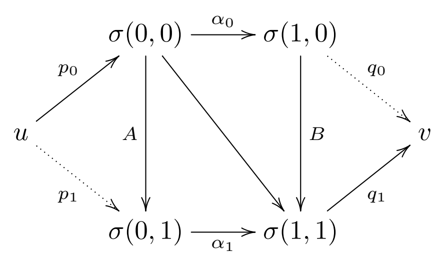
in K has a number r (which could be -1 or k) such that $v_i \in A$ for $i \leq r$ and $v_i \in B$ for $i \geq r+1$. The *frontier subcomplex* L is generated by 1-cells and 2-cells which have vertices in A and B.

Suppose that $u \in A$ and $v \in B$. Suppose that the 1-cell $\sigma : x \to y$ has $x \in A$ and $y \in B$. Then composition with σ defines a map

$$\sigma_* : P(A)(u, x) \times P(B)(y, v) \to P(K)(u, v).$$

Suppose that $\omega : \Delta^1 \times \Delta^1 \to K$ is defined by 2-simplices ω_0 and ω_1 such that $d_1\omega_0 = d_1\omega_1$ and $d_2(\omega_0) \in A$ and $d_0(\omega_1) \in B$. One of the 2-simplices ω_0 or ω_1 could be degenerate.

Consider the picture:



There are induced maps

$$\omega_0 : P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,1), v) \to P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,0), v)$$

and

$$\omega_0 : P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,1), v) \to P(A)(u, \sigma(0,1)) \times P(B)(\sigma(1,1), v)$$

These maps define the displayed parallel pair of arrows in the diagram

$$\bigsqcup_{\omega \text{ as above}} P(A)(u, \sigma(0,0)) \times P(B)(\sigma(1,1), v) \rightrightarrows \bigsqcup_{x \xrightarrow{\sigma} y \,\in\, L} P(A)(u, x) \times P(B)(y, v) \to P(K)(u, v). \tag{4}$$

Lemma 16. *The diagram (4) is a coequalizer.*

The proof of Lemma 16 is essentially by inspection.

In practical terms, Lemma 16 says that one can compute $P(K)$ by first computing $P(A)$ and $P(B)$ (in parallel), and then by stitching these calculations together with the coequalizer (4). This coequalizer defines $P(K)(u,v)$ as a set of equivalence classes on a set that we've computed, namely

$$\bigsqcup_{x \xrightarrow{\sigma} y \,\in\, L} P(A)(u,x) \times P(B)(y,v),$$

for an equivalence relation that is defined by the parallel pair of functions in the coequalizer picture (4).

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