E_2 model structures for presheaf categories

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Introduction

The purpose of this paper is to develop analogs of the E_2 model structures of Dwyer, Kan and Stover for categories related to pointed bisimplicial presheaves and simplicial presheaves of spectra. The development is by analogy with and builds on the work of Dwyer, Kan and Stover [1], [2], along with later work of Goerss and Hopkins [3].

The technical challenge met in the present paper is that not all objects in the categories under consideration are fibrant. This is overcome with the introduction of a bounded cofibration approximation technique which builds on an approach to constructing localizations that appears in [4]. The results proven here are completely combinatorial and apply, in particular, to pointed bisimplicial presheaves, as well as simplicial spectra, simplicial symmetric spectra and their motivic analogs.

The first section of this paper gives a list of general results which hold for simplicial objects in a proper closed simplicial model category $\mathcal{M}$. The basic notions of $\mathcal{A}$-fibration and $\mathcal{A}$-equivalence are given there, where $\mathcal{A}$ is a small diagram consisting of homotopy cogroup objects A_i of $\mathcal{M}$. Generally a map $g : X \to Y$ of simplicial objects of $\mathcal{M}$ is defined to be an $\mathcal{A}$-equivalence if and only if the induced map of simplicial groups

$$g_* : [A_i \wedge S^j, X] \to [A_i \wedge S^j, Y] \tag{1}$$

is a weak equivalence for suspensions $A_i \wedge S^j$ of all homotopy cogroup objects A_i in the diagram $\mathcal{A}$.

These definitions were introduced by Goerss and Hopkins [3]. In the special case where $\mathcal{M}$ is the category of pointed spaces and the diagram $\mathcal{A}$ is the list of spheres, the $\mathcal{A}$-equivalences and the $\mathcal{A}$-fibrations are respectively the E_2-equivalences and E_2-fibrations of Dwyer, Kan and Stover [1]. In general, following [3], the diagram $\mathcal{A}$ is not required to be closed under suspension.

The $\mathcal{A}$-structure is constructed within the Reedy model structure for the category $s\mathcal{M}$ of simplicial objects in $\mathcal{M}$; in particular a map $g : X \to Y$ is said to be an $\mathcal{A}$-fibration if g is a Reedy fibration and all simplicial group maps g_* displayed above are fibrations of simplicial groups.

The $\mathcal{A}$-equivalences and $\mathcal{A}$-fibrations define the model structures which appear in this paper, but the demonstration of the full list of closed model axioms requires extra structure in the underlying category $\mathcal{M}$. It is a general result (Lemma 6) that a Reedy fibration $g : X \to Y$ is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to the maps

$$\Lambda^n_k \otimes (A_i \wedge S^j) \to \Delta^n \otimes (A_i \wedge S^j)$$

of simplicial objects which arise from tensoring externally with the standard anodyne inclusions $\Lambda^n_k \subset \Delta^n$, provided that Y is Reedy fibrant. Going further (ie. removing the condition that Y is Reedy fibrant) requires a technique for varying the A_i up to weak equivalence in a controlled way, so that $\mathcal{A}$-fibrations can still be described by a right lifting property with respect to some set of maps. In all of the examples studied in this paper there is an ambient notion of cardinality, and the control arises from the imposition of cardinality bounds within weak equivalence classes. This is the bounded cofibration approximation technique referred to above, and appears here as Lemma 14 for pointed simplicial presheaves in Section 2, Lemma 20 for presheaves of spectra in Section 3, and Lemma 29 for T-spectra (representing the motivic stable category) in Section 4. All of these results follow from the bounded cofibration condition, which is a feature of localization theory for simplicial presheaves, and is part of the structure of all of the respective underlying model categories [4], [11], [12].

The main results of this paper assert the existence of $\mathcal{A}$-model structures incorporating the $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences as defined above for simplicial objects in ordinary and f-local categories of pointed simplicial presheaves (Theorem 16), presheaves of spectra (Theorem 22), and the motivic stable category (Theorem 32). It is shown that the $\mathcal{A}$-model structures for presheaves of spectra and T-spectra induce $\mathcal{A}$-structures on the corresponding categories of symmetric spectrum objects (Theorem 24, Theorem 34), such that the underlying Quillen equivalences relating symmetric spectrum and spectrum objects induce Quillen equivalences on the respective $\mathcal{A}$-model structures (Theorem 25, Theorem 35).

Theorem 16 specializes to an $\mathcal{A}$-model structure for pointed bisimplicial sets, and to an $\mathcal{A}$-model structure for simplicial objects in the unstable motivic category. Theorem 22, Theorem 24 and Theorem 25, respectively, specialize to $\mathcal{A}$-model structures for simplicial spectra and simplicial symmetric spectra, and give a Quillen equivalence relating them.

The moral seems to be that any proper closed simplicial model category which can be represented in the category of sets and satisfies a bounded cofibration condition admits E_2-style model structures on its corresponding category of simplicial objects.

1 Generalities

Throughout this section $\mathcal{M}$ will denote a proper closed simplicial model category having an object $*$ which is both initial and terminal. The basic examples of such categories $\mathcal{M}$ for the purposes of this paper will include the category $\mathbf{S}_*(\mathcal{C})$ of pointed simplicial presheaves and the category $\mathbf{Spt}(\mathcal{C})$ of presheaves of spectra on a small Grothendieck site $\mathcal{C}$. These examples specialize, respectively, to pointed simplicial sets and spectra within the Bousfield-Friedlander model for the stable category.

A homotopy cogroup object in $\mathcal{M}$ is a cofibrant object A with augmentation $\epsilon : A \to *$ such that the set of morphisms $[A, X]$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$ naturally has the structure of a group, with identity e represented by the composite

$$A \xrightarrow{\epsilon} * \to X.$$

Every presheaf of spectra is a homotopy cogroup object for the category of presheaves of spectra. In general, if A is a homotopy cogroup object for the category of pointed simplicial sets (eg. $A = S^n$), then the pointed simplicial presheaf $L_U A$ given by applying the left adjoint L_U to the U-sections functor is a homotopy cogroup object for pointed simplicial presheaves.

Since A is a cogroup object, and in particular has augmentation $\epsilon : A \to *$, there is an isomorphism of simplicial groups

$$\pi_1 \mathbf{hom}(A, X) \cong [A \wedge S^1, X]$$

if X is a fibrant object of $\mathcal{M}$. Here, $A \wedge S^1$ is defined by the pushout square

$$\begin{array}{ccc} A \otimes \partial\Delta^1 & \xrightarrow{\epsilon_*} & * \\ \downarrow & & \downarrow \\ A \otimes \Delta^1 & \longrightarrow & A \wedge S^1 \end{array}$$

where the map labelled ϵ_* is the composite

$$A \otimes \partial\Delta^1 \cong A \vee A \xrightarrow{(\epsilon,\epsilon)} *.$$

Then $A \wedge S^1$ is a cogroup object with augmentation induced by the composite

$$A \otimes \Delta^1 \to A \otimes \Delta^0 \cong A \xrightarrow{\epsilon} *.$$

Continuing inductively, we see that there is an isomorphism

$$\pi_j \mathbf{hom}(A, X) \cong [A \wedge S^1 \wedge \cdots \wedge S^1, X],$$

where there are j copies of S^1 appearing as smash factors, and

$$A \wedge S^j = A \wedge S^1 \wedge \cdots \wedge S^1$$

is a cogroup object.

Lemma 1. *Suppose given a pullback diagram*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{pr_R} & X \\ {\scriptstyle pr_L}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \longrightarrow & Y \end{array}$$

in which p is a fibration, and let A be a homotopy cogroup object in $\mathcal{M}$. Suppose that $i : F \to X$ is the inclusion of the fibre of p over the map $* \to Y$, and suppose that the homomorphism $i_* : [A, F] \to [A, X]$ is a monomorphism. Then the canonical homomorphism

$$pr_* : [A, Z \times_Y X] \to [A, Z] \times_{[A,Y]} [A, X]$$

is an isomorphism.

Proof. We can assume that Z, X and Y are fibrant since $\mathcal{M}$ is proper. The map pr_* is surjective by the homotopy extension property. Suppose that $[\alpha]$ is in the kernel of pr_*. Then $pr_{L*}[\alpha] = e$ so that $[\alpha]$ lifts to an element $[\beta] \in [A, F]$. But $pr_{R*}[\alpha] = e$ so that $i_*[\beta] = pr_{R*}[\alpha] = e$, and so $[\beta] = e$. It follows that $[\alpha] = e$. □

Recall [5, VII.2] that the category $s\mathcal{M}$ of simplicial objects in $\mathcal{M}$ has a Reedy model structure. In particular, a Reedy weak equivalence is a map $f : X \to Y$ such that all maps $X_n \to Y_n$ are weak equivalences of $\mathcal{M}$, the map $p : Z \to W$ is a Reedy fibration if all induced maps

$$Z_n \to W_n \times_{M_n W} M_n Z$$

are fibrations of $\mathcal{M}$ for $n \geq 0$, and Reedy cofibrations are defined by a left lifting property. If $Z_p = Z_{p,*}$, then the matching space $M_n Z$ is the simplicial set specified by

$$M_n Z = \operatorname{cosk}_{n-1} Z_n.$$

for $n \geq 0$ and $M_{-1} Z = *$. In other words, $M_n Z$ is a piece of a coskeleton constructed in horizontal degrees. If f is a Reedy fibration, then all maps $f : X_n \to Y_n$, $n \geq 0$, are fibrations of $\mathcal{M}$.

In general, a map $i : U \to V$ of simplicial objects in $\mathcal{M}$ is a Reedy cofibration if and only if all maps

$$L_n V \cup_{L_n U} U_{n+1} \to V_{n+1}$$

are cofibrations of $\mathcal{M}$. Here, $L_n U = \operatorname{sk}_n U_n$. If $i : U \to V$ is a Reedy cofibration of $s\mathcal{M}$, then all maps $i : U_n \to V_n$, $n \geq 0$, are cofibrations of $\mathcal{M}$.

Recall that if K is a finite simplicial set and X is a simplicial object in $\mathcal{M}$ then $M_K X$ is the object of $\mathcal{M}$ defined by

$$M_K X = \varprojlim_{\Delta^n \to K} X_n.$$

It is a standard observation that there are natural isomorphisms $M_{\Delta^n}X \cong X_n$ and $M_{\partial\Delta^n}X = M_nX$ for all $X \in s\mathcal{M}$.

Under the stated assumption that the underlying model category $\mathcal{M}$ is proper, a revised statement of Lemma 1.8 in [3] can read as follows:

Lemma 2. *Consider a diagram in $s\mathcal{M}$*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow[i_Y]{} & Y' \end{array}$$

where q and q' are Reedy fibrations and i_X and i_Y are Reedy weak equivalences. Then for any cofibrant $A \in \mathcal{M}$ and any $K \subset \partial\Delta^n$ the induced map

$$Y_n \times_{M_KY} M_KX \to Y'_n \times_{M_KY'} M_KX'$$

is a weak equivalence of $\mathcal{M}$.

Proof. If $K = \emptyset$ the map is $Y_n \to Y'_n$, which is a weak equivalence by the assumption that i_Y is a Reedy weak equivalence. Suppose that L is obtained by attaching a non-degenerate k-simplex and we know that the result is true for all complexes K of dimension at most k and having fewer non-degenerate k-simplices. Then the inductive assumption implies that a comparison of the fibre square

$$\begin{array}{ccc} Y_n \times_{M_LY} M_LX & \longrightarrow & X_k \\ \downarrow & & \downarrow{\scriptstyle d} \\ Y_n \times_{M_KY} M_KX & \longrightarrow & Y_k \times_{M_kY} M_kX \end{array}$$

with the corresponding square for q' induces a weak equivalence

$$Y_n \times_{M_LY} M_LX \to Y'_n \times_{M_LY'} M_LX',$$

since the respective maps d are fibrations and $\mathcal{M}$ is proper. ☐

Suppose given integers $s_0, \dots, s_r$ such that $0 \leq s_0 < \dots < s_r \leq n$, and let $\Delta^n\langle s_0, \dots, s_r\rangle$ be the subcomplex of Δ^n which is generated by the simplices $d_{s_j}\iota_n$. Then [5, p.218] there are pushout diagrams

$$\begin{array}{ccc} \Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle & \xrightarrow{d_*^{s_r-1}} & \Delta^n\langle s_0, \dots, s_{r-1}\rangle \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow[d^{s_r}]{} & \Delta^n\langle s_0, \dots, s_r\rangle \end{array} \qquad (2)$$

The map $d_*^{s_r-1}$ is induced by applying the coface d^{s_r-1} to all generating simplices of $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$.

Write

$$M^n_{\langle s_0, \dots s_r\rangle} X = M_{\Delta^n\langle s_0,\dots,s_r\rangle} X$$

for an object X of $s\mathcal{M}$.

Lemma 3. *Suppose that X is a Reedy fibrant object of $s\mathcal{M}$, and that that $r < n$. Suppose that A is a homotopy cogroup object of $\mathcal{M}$.*

1) There are isomorphisms

$$\pi_j \mathbf{hom}(A, M^n_{\langle s_0,\dots,s_r\rangle} X) \cong M^n_{\langle s_0,\dots,s_r\rangle} \pi_j \mathbf{hom}(A, X), \ \ j \geq 0.$$

2) Suppose that there is a pushout diagram of finite simplicial sets

$$\begin{array}{ccc} \Delta^n\langle s_0, \dots, s_r\rangle & \longrightarrow & K \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

Then the group homomorphism

$$\begin{array}{c} \pi_j \mathbf{hom}(A, M_L X) \\ \downarrow \\ \pi_j \mathbf{hom}(A, M_K X) \times_{\pi_j \mathbf{hom}(A, M^n_{\langle s_0,\dots,s_r\rangle} X)} \pi_j \mathbf{hom}(A, X) \end{array}$$

is an isomorphism for $j \geq 0$.

Remark 4. The assumption that $r < n$ in the statement of Lemma 3 means that there is at least one face missing from $\Delta^n\langle s_0, \dots, s_r\rangle \subset \partial\Delta^n$. One can show inductively (using the pushouts (2)), under this assumption, that the inclusion

$$\Delta^n\langle s_0, \dots, s_r\rangle \subset \Delta^n$$

is a trivial cofibration.

Proof of Lemma 3. Inductively (in r), the canonical map

$$\pi_j \mathbf{hom}(A, M^{n-1}_{\langle s_0,\dots,s_{r-1}\rangle} X) \to M^{n-1}_{\langle s_0,\dots s_{r-1}\rangle} \pi_j \mathbf{hom}(A, X)$$

is an isomorphism. Also, $\pi_j \mathbf{hom}(A, X)$ is a simplicial group and $r - 1 < n - 1$, so the restriction map

$$\pi_j \mathbf{hom}(A, X_{n-1}) \to M^{n-1}_{\langle s_0,\dots,s_{r-1}\rangle} \pi_j \mathbf{hom}(A, X)$$

is surjective for all $p \geq 0$. Thus, if F is the fibre of the map $X_{n-1} \to M^{n-1}_{\langle s_0,\dots,s_{r-1}\rangle} X$, the homomorphism

$$\pi_j \mathbf{hom}(A, F) \to \pi_j \mathbf{hom}(A, X_{n-1})$$

is a monomorphism for $j \geq 0$. It therefore follows from Lemma 1 that the homomorphism

$$\pi_j \mathbf{hom}(A, M^n_{\langle s_0, \dots, s_r \rangle} X) \to M^n_{\langle s_0, \dots s_r \rangle} \pi_j \mathbf{hom}(A, X)$$

is an isomorphism for $j \geq 0$, giving statement 1).

The second statement is a consequence of the first, along with Lemma 1. In effect, there is a pullback square

$$\begin{array}{ccc} M_L X & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ M_K X & \longrightarrow & M^n_{\langle s_0, \dots, s_r \rangle} X \end{array}$$

Statement 1) implies that the group homomorphisms

$$\pi_j \mathbf{hom}(A, X_n) \to \pi_j \mathbf{hom}(A, M^n_{\langle s_0, \dots, s_r \rangle} X)$$

are surjective in all degrees, so that all morphisms

$$\pi_j \mathbf{hom}(A, F) \to \pi_j \mathbf{hom}(A, X_n)$$

are monomorphisms. □

Lemma 3 also implies Lemma 1.11 of [3], which effectively says (under the assumptions that A is a homotopy cogroup object and X is Reedy fibrant) that the homomorphism

$$\pi_j \mathbf{hom}(A, M_K X) \to M_K \pi_j \mathbf{hom}(A, X)$$

is an isomorphism if K is a finite simplicial set which is constructed by a finite string of anodyne extensions. We shall mostly be interested in the following special case:

Corollary 5. *Suppose that A is a homotopy cogroup object of $\mathcal{M}$ and that X is a Reedy fibrant object of $s\mathcal{M}$. Then the canonical group homomorphism*

$$\pi_j \mathbf{hom}(A, M^k_n X) \to M^k_n \pi_j \mathbf{hom}(A, X)$$

is an isomorphism for $j \geq 0$.

A fibrant replacement f' for a map $f : X \to Y$ in a closed model category is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

in which the horizontal maps are weak equivalences, and X' and Y' are fibrant. We shall repeatedly use fibrant replacements within Reedy model structures, and refer to them as Reedy fibrant replacements.

Suppose that I is a small category, and that $\mathcal{A} : I \to \mathcal{M}$ is a functor taking values in cogroup objects of $\mathcal{M}$ with $i \mapsto \mathcal{A}(i) = A_i$.

Say that a map $f : X \to Y$ of $s\mathcal{M}$ is an $\mathcal{A}$-fibration if f is a Reedy fibration, and there is some Reedy fibrant replacement f' such that the induced map

$$\pi_j \mathbf{hom}(A_i, X') \to \pi_j \mathbf{hom}(A_i, Y') \tag{3}$$

is a fibration of simplicial groups for all $i \in I$ and all $j \geq 0$. It's an exercise to show that there is some Reedy fibrant replacement f' such that the above map is a fibration of simplicial groups if and only if the corresponding map is a fibration of simplicial groups for any Reedy fibrant replacement f'. The map f is said to be an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement f' such that the induced map (3) is a weak equivalence of simplicial groups. An $\mathcal{A}$-cofibration is a map of $s\mathcal{M}$ which has the left lifting property with respect to all maps which are simultaneously $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences.

It's an observation that every Reedy weak equivalence is an $\mathcal{A}$-equivalence. In particular, every trivial Reedy cofibration is a trivial $\mathcal{A}$-cofibration.

One can also see that a Reedy fibration $f : X \to Y$ is an $\mathcal{A}$-fibration (respectively $\mathcal{A}$-equivalence) if and only if the simplicial group maps

$$[A_i \wedge S^j, X] \to [A_i \wedge S^j, Y]$$

are fibrations (respectively weak equivalences) for all objects i of I and all $j \geq 0$. It follows in particular that every trivial Reedy fibration is an $\mathcal{A}$-trivial fibration.

Lemma 6. *Suppose that $f : X \to Y$ is a Reedy fibration and that Y is Reedy fibrant. Then*

1) f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps

$$\Lambda^n_k \otimes (A_i \wedge S^j) \to \Delta^n \otimes (A_i \wedge S^j)$$

2) f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if it has the right lifting property with respect to all maps

$$\partial\Delta^n \otimes (A_i \wedge S^j) \to \Delta^n \otimes (A_i \wedge S^j)$$

Proof. The Reedy fibration $f : X \to Y$ is an $\mathcal{A}$-fibration if and only if all group homomorphisms

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n] \times_{M^k_n[A_i \wedge S^j, Y]} M^k_n[A_i \wedge S^j, X]$$

are surjective.

The map f has the right lifting property with respect to all maps

$$\Lambda^n_k \otimes (A_i \wedge S^j) \to \Delta^n \otimes (A_i \wedge S^j)$$

if and only if the maps

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n \times_{M^k_n Y} M^k_n X]$$

are surjective. In effect, f has the stated lifting property if and only if the fibrations

$$\mathbf{hom}(A_i \wedge S^j, X_n) \to \mathbf{hom}(A_i \wedge S^j, Y_n \times_{M_n^k Y} M_n^k X)$$

are surjective maps of Kan complexes. The map $X_n \to Y_n \times_{M_n^k Y} M_n^k X$ is a fibration by Lemma 1.7 of [3], and X is Reedy fibrant so that all objects $M_K X$ and in particular the objects $M_n^k X$ are fibrant — again by Lemma 1.7 of [3].

Thus to show that f is an $\mathcal{A}$-fibration if and only if it has the advertised lifting property, it suffices to show that the canonical group homomorphisms

$$[A_i \wedge S^j, Y_n \times_{M_n^k Y} M_n^k X] \to [A_i \wedge S^j, Y_n] \times_{M_n^k [A_i \wedge S^j, Y]} M_n^k [A_i \wedge S^j, X]$$

are isomorphisms. This follows from the comparison of long exact sequences arising from the comparison of fibrations

$$\begin{array}{ccc} Y_n \times_{M_n^k Y} M_n^k X & \longrightarrow & M_n^k X \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & M_n^k Y \end{array}$$

and the isomorphisms

$$\pi_j \mathbf{hom}(A_i, M_n^k Y) \cong M_n^k \pi_j \mathbf{hom}(A_i, Y)$$

of Corollary 5 and Lemma 1.

The proof of part 2) is similar, and ultimately depends on showing that the comparison homomorphisms

$$[A_i \wedge S^j, Y_n \times_{M_n Y} M_n X] \to [A_i \wedge S^j, Y_n] \times_{M_n [A_i \wedge S^j, Y]} M_n [A_i \wedge S^j, X] \quad (4)$$

are isomorphisms under the assumption that either f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, or that f satisfies the lifting property. Observe that if $n = 0$ there is nothing to prove.

One can check directly that there are pullback squares

$$\begin{array}{ccc} Y_{n+1} \times_{M_{n+1} Y} M_{n+1} X & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ Y_{n+1} \times_{M_{n+1}^0 Y} M_{n+1}^0 X & \longrightarrow & Y_n \times_{M_n Y} M_n X \end{array} \quad (5)$$

for $n \geq 0$, where the bottom horizontal map is defined by

$$(y, (x_1, \dots, x_{n+1})) \mapsto (d_0 y, (d_0 x_1, \dots, d_0 x_{n+1})).$$

If all homomorphisms $[A_i \wedge S^j, Y_n] \to [A_i \wedge S^j, Y_n \times_{M_n Y} M_n X]$ are surjective (ie. if f satisfies the lifting property), then one uses the pullback squares (5), part 1) and Lemma 1 to show inductively that the maps (4) are isomorphisms. It follows that all homomorphisms

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n] \times_{M_n[A_i \wedge S^j, Y]} M_n[A_i \wedge S^j, X]$$

are surjective, so that f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence.

Conversely, assume that all homomorphisms

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n] \times_{M_n[A_i \wedge S^j, Y]} M_n[A_i \wedge S^j, X]$$

are surjective. Assume inductively that the morphism

$$[A_i \wedge S^j, Y_n \times_{M_n Y} M_n X] \to [A_i \wedge S^j, Y_n] \times_{M_n[A_i \wedge S^j, Y]} M_n[A_i \wedge S^j, X]$$

is an isomorphism for all $j \geq 0$. Then the map

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n \times_{M_n Y} M_n X]$$

is surjective for all $j \geq 0$, and so all squares

$$\begin{array}{ccc} [A_i \wedge S^j, Y_{n+1} \times_{M_{n+1} Y} M_{n+1} X] & \longrightarrow & [A_i \wedge S^j, X_n] \\ \downarrow & & \downarrow \\ [A_i \wedge S^j, Y_{n+1} \times_{M^0_{n+1} Y} M^0_{n+1} X] & \longrightarrow & [A_i \wedge S^j, Y_n \times_{M_n Y} M_n X] \end{array}$$

are pullbacks in the category of abelian groups by Lemma 1. It follows that the maps

$\to$

$$[A_i \wedge S^j, Y_{n+1}] \times_{M_{n+1}[A_i \wedge S^j, Y]} M_{n+1}[A_i \wedge S^j, X]$$

are isomorphisms. All homomorphisms

$$[A_i \wedge S^j, X_n] \to [A_i \wedge S^j, Y_n \times_{M_n Y} M_n X]$$

are therefore surjective, by induction on n. □

Corollary 7. *Every Reedy fibrant object X is an $\mathcal{A}$-fibrant object of $s\mathcal{M}$.*

Recall that for a simplicial object X in $\mathcal{M}$ and a simplicial set K, the object $M_K X \in \mathcal{M}$ is defined by

$$M_K X = \varinjlim_{\Delta^m \to K} X_m.$$

Recall further that there is a natural bijection

$$\hom(K \otimes A, X) \cong \hom(A, M_K X).$$

There is a simplicial object $\mathbf{M}_K X$ which is specified in horizontal degree n by

$$\mathbf{M}_K X_n = M_{K \times \Delta^n} X,$$

and there is a natural bijection

$$\hom(L \otimes A, \mathbf{M}_K X) \cong \hom((K \times L) \otimes A, X).$$

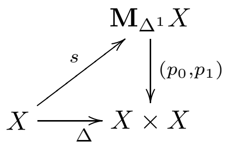
An adjointness argument then gives the following:

Corollary 8. *Suppose that $p : X \to Y$ is an $\mathcal{A}$-fibration, Y is Reedy fibrant, and that $i : K \subset L$ is an inclusion of finite simplicial sets. Then the map*

$$\mathbf{M}_L X \to \mathbf{M}_L Y \times_{\mathbf{M}_K Y} \mathbf{M}_K X$$

is an $\mathcal{A}$-fibration. which is an $\mathcal{A}$-equivalence if p is an $\mathcal{A}$-equivalence or i is a weak equivalence of simplicial sets.

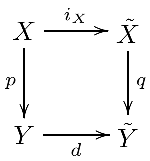
Corollary 9. *Suppose that X is Reedy fibrant. Then the inclusion $\partial\Delta^1 \subset \Delta^1$ induces a commutative diagram*



in which the map (p_0, p_1) is an $\mathcal{A}$-fibration and the maps p_0 and p_1 are trivial $\mathcal{A}$-fibrations.

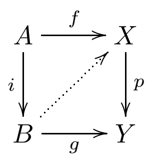
We shall also need the following general facts in later sections.

Lemma 10. *Suppose that $p : X \to Y$ is a fibration and that*

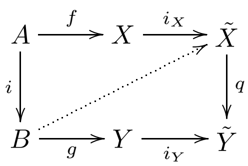


is a fibrant replacement in some proper closed model category, where q is a fibration. If q has the right lifting property with respect to a cofibration $i : A \to B$, then so does p.

Proof. Consider the lifting problem



In order to solve this problem it suffices to show that the dotted arrow exists in the diagram



In effect, the canonical map $\theta : X \to Y \times_{\tilde{Y}} \tilde{X}$ is a weak equivalence since the model structure is proper. Factor θ as

$$\begin{array}{ccc} X & \xrightarrow{\omega} & Z \\ & \searrow^{\theta} & \downarrow{\scriptstyle \pi} \\ & & Y \times_{\tilde{Y}} \tilde{X} \end{array}$$

where ω is a trivial cofibration and π is a trivial cofibration. Then the indicated lift induces a lift $\sigma : C \to Z$. There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle \omega}\downarrow & \nearrow^{r} & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[q_*\pi]{} & Y \end{array}$$

and the composite $r \cdot \sigma$ is the desired lift. $\square$

Lemma 11. *Suppose that $p : Z \to W$ is a fibration and that W is a fibrant object in some closed model category with initial object $\emptyset$. Suppose that $f : A \to B$ is a weak equivalence of cofibrant objects. Then p has the right lifting property with respect to $\emptyset \to A$ if and only if it has the right lifting property with respect to $\emptyset \to B$.*

Proof. Precomposition with f defines bijections

$$[B, W] \cong [A, W], \quad [B, Z] \cong [A, Z]$$

and all sets of morphisms in the homotopy category coincide with homotopy classes of maps since A and B are cofibrant and Z and W are fibrant.

Suppose that p has the right lifting property with respect to $\emptyset \to A$ and consider the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\alpha]{} & W \end{array}$$

Then there is a map $\theta_1 : A \to Z$ such that $p \cdot \theta_1 = \alpha \cdot f$. But then there is a map $\theta_2 : B \to Z$ such that $\theta_2 \cdot f \simeq \theta_1$. Then $p \cdot \theta_2 \cdot f \simeq \alpha \cdot f$ so that there is a homotopy $p \cdot \theta_2 \simeq \alpha$. In other words, α lifts up to homotopy, so α has a lifting by the usual homotopy lifting argument.

Suppose that p has the right lifting property with respect to $\emptyset \to B$ and consider the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\beta]{} & W \end{array}$$

There is a map $\gamma : B \to W$ such that $\gamma \cdot f \simeq \beta$. At the same time γ lifts to Z by assumption. It follows that β lifts to Z up to homotopy and hence lifts on the nose. □

2 Pointed simplicial presheaves

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let $\mathbf{S}(\mathcal{C})$ denote the category of simplicial presheaves on $\mathcal{C}$. This category has a standard proper closed simplicial model structure for which the cofibrations are the monomorphisms and the weak equivalences (usually called local weak equivalences) are maps which induce isomorphisms in all homotopy groups for all local choices of base points. In the presence of stalks, a map $f : X \to Y$ is a weak equivalence if and only if f induces weak equivalences $f_* : X_x \to Y_x$ of simplicial sets in all stalks. The fibrations, or global fibrations, are those maps which have the right lifting property with respect to all maps which are simultaneously cofibrations and local weak equivalences.

Suppose that α is an infinite cardinal which is greater than the cardinality of the set of morphisms of $\mathcal{C}$. One shows that a map is a global fibration if and only if it has the right lifting property with respect to all cofibrations $i : A \to B$ which are local weak equivalences and are α-bounded in the sense that the cardinality of all sets of simplices $B_n(U)$, $U \in \mathcal{C}$, $n \geq 0$, is bounded above by α. This result is now part of the usual approach to constructing the standard closed model structure for $\mathbf{S}(\mathcal{C})$, in that it produces a set of generating trivial cofibrations for the structure. In turn (see [4]) it is a consequence of the bounded cofibration condition (**BC**), which can be stated as follows:

BC: For every diagram of simplicial presheaf monomorphisms

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

with i a weak equivalence as well as a cofibration and A α-bounded, there is a subobject $B \subset Y$ such that $A \subset B$, the object B is α-bounded, and the inclusion $B \cap X \to B$ is a weak equivalence and a cofibration.

One says that A is α-bounded if the cofibration $\emptyset \to A$ is α-bounded. The classes of cofibrations and local weak equivalences in the category of simplicial presheaves together satisfy this condition.

Formally inverting a cofibration $f : A \to B$ of $\mathbf{S}(\mathcal{C})$ gives rise to the f-local theory for the category of simplicial presheaves, as in [4]. The cofibrations are just monomorphisms again, but the weak equivalences are more difficult to define. Explicitly, an f-injective object is defined to be a globally fibrant simplicial presheaf Z such that the global fibration $Z \to *$ has the right lifting property with respect to all inclusions

$$(B \times Y) \cup_{(A\times Y)} (A \times L_U \Delta^n) \to B \times L_U \Delta^n$$

which are induced by the map f and all generating cofibrations arising from subobjects $Y \subset L_U\Delta^n$ of the simplicial presheaves $L_U\Delta^n$. The objects $L_U\Delta^n$ arise from the standard n-simplices by applying the left adjoints L_U of the U-sections functors $X \mapsto X(U)$, $U \in \mathcal{C}$. A map $g : X \to Y$ is an f-local weak equivalence if and only if it induces a bijection

$$[Y, Z] \cong [X, Z]$$

of simplicial presheaf homotopy classes of maps for all f-injective objects Z. Then there is an f-local closed simplicial model category structure on the category of simplicial presheaves for which the cofibrations are the monomorphisms as before, and for which the weak equivalences are the f-local weak equivalences. The f-local fibrations are defined by a lifting property in the usual way, and then one shows after the fact that the classes of f-injective and f-local fibrant objects coincide.

There are two remarks to be made:

1) The f-local theory for $\mathbf{S}(\mathcal{C})$ satisfies the bounded cofibration condition **BC** for some choice of large cardinal α — this is proved in Lemma 4.7 of [4].

2) The f-local theory if proper if f is of the form $f : * \to I$, ie. some global choice of base point for a simplicial presheaf I. This is proved in Appendix A of [12]. The method of proof for this result leads to a general criterion: the f-local theory is proper if for every diagram

$$\begin{array}{ccccc} & & & & U \\ & & & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B & \longrightarrow & X \end{array}$$

with p a global fibration and X an f-local fibrant object, the induced map $f_* : A \times_X U \to B \times_X U$ is an f-local equivalence. Lemma A.1 of [4] amounts to a demonstration of this criterion for $f : * \to I$.

We shall henceforth refer to cofibrations, weak equivalences and fibrations within a proper closed simplicial model structure for the category of simplicial presheaves; this structure will be either the standard theory, or a proper f-local theory. The unstable motivic homotopy theory [12] of Morel and Voevodsky [13] is an example of the latter. All of these theories satisfy a bounded cofibration condition.

The category $\mathbf{S}_*(\mathcal{C})$ of pointed simplicial presheaves $* \to X$ inherits a proper closed model structure from any proper closed simplicial model structure on the category of simplicial presheaves. In particular, a pointed map $f : X \to Y$ is a weak equivalence (respectively fibration, cofibration) if and only if f is a weak equivalence (respectively fibration, cofibration) of $\mathbf{S}(\mathcal{C})$. The pointed function complexes $\mathbf{hom}_*(X, Y)$ appear as fibres of ordinary function complexes in the usual way, meaning that there is a defining pullback diagram

$$\begin{array}{ccc} \mathbf{hom}_*(X,Y) & \longrightarrow & \mathbf{hom}(X,Y) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{hom}(*,Y) \end{array}$$

We have the following simple consequences of the bounded cofibration condition.

Lemma 12. *Suppose given cofibrations $A \to B \to X$ such that the composite $A \to X$ is a weak equivalence and B is α-bounded. Then there is a trivial cofibration $C \subset X$ such that $B \subset C$ and C is α-bounded.*

Proof. This result is a consequence of the bounded cofibration condition.

Under the stated assumptions, and for the diagram of cofibrations

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & X \end{array}$$

there is an α-bounded object C with $B \subset C \subset X$ and such that the induced cofibration $C \cap A \to C$ is a weak equivalence. Then $A = C \cap A$ because $A \subset C$. □

Corollary 13. *Suppose given a diagram of trivial cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

such that A is α-bounded. Then there is an α-bounded subobject $C \subset Y$ such that the inclusions $A \subset C$ and $C \cap X \to C$ are trivial cofibrations.

Proof. By the bounded cofibration condition, there is an α-bounded subobject $D_1 \subset Y$ with $A \subset D_1$ and such that $D_1 \cap X \to D_1$ is a trivial cofibration. By the second lemma, there is a trivial cofibration $C_1 \subset Y$ with C_1 α-bounded and such that $D_1 \subset C_1$. Inductively form the string of cofibrations

$$A \subset D_1 \subset C_1 \subset D_2 \subset C_2 \subset \dots$$

such that all objects are α-bounded, all maps $A \to C_i$ and $D_i \cap X \to D_i$ are trivial cofibrations. Let $C = \varinjlim C_i \subset Y$. Then C is α-bounded, the map $A \to C$ is a trivial cofibration, and $C \cap X \to C$ is a trivial cofibration. □

Suppose that $i : A \to B$ and $j : A \to C$ are α-bounded cofibrations. A bounded relation from i to j is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ {\scriptstyle i}\big\downarrow & \searrow & \big\downarrow{\scriptstyle u} \\ B & \xrightarrow[v]{} & D \end{array} \tag{6}$$

where u and v are α-bounded trivial cofibrations. I shall also say that j is an α-bounded approximation of i. In the pointed category, if $A = *$ one says that C is an α-bounded approximation of B.

Lemma 14. *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \tilde{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Y & \xrightarrow[i_Y]{} & \tilde{Y} \end{array}$$

is a fibrant replacement of a fibration p in the sense that i_X and i_Y are trivial cofibrations, $\tilde{Y}$ is fibrant, and q is a fibration. Suppose that $i : A \to B$ is an α-bounded cofibration. Then the dotted arrow exists in all diagrams of the form

$$\begin{array}{ccccc} A & \xrightarrow{\gamma} & X & \xrightarrow{i_X} & \tilde{X} \\ {\scriptstyle i}\big\downarrow & & \cdot^{\cdot^{\cdot}} & & \big\downarrow{\scriptstyle q} \\ B & & \xrightarrow[\beta]{} & & \tilde{Y} \end{array} \tag{7}$$

if and only if p has the right lifting property with respect to all α-bounded cofibrations $j : A \to C$ related to i.

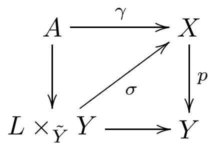
Proof. Suppose that p has the right lifting property with respect to all α-bounded cofibrations related to i, and consider the lifting problem in the diagram (7). The map β has a factorization

$$\begin{array}{ccc} B & \xrightarrow{\omega} & Z \\ & {\scriptstyle \beta}\searrow & \big\downarrow{\scriptstyle \pi} \\ & & \tilde{Y} \end{array}$$

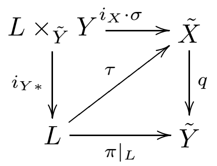
where π is a fibration and ω is a trivial cofibration. Then the induced map $i_{Y*} : Z \times_{\tilde{Y}} Y \to Z$ is a trivial cofibration by properness since π is a fibration. Now apply Corollary 13 to the trivial cofibrations

$$\begin{array}{ccc} & & Z \times_{\tilde{Y}} Y \\ & & \big\downarrow{\scriptstyle i_{Y*}} \\ B & \xrightarrow[\omega]{} & Z \end{array}$$

to find a factorization $B \subset L \subset Z$ of ω with L α-bounded, such that the inclusion $B \subset L$ and the map $i_{Y*} : L \times_{\tilde{Y}} Y \to L$ are trivial cofibrations. Then there is an induced commutative diagram

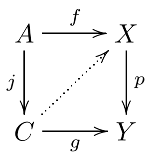


in which the lifting σ exists by the hypothesis on p. But then the lift τ exists in the diagram

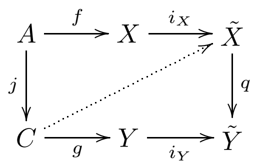


since i_{Y*} is a trivial cofibration. The restriction $\tau|_B : B \to \tilde{X}$ is the required lift.

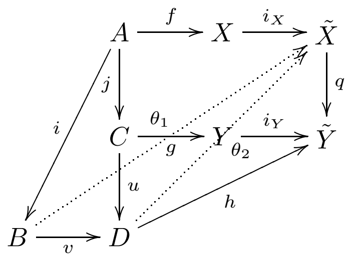
Suppose that the dotted arrow exists in all diagrams (7) and that the diagram (6) gives $j : A \to C$ the structure of an α-bounded cofibration related to i. Consider a lifting problem



By the proof of Lemma 10 it suffices to show that the dotted arrow exists in the diagram



There is a diagram



where the map h exists because u is a trivial cofibration and $\tilde{Y}$ is fibrant. Then, in order, the dotted arrows θ_1 and θ_2 exist making the diagram commute, because q has the lifting property (7), and then v is a trivial cofibration. The required lift is the composite $\theta_2 \cdot u$. □

In general, if $i : U \to V$ is an inclusion of simplicial sets, then the map

$$\mathrm{sk}_n V_{n+1} \cup_{\mathrm{sk}_n U_{n+1}} U_{n+1} \to V_{n+1}$$

is a monomorphism, since one can show that

$$\mathrm{sk}_n U_{n+1} = \mathrm{sk}_n V_{n+1} \cap U_{n+1}$$

in V_{n+1}. It follows that every levelwise cofibration of bisimplicial presheaves is a Reedy cofibration [5, VII.2], so that the Reedy structure on the category of bisimplicial presheaves is defined by degreewise cofibrations and degreewise weak equivalences in the category of simplicial presheaves.

Suppose that

$$\mathcal{A} : I \to \mathbf{S}_*(\mathcal{C}), \ \ i \mapsto A_i$$

is an α-bounded diagram of α-bounded homotopy cogroup objects in the category of pointed simplicial presheaves, where α is some large cardinal for which $\mathbf{S}(\mathcal{C})$ satisfies the bounded cofibration condition. The requirement that $\mathcal{A}$ is an α-bounded diagram means that the cardinality of the morphism set of the index category I should be bounded above by α. As in the first section, a map $f : X \to Y$ of bisimplicial presheaves is said to be an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement $f' : X' \to Y'$ such that the induced maps

$$\pi_j \mathbf{hom}_*(A_i, X') \to \pi_j \mathbf{hom}_*(A_i, Y'), \ \ i \in I, j \geq 0, \tag{8}$$

are weak equivalences of simplicial groups. The map f is an $\mathcal{A}$-fibration if f is a Reedy fibration and the maps (8) are fibrations of simplicial groups.

An $\mathcal{A}$-cofibration is a map which has the left lifting property with respect to all maps which are $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences. Every degreewise trivial cofibration has the left lifting property with respect to all Reedy fibrations, and is therefore a $\mathcal{A}$-cofibration (as well as a $\mathcal{A}$-equivalence).

Lemma 15. *Suppose that the map $f : X \to Y$ is a Reedy fibration of bisimplicial presheaves. Then*

1) *f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps*

$$\Lambda^m_k \otimes B \to \Delta^m \otimes B$$

where B is any α-bounded approximation of some $A_i \wedge S^j$, $i \in I, j \geq 0$.

2) *f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if f has the right lifting property with respect to all maps*

$$\partial\Delta^m \otimes B \to \Delta^m \otimes B$$

where B is any α-bounded approximation of some $A_i \wedge S^j$, $i \in I, j \geq 0$.

Proof. Suppose that

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[i_Y]{} & \tilde{Y} \end{array}$$

is a Reedy fibrant replacement for f. Then the diagrams

$$\begin{array}{ccc} X_m & \longrightarrow & \tilde{X}_m \\ \downarrow & & \downarrow \\ Y_m \times_{M_m^k Y} M_m^k X & \longrightarrow & \tilde{Y}_m \times_{M_m^k \tilde{Y}} M_m^k \tilde{X} \end{array}$$

$$\begin{array}{ccc} X_m & \longrightarrow & \tilde{X}_m \\ \downarrow & & \downarrow \\ Y_m \times_{M_m Y} M_m X & \longrightarrow & \tilde{Y}_m \times_{M_m \tilde{Y}} M_m \tilde{X} \end{array}$$

are fibrant replacements in the category of pointed simplicial presheaves by Lemma 2, where in particular all vertical maps are fibrations. Now use Lemma 6 and Lemma 14. □

Theorem 16. *Suppose that $\mathcal{A} : I \to \mathbf{S}_*(\mathcal{C})$ is an α-bounded diagram of α-bounded cogroup objects. Then with the definitions of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration and $\mathcal{A}$-cofibration given above, the category $s\mathbf{S}_*(\mathcal{C})$ of pointed bisimplicial presheaves on a small site $\mathcal{C}$ satisfies the axioms for a closed model category.*

Proof. The axioms **CM1** and **CM2** are trivially verified. The class of $\mathcal{A}$-fibrations is closed under retract on account of Lemma 15, giving the non-trivial part of **CM3**.

Observe that a map $f : X \to Y$ of bisimplicial presheaves is a Reedy fibration if and only if it has the right lifting property with respect to the maps

$$(\partial\Delta^n \otimes B) \cup_{(\partial\Delta^n \otimes A)} (\Delta^n \otimes A) \subset \Delta^n \otimes B$$

where $A \to B$ varies over a set of α-bounded cofibrations of simplicial presheaves. It follows from Lemma 15 that there are factorizations

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

of a map f, where i is an $\mathcal{A}$-cofibration and p is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations.

Suppose that $j : A \to B$ is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations. We will show that j is an $\mathcal{A}$-equivalence — this will complete the proof of the factorization axiom **CM5**, and then **CM4** is a standard consequence. This will be achieved essentially with the Quillen argument [5, p.114] that starts with the existence of a natural fibrant model, which is given by Corollary 7.

Use Corollary 9 to form the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{i_A} & \tilde{A} \\
{\scriptstyle j}\downarrow & \nearrow{\scriptstyle u} & \\
B & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{s i_B j} & \mathbf{M}_{\Delta^1}\tilde{B} \\
{\scriptstyle j}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle (p_0,p_1)} \\
B & \xrightarrow[(i_B, \tilde{j}\cdot u)]{} & \tilde{B} \times \tilde{B}
\end{array}
$$

Then the simplicial group map

$$u_* : [A_i \wedge S^j, B] \to [A_i \wedge S^j, \tilde{A}]$$

induces an epimorphism on homotopy groups by the first diagram since i_A is an $\mathcal{A}$-equivalence, while the existence of the homotopy h implies that u_* is a monomorphism on homotopy groups. Thus u is an $\mathcal{A}$-equivalence, so j is an $\mathcal{A}$-equivalence. □

3 Presheaves of spectra

The weak equivalences for the stable model category structure for the category $\mathbf{Spt}(\mathcal{C})$ of presheaves of spectra are those maps $f : X \to Y$ which induce isomorphisms $f_* : \pi_* X \to \pi_* Y$ for all sheaves of stable homotopy groups. A map $i : A \to B$ is a cofibration if the map $i : A^0 \to B^0$ and all maps

$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$

are cofibrations of pointed simplicial presheaves. The fibrations of the category are the stable fibrations, which are those maps which have the right lifting property with respect to morphisms which are cofibrations and stable equivalences. The stable model structure is a proper closed simplicial model category. It also satisfies a stable bounded cofibration condition:

sBC: There is an infinite cardinal α such that for every diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow{\scriptstyle i} \\
A & \longrightarrow & Y
\end{array}
$$

of level cofibrations with i a stable equivalence and A α-bounded, there is a subobject $B \subset Y$ such that $A \subset B$, the object B is α-bounded, and the inclusion $B \cap X \to B$ is a stable equivalence and a level cofibration.

A level cofibration is a map of presheaves of spectra $i : A \to B$ such that all maps $i : A^n \to B^n$ are cofibrations. Note that the statement **sBC** is not the version of the bounded cofibration condition given in [4], since we do not require the map i to be a cofibration; it does, however, have the same proof. This version of the stable bounded cofibration condition implies the former, since cofibrations pull back along inclusions of subobjects by Lemma 3.1 of [4].

The following two results follow from the stable bounded cofibration condition **sBC** in the same way that Lemma 12 and Corollary 13 follow from the bounded cofibration condition for simplicial presheaves:

Lemma 17. *Suppose given inclusions (ie. level cofibrations) $A \subset B \subset X$ such that the composite $A \to X$ is a stable equivalence and B is α-bounded. Then there is a subobject $C \subset X$ such that $B \subset C$, C is α-bounded, and the inclusion $C \subset X$ is a stable equivalence.*

Corollary 18. *Suppose given a diagram of level cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

of a presheaf of spectra X such that A is α-bounded and both maps are stable equivalences. Then there is an α-bounded subobject $C \subset Y$ such that the inclusions $A \subset C$ and $C \cap X \to C$ are stable equivalences.

Suppose that K is a cofibrant and α-bounded presheaf of spectra. Say that A is an α-bounded approximation of K if there is a sequence of level cofibrations

$$A \xrightarrow{u} B \xleftarrow{v} K$$

such that B is α-bounded, and both α and β are stable equivalences.

Lemma 19. *Suppose given a diagram*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ K & \xrightarrow[\gamma]{} & Y \end{array}$$

of maps of presheaves of spectra, where K is cofibrant and α-bounded, and f is a stable equivalence. Then there is an extended diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \\ {\scriptstyle v}\uparrow & \nearrow_{\gamma} & \\ K & & \end{array}$$

in which A and B are α-bounded, and the morphisms β and β' are level cofibrations and stable equivalences.

Proof. It is harmless to suppose that the map γ is a level cofibration and a stable equivalence, since the stable model structure is proper.

Factorize f as

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & \searrow^{f} & \downarrow{\pi} \\ & & Y \end{array}$$

where j is a cofibration, π is a stable fibration, and both maps are stable equivalences. The map γ lifts to a monomorphism $m : K \to Z$ since K is cofibrant, and then (by Corollary 18) there is an α-bounded subobject $B \subset Z$ containing K such that the induced monomorphism

$$u : A = B \times_Z X \to B$$

is a stable equivalence. The inclusion $K \subset B$ is v. □

It is shown, for example in [4] that the stable model structure for presheaves of spectra is cofibrantly generated. In particular, the structure has functorial factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_f} & M(f) \\ & \searrow^{f} & \downarrow{p_f} \\ & & Y \end{array}$$

for arbitrary maps f, such that i_f is a cofibration and p_f is a trivial stable fibration. Write $M(A)$ for the cofibrant model for a presheaf of spectra A which arises from the canonical map $* \to A$ in this way, and let $p_A : M(A) \to A$ be the corresponding trivial stable fibration.

Lemma 20. *Suppose that the presheaf of spectra K is α-bounded and cofibrant. Suppose also that*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[i_Y]{} & \tilde{Y} \end{array}$$

is a stably fibrant model for the stable fibration p. Then q has the right lifting property with respect to $* \to K$ if and only if p has the right lifting property with respect to all cofibrations $* \to M(A)$ arising from cofibrant models for α-bounded approximations A of K.

Proof. First of all q has the right lifting property with respect to $* \to K$ if and only if it has the right lifting property with respect to all $* \to M(A)$, by Lemma

11. It therefore follows from Lemma 10 that p has the right lifting property with respect to all $* \to M(A)$ if q has the right lifting property with respect to $* \to K$.

Suppose that p has the right lifting property with respect to all maps $* \to M(A)$, and consider the lifting problem

$$\begin{array}{ccc} * & \longrightarrow & \tilde{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle q} \\ K & \xrightarrow[\alpha]{} & \tilde{Y} \end{array}$$

Use Lemma 19 to find stably trivial level cofibrations $A \to B \leftarrow K$ such that B is α-bounded, and which fit into a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle i_Y} \\ B & \longrightarrow & \tilde{Y} \\ {\scriptstyle v}\uparrow & \nearrow{\scriptstyle \alpha} & \\ K & & \end{array}$$

Then the resulting map $M(A) \to Y$ lifts to X by assumption so that the composite $M(A) \to Y \to \tilde{Y}$ lifts to $\tilde{X}$. It follows from Lemma 11 that the map $M(B) \to \tilde{Y}$ lifts to $\tilde{X}$, and so the map $K \to \tilde{Y}$ lifts to $\tilde{X}$. □

Suppose that

$$\mathcal{A} : I \to \mathbf{Spt}(\mathcal{C}), \quad i \mapsto A_i$$

is an α-bounded diagram of α-bounded, cofibrant presheaves of spectra. Following the pattern of definitions of the first section, we say that map $f : X \to Y$ of simplicial presheaves of spectra an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement $f' : X' \to Y'$ for f such that the induced maps

$$\pi_j \mathbf{hom}(A_i, X') \to \pi_j \mathbf{hom}(A_i, Y') \tag{9}$$

are weak equivalences of simplicial abelian groups for all $i \in I$ and all $j \in \mathbb{Z}$. The map f is an $\mathcal{A}$-fibration if it is a Reedy fibration and all maps (9) are fibrations of simplicial abelian groups. An $\mathcal{A}$-cofibration is a map of simplicial presheaves of spectra which has the left lifting property with respect to all maps which are $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences.

Lemma 21. *Suppose that the map $f : X \to Y$ is a Reedy fibration of simplicial presheaves of spectra. Then*

1) *the map f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps*

$$\Lambda^m_k \otimes M(A) \to \Delta^m \otimes M(A)$$

where A runs through the α-bounded approximations of the objects $A_i \wedge S^j$.

2) the map f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if f has the right lifting property with respect to all maps

$$\partial\Delta^m \otimes M(A) \to \Delta^m \otimes M(A)$$

where A is any α-bounded approximation of some $A_i \wedge S^j$.

Proof. The proof is by analogy with that of Lemma 15. Use Lemma 20 in place of Lemma 14. □

Theorem 22. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}(\mathcal{C})$ is an α-bounded diagram of α-bounded, cofibrant presheaves of spectra A_i. Then with the definitions of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration and $\mathcal{A}$-cofibration given above, the category $s\mathbf{Spt}(\mathcal{C}$ of simplicial presheaves of spectra on a small site $\mathcal{C}$ satisfies the axioms for a closed model category.*

Proof. The axioms **CM1** and **CM2** are trivially verified. The class of $\mathcal{A}$-fibrations is closed under retract on account of Lemma 21, giving the non-trivial part of **CM3**.

It follows from the bounded cofibration condition **sBC** that a map of presheaves of spectra is a stable fibration if and only if it has the right lifting property with respect to all cofibrations which are α-bounded and stable equivalences [4]. It follows that a map $f : X \to Y$ of presheaves of spectra is a Reedy fibration if and only if it has the right lifting property with respect to the maps

$$(\partial\Delta^n \otimes B) \cup_{(\partial\Delta^n \otimes A)} (\Delta^n \otimes A) \subset \Delta^n \otimes B$$

where $A \to B$ varies over a set of α-bounded stably trivial cofibrations. Lemma 21 then implies that there are factorizations

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

of a map f, where i is an $\mathcal{A}$-cofibration and p is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations.

The proof is finished by showing that every $\mathcal{A}$-cofibration $j : A \to B$ which has the left lifting property with respect to all $\mathcal{A}$-fibrations must be an $\mathcal{A}$-equivalence. This is done as in the proof of Theorem 16. □

Write $\mathbf{Spt}^{\Sigma}(\mathcal{C})$ for the category of presheaves of symmetric spectra on the site $\mathcal{C}$. It is shown in [11, Th. 12] that this category has a proper closed simplicial model structure. The fibrations for this theory, or stable fibrations, are those morphisms $p : X \to Y$ of presheaves of symmetric spectra such that the underlying map $p_* : UX \to UY$ of presheaves of spectra are stable fibrations in the sense described above. A stable weak equivalence $f : X_1 \to X_2$ of $\mathbf{Spt}^{\Sigma}(\mathcal{C})$ is a map which induces a weak equivalence of function complexes

$$\mathbf{hom}(X_2, W) \to \mathbf{hom}(X_1, W)$$

where W is stably fibrant and injective. An injective object is a fibrant object in a model structure of level cofibrations and level weak equivalences. The function complexes displayed above are formed in the expected way, and are part of the data for the simplicial model structure.

Observe that $p : X \to Y$ is a stable fibration of presheaves of symmetric spectra if and only if it has the right lifting property with respect to all maps $i_* : VA \to VB$ arising from α-bounded trivial stable cofibrations $i : A \to B$ of presheaves of spectra, by application of the left adjoint $V : \mathbf{Spt}(\mathcal{C}) \to \mathbf{Spt}^{\Sigma}(\mathcal{C})$ to the functor $U : \mathbf{Spt}^{\Sigma}(\mathcal{C}) \to \mathbf{Spt}(\mathcal{C})$ which forgets the symmetric structure.

Suppose that

$$\mathcal{A} : I \to \mathbf{Spt}(\mathcal{C}), \quad i \mapsto A_i,$$

is an α-bounded diagram of α-bounded, cofibrant presheaves of spectra as above. Say that a map $f : X \to Y$ of simplicial presheaves of symmetric spectra is an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement $f' : X' \to Y'$ for f such that the induced maps

$$\pi_j \mathbf{hom}(VA_i, X') \to \pi_j \mathbf{hom}(VA_i, Y') \tag{10}$$

are weak equivalences of simplicial abelian groups for all $i \in I$ and $j \in \mathbb{Z}$. The map f is an $\mathcal{A}$-fibration if it is a Reedy fibration and all maps (10) are fibrations of simplicial abelian groups. An $\mathcal{A}$-cofibration is a map of simplicial presheaves of symmetric spectra which has the left lifting property with respect to all maps which are $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences.

The following is a consequence of Lemma 21, by an adjointness argument:

Lemma 23. *Suppose that the map $f : X \to Y$ is a Reedy fibration of simplicial presheaves of symmetric spectra. Then*

1) *the map f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps*

$$\Lambda^m_k \otimes VM(A) \to \Delta^m \otimes VM(A)$$

where A runs through the α-bounded approximations of the objects $A_i \wedge S^j$ in $\mathbf{Spt}(\mathcal{C})$.

2) *the map f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if f has the right lifting property with respect to all maps*

$$\partial\Delta^m \otimes VM(A) \to \Delta^m \otimes VM(A)$$

where A is any α-bounded approximation of some $A_i \wedge S^j$ in $\mathbf{Spt}(\mathcal{C})$.

Theorem 24. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}(\mathcal{C})$ is an α-bounded diagram of α-bounded, cofibrant presheaves of spectra A_i. Then with the definitions of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration and $\mathcal{A}$-cofibration given above, the category $s\mathbf{Spt}^{\Sigma}(\mathcal{C})$ of simplicial presheaves of symmetric spectra on a small site $\mathcal{C}$ satisfies the axioms for a closed model category.*

Proof. The axioms **CM1** and **CM2** are trivially verified. The class of $\mathcal{A}$-fibrations is closed under retract on account of Lemma 23, giving the non-trivial part of **CM3**.

Observe that a map $f : X \to Y$ of presheaves of symmetric spectra is a Reedy fibration if and only if it has the right lifting property with respect to the maps

$$(\partial\Delta^n \otimes VB) \cup_{(\partial\Delta^n \otimes VA)} (\Delta^n \otimes VA) \subset \Delta^n \otimes VB$$

where $A \to B$ varies over a set of α-bounded stably trivial cofibrations of $\mathbf{Spt}(\mathcal{C})$. It follows from Lemma 23 that there are factorizations

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

of a map f, where i is an $\mathcal{A}$-cofibration and p is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations.

The proof is finished by showing that every $\mathcal{A}$-cofibration $j : A \to B$ which has the left lifting property with respect to all $\mathcal{A}$-fibrations must be an $\mathcal{A}$-equivalence. This is achieved by the usual argument, as in the proof of Theorem 16. □

The forgetful functor $U : \mathbf{Spt}^{\Sigma}(\mathcal{C}) \to \mathbf{Spt}(\mathcal{C})$ preserves stable fibrations and trivial stable fibrations, and so U and its left adjoint V form a Quillen adjunction [6]. One knows further that the functor U reflects stable equivalences between fibrant objects [11, Lem. 10], and that the composite

$$X \xrightarrow{\eta} UVX \xrightarrow{Uj} U(VX)_s$$

is a stable equivalence of presheaves of spectra, where $j : VX \to VX_s$ is any choice of stably fibrant model in the category of presheaves of symmetric spectra [11, p. 154]. Then it follows by a standard argument given in [6] that the functors U and V induce an adjoint equivalence of stable homotopy categories

$$\mathrm{Ho}(\mathbf{Spt}(\mathcal{C})) \simeq \mathrm{Ho}(\mathbf{Spt}^{\Sigma}(\mathcal{C})).$$

Similar statements obtain for simplicial objects. The functor U induces a forgetful functor

$$U : s\mathbf{Spt}^{\Sigma}(\mathcal{C}) \to s\mathbf{Spt}(\mathcal{C}).$$

This functor preserves and reflects Reedy fibrations, and also preserves and reflects $\mathcal{A}$-fibrations and trivial $\mathcal{A}$-fibrations. The functor U further reflects $\mathcal{A}$-equivalences between $\mathcal{A}$-fibrant objects. Any Reedy fibrant model $j : VX \to (VX)_R$ is an $\mathcal{A}$-fibrant model for any simplicial presheaf of spectra X, so the component maps $j : VX_n \to ((VX)_R)_n$ are stably fibrant models. Finally, all composites

$$X_n \xrightarrow{\eta} UVX_n \xrightarrow{j_*} U((VX)_R)_n$$

are stable equivalences so that the map

$$X \xrightarrow{\eta} UVX \xrightarrow{j_*} U(VX)_R$$

is an $\mathcal{A}$-equivalence. This last statement holds in particular when X is $\mathcal{A}$-cofibrant, so [6, Cor. 1.3.16] implies the following:

Theorem 25. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}(\mathcal{C})$ is an α-bounded diagram of α-bounded, cofibrant presheaves of spectra A_i. Then the adjoint functors*

$$U : s\mathbf{Spt}^{\Sigma}(\mathcal{C}) \leftrightarrows s\mathbf{Spt}(\mathcal{C}) : V$$

form a Quillen equivalence between the model structures associated to the diagram $\mathcal{A}$ on the respective categories of simplicial objects, and hence induce an adjoint equivalence

$$\mathrm{Ho}_{\mathcal{A}}(s\mathbf{Spt}^{\Sigma}(\mathcal{C})) \simeq \mathrm{Ho}_{\mathcal{A}}(s\mathbf{Spt}(\mathcal{C}))$$

of the associated homotopy categories.

4 The motivic stable category

Suppose that $(Sm|_S)_{Nis}$ denotes the category of smooth schemes of finite type over a scheme S of finite dimension, and suppose that this category is equipped with the Nisnevich topology. Choose a fixed pointed simplicial presheaf T on $(Sm|_S)_{Nis}$ which is compact in the sense of [12, Sec. 2.2] — examples of such include all constant simplicial presheaves associated to pointed finite simplicial sets such as S^1, all constant simplicial presheaves represented by pointed smooth S-schemes. The collection compact objects is closed under taking homotopy cofibres, and therefore includes the Morel-Voevodsky object $\mathbb{A}^1/(\mathbb{A}^1 - 0)$.

A T-spectrum X consists of pointed simplicial presheaves X^n, $n \geq 0$ together with bonding map $T \wedge X^n \to X^{n+1}$, and maps between T-spectra $f : X \to Y$ consist of collections of pointed simplicial presheaf maps $f : X^n \to Y^n$ which preserve structure in the obvious way.

The corresponding category of T-spectra is denoted by $\mathbf{Spt}_T(Sm|_S)_{Nis}$. It is shown in [12, Th. 2.9] that this category carries a proper closed simplicial model structure whose associated homotopy category is the motivic stable category of Morel and Voevodsky. The cofibrations for that theory are defined by analogy with cofibrations of spectra: a map $i : A \to B$ is a cofibration of T-spectra if $i : A^0 \to B^0$ and all morphisms

$$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1}$$

are monomorphisms of simplicial presheaves. A morphism $f : X \to Y$ of T-spectra is said to be a stable equivalence if the induced map $f_* : Q_T \mathcal{L}X \to Q_T \mathcal{L}Y$ is a level motivic weak equivalence, where $X \to \mathcal{L}X$ is a natural level motivic fibrant model and the stabilization functor $Y \mapsto Q_T Y$ is the T-stabilization functor which is constructed by analogy with the stabilization functor for ordinary spectra by iterating a T-loop construction.

Level cofibrations, level motivic weak equivalences and level motivic fibrations have meanings analogous to those in the previous section: a level cofibration (respectively level equivalence, level fibration) is a map $f : X \to Y$ such that all component maps $f : X^n \to Y^n$ are cofibrations (respectively motivic weak equivalences, motivic fibrations) of simplicial presheaves on the smooth Nisnevich site $(Sm|_S)_{Nis}$. Recall that a motivic weak equivalence (respectively motivic fibration) is an f-weak equivalence (repectively f-fibration) of simplicial presheaves on $(Sm|_S)_{Nis}$, where we have formally inverted a map $f : * \to \mathbb{A}^1$ in the simplicial presheaf closed model structure arising from the Nisnevich topology.

The motivic stable model structure satisfies a bounded cofibration condition:

sBC: There is an infinite cardinal α such that for every diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

of level cofibrations with i a stable equivalence and A α-bounded, there is a subobject $B \subset Y$ such that $A \subset B$, the object B is α-bounded, and the inclusion $B \cap X \to B$ is a stable equivalence and a cofibration.

The proof of this result is given in [12, Sec. 2.5], but the result itself is not stated there in this form. As with the statement of the bounded cofibration condition for presheaves of spectra (and since cofibrations are preserved by pullback), the version given here is slightly stronger but has the same proof.

As in the case of presheaves of spectra, the following two results follow from the stable bounded cofibration condition **sBC** in the way that Lemma 12 and Corollary 13 follow from the bounded cofibration condition for simplicial presheaves. For Corollary 27, one has to know that an inductive limit of stable equivalences is a stable equivalence, but this is the result of a standard argument.

Lemma 26. *Suppose given level cofibrations $A \subset B \subset X$ of T-spectra such that the composite $A \to X$ is a stable equivalence and B is α-bounded. Then there is a subobject $C \subset X$ such that $B \subset C$, C is α-bounded, and the inclusion $C \subset X$ is a stable equivalence.*

Corollary 27. *Suppose given a diagram of level cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

such that A is α-bounded, and both maps are stable equivalences. Then there is an α-bounded subobject $C \subset Y$ such that the inclusions $A \subset C$ and $C \cap X \to C$ are stable equivalences.

As before, Corollary 27 give rise to a bounded approximation technique. Suppose that K is a cofibrant and α-bounded T-spectrum. Say that A is an α-bounded approximation of K if there is a sequence of level cofibrations

$$A \xrightarrow{u} B \xleftarrow{v} K$$

such that B is α-bounded, and both u and v are stable equivalences. Then we have the following analogue of Lemma 19, having formally the same proof:

Lemma 28. *Suppose given a diagram*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ K & \xrightarrow[\gamma]{} & Y \end{array}$$

of maps of T-spectra, where K is cofibrant and α-bounded, and f is a stable equivalence. Then there is an extended diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \\ {\scriptstyle v}\uparrow & \nearrow_{\gamma} & \\ K & & \end{array}$$

in which A and B are α-bounded, and the morphisms u and v are level cofibrations and stable equivalences.

The bounded cofibration condition **sBC** implies that a map $p : X \to Y$ of T-spectra is a stable fibration if and only if it has the right lifting property with respect to all α-bounded stably trivial cofibrations. Also, since trivial stable and trivial level fibrations coincide, the map p is stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all cofibrations

$$\Sigma_T^\infty Y_+[-n] \to \Sigma_T^\infty L_U \Delta_+^n[-n]$$

of shifted suspension T-spectrum objects arising from inclusions of simplicial presheaves $Y \subset L_U \Delta^n$. It follows that the motivic stable model structure is cofibrantly generated. In particular, as in ordinary stable homotopy theory, the structure has functorial factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_f} & M(f) \\ & \searrow^{f} & \downarrow{\scriptstyle p_f} \\ & & Y \end{array}$$

for arbitrary maps f, such that i_f is a cofibration and p_f is a trivial stable fibration. Write $M(A)$ for the cofibrant model for a T-spectrum X which arises from the canonical map $* \to A$ in this way, and let $p_A : M(A) \to A$ be the corresponding trivial stable fibration.

Lemma 29. *Suppose that the T-spectrum K is α-bounded and cofibrant. Suppose also that*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[i_Y]{} & \tilde{Y} \end{array}$$

is a stably fibrant model for the stable fibration p. Then q has the right lifting property with respect to $* \to K$ if and only if p has the right lifting property with respect to all cofibrations $* \to M(A)$ arising from cofibrant models for α-bounded approximations A of K.

Proof. The proof of Lemma 29 is formally the same as the proof of Lemma 20. ∎

Suppose that

$$\mathcal{A} : I \to \mathbf{Spt}_T(Sm|_S)_{Nis}, \;\; I \mapsto A_i$$

is an α-bounded diagram of α-bounded, cofibrant T-spectra. We say that map $f : X \to Y$ of simplicial T-spectra an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement $f' : X' \to Y'$ for f such that the induced maps

$$\pi_j \mathbf{hom}(A_i, X') \to \pi_j \mathbf{hom}(A_i, Y') \tag{11}$$

are weak equivalences of simplicial abelian groups for all $i \in I$ and all $j \in \mathbb{Z}$. The map f is an $\mathcal{A}$-fibration if it is a Reedy fibration and all maps (11) are fibrations of simplicial abelian groups. An $\mathcal{A}$-cofibration is a map of simplicial T-spectra which has the left lifting property with respect to all maps which are $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences.

Remark 30. It's worth pointing out once again that there are isomorphisms

$$\pi_j \mathbf{hom}(A_i, X') \cong [A_i \wedge S^j, X'] \cong [A_i \wedge S^j, X],$$

which arise from the simplicial model structure on the category of T-spectra. It follows, for example, that the map $f : X \to Y$ of simplicial T-spectra is an $\mathcal{A}$-equivalence if and only if it induces a weak equivalence of simplicial abelian groups

$$[A_i \wedge S^j, X] \to [A_i \wedge S^j, Y]$$

for all i and j. Note that only the topological suspensions are involved, and that they are necessary for the fibre sequence arguments for Lemma 6 to work. One can build in a dependence on T-suspensions by insisting, for example, that the diagram $\mathcal{A}$ is closed under the functor $A_i \mapsto A_i \wedge T$.

The following is a consequence of Lemma 29:

Lemma 31. *Suppose that the map $f : X \to Y$ is a Reedy fibration of simplicial T-spectra. Then*

1) *the map f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps*

$$\Lambda^m_k \otimes M(A) \to \Delta^m \otimes M(A)$$

where A runs through the α-bounded approximations of the objects $A_i \wedge S^j$.

2) *the map f is a A-fibration and a A-equivalence if and only if f has the right lifting property with respect to all maps*

$$\partial\Delta^m \otimes M(A) \to \Delta^m \otimes M(A)$$

where A is any α-bounded approximation of some $A_i \wedge S^j$.

We then have the following analog of Theorem 22:

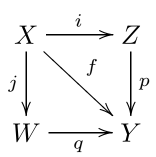
Theorem 32. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ is an α-bounded diagram of α-bounded, cofibrant T-spectra A_i. Then with the definitions of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration and $\mathcal{A}$-cofibration given above, the category $s\mathbf{Spt}_T(Sm|_S)_{Nis}$ of simplicial T-spectra satisfies the axioms for a closed model category.*

Proof. The axioms **CM1** and **CM2** are trivially verified. The class of $\mathcal{A}$-fibrations is closed under retract on account of Lemma 31, giving the non-trivial part of **CM3**.

Observe that a map $f : X \to Y$ of T-spectra is a Reedy fibration if and only if it has the right lifting property with respect to the maps

$$(\partial\Delta^n \otimes B) \cup_{(\partial\Delta^n \otimes A)} (\Delta^n \otimes A) \subset \Delta^n \otimes B$$

where $A \to B$ varies over a set of α-bounded stably trivial cofibrations. It follows from Lemma 31, that there are factorizations



of a map f, where i is an $\mathcal{A}$-cofibration and p is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations.

The proof is finished by showing that every $\mathcal{A}$-cofibration $j : A \to B$ which has the left lifting property with respect to all $\mathcal{A}$-fibrations must be an $\mathcal{A}$-equivalence, but this is proved just like the corresponding step in the proof of Theorem 16. □

Write $\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$ for the category of presheaves of symmetric T-spectra. It is shown in [12, Th. 4.15] that this category has a proper closed simplicial model structure. The fibrations for this theory, or stable fibrations, are those morphisms $p : X \to Y$ of symmetric T-spectra such that the underlying map $p_* : UX \to UY$ of T-spectra are stable fibrations in the sense described above. A stable weak equivalence $f : X_1 \to X_2$ of $\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$ is a map which induces a weak equivalence of function complexes

$$\mathbf{hom}(X_2, W) \to \mathbf{hom}(X_1, W)$$

where W is stably fibrant and injective. An injective object is a fibrant object in a model structure of level cofibrations and level weak equivalences. The function complexes are formed in the expected way.

Observe that $p : X \to Y$ is a stable fibration of symmetric T-spectra if and only if it has the right lifting property with respect to all maps $i_* : VA \to VB$ arising from α-bounded trivial stable cofibrations $i : A \to B$ of T-spectra, by application of the left adjoint $V : \mathbf{Spt}_T(Sm|_S)_{Nis} \to \mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$ to the functor $U : \mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis} \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ which forgets the symmetric structure. Recall further [12, Cor. 4.14] that a map $p : X \to Y$ of symmetric T-spectra is a stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all maps

$$F_n(A) \to F_n(L_U \Delta^r_+)$$

induced by inclusions $A \subset L_U \Delta^r$ by the left adjoint to the level n functor $X \mapsto X^n$ which takes values in pointed simplicial presheaves on the smooth Nisnevich site. It follows that the symmetric T-spectrum category $\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}$ is cofibrantly generated.

Suppose that

$$\mathcal{A} : I \to \mathbf{Spt}_T(Sm|_S)_{Nis}, \quad i \mapsto A_i,$$

is an α-bounded diagram of α-bounded, cofibrant T-spectra as above. Say that a map $f : X \to Y$ of simplicial symmetric T-spectra is an $\mathcal{A}$-equivalence if there is a Reedy fibrant replacement $f' : X' \to Y'$ for f such that the induced maps

$$\pi_j \mathbf{hom}(VA_i, X') \to \pi_j \mathbf{hom}(VA_i, Y') \tag{12}$$

are weak equivalences of simplicial abelian groups for all $i \in I$ and $j \in \mathbb{Z}$. The map f is an $\mathcal{A}$-fibration if it is a Reedy fibration and all maps (12) are fibrations of simplicial abelian groups. An $\mathcal{A}$-cofibration is a map of simplicial symmetric

T-spectra which has the left lifting property with respect to all maps which are $\mathcal{A}$-fibrations and $\mathcal{A}$-equivalences.

The following is a consequence of Lemma 29, by an adjointness argument:

Lemma 33. *Suppose that the map $f : X \to Y$ is a Reedy fibration of simplicial symmetric T-spectra. Then*

1) *the map f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all maps*

$$\Lambda^m_k \otimes VM(A) \to \Delta^m \otimes VM(A)$$

where A runs through the α-bounded approximations of the objects $A_i \wedge S^j$ in $\mathbf{Spt}_T(Sm|_S)_{Nis}$.

2) *the map f is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if f has the right lifting property with respect to all maps*

$$\partial\Delta^m \otimes VM(A) \to \Delta^m \otimes VM(A)$$

where A is any α-bounded approximation of some $A_i \wedge S^j$ in the category of T-spectra.

Theorem 34. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ is an α-bounded diagram of α-bounded, cofibrant T-spectra A_i. Then with the definitions of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration and $\mathcal{A}$-cofibration given above, the category $s\mathbf{Spt}^{\Sigma}_T(Sm|_S)_{Nis}$ of simplicial symmetric T-spectra satisfies the axioms for a closed model category.*

Proof. The axioms **CM1** and **CM2** are trivially verified. The class of $\mathcal{A}$-fibrations is closed under retract on account of Lemma 33, giving the non-trivial part of **CM3**.

Observe that a map $f : X \to Y$ of symmetric T-spectra is a Reedy fibration if and only if it has the right lifting property with respect to the maps

$$(\partial\Delta^n \otimes VB) \cup_{(\partial\Delta^n \otimes VA)} (\Delta^n \otimes VA) \subset \Delta^n \otimes VB$$

where $A \to B$ varies over a set of α-bounded stably trivial cofibrations of $\mathbf{Spt}_T(Sm|_S)_{Nis}$. It follows from Lemma 33 that there are factorizations

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

of a map f, where i is an $\mathcal{A}$-cofibration and p is an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence, q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration which has the left lifting property with respect to all $\mathcal{A}$-fibrations.

The proof is finished by showing that every $\mathcal{A}$-cofibration $j : A \to B$ which has the left lifting property with respect to all $\mathcal{A}$-fibrations must be an $\mathcal{A}$-equivalence. This is accomplished by the usual argument. □

The forgetful functor $U : \mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis} \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ preserves stable fibrations and trivial stable fibrations, and so U and its left adjoint V form a Quillen adjunction. One knows further that the functor U reflects stable equivalences between fibrant objects [12, Cor. 4.6], and that the composite

$$X \xrightarrow{\eta} UVX \xrightarrow{Uj} U(VX)_s$$

is a stable equivalence of T-spectra, where $j : VX \to VX_s$ is any choice of stably fibrant model in the category of symmetric T-spectra [12, Prop 4.30]. Then it follows that the functors U and V induce an adjoint equivalence of stable homotopy categories

$$\mathrm{Ho}(\mathbf{Spt}_T(Sm|_S)_{Nis}) \simeq \mathrm{Ho}(\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}).$$

The functor U induces a forgetful functor

$$U : s\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis} \to s\mathbf{Spt}_T(Sm|_S)_{Nis}.$$

This functor preserves and reflects Reedy fibrations, and also preserves and reflects $\mathcal{A}$-fibrations and trivial $\mathcal{A}$-fibrations. The functor U also reflects $\mathcal{A}$-equivalences between $\mathcal{A}$-fibrant objects. Any Reedy fibrant model $j : VX \to (VX)_R$ is an $\mathcal{A}$-fibrant model for any T-spectrum X, so in particular the component maps $j : VX_n \to ((VX)_R)_n$ are stably fibrant models. Then all composites

$$X_n \xrightarrow{\eta} UVX_n \xrightarrow{j_*} U((VX)_R)_n$$

are stable equivalences so that the map

$$X \xrightarrow{\eta} UVX \xrightarrow{j_*} U(VX)_R$$

is an $\mathcal{A}$-equivalence. As in the case of presheaves of spectra, we then have the following:

Theorem 35. *Suppose that $\mathcal{A} : I \to \mathbf{Spt}_T(Sm|_S)_{Nis}$ is an α-bounded diagram of α-bounded, cofibrant T-spectra A_i. Then the adjoint functors*

$$U : s\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis} \leftrightarrows s\mathbf{Spt}_T(Sm|_S)_{Nis} : V$$

form a Quillen equivalence between the model structures associated to the diagram $\mathcal{A}$ on the respective categories of simplicial objects, and hence induce an adjoint equivalence

$$\mathrm{Ho}_{\mathcal{A}}(s\mathbf{Spt}_T^{\Sigma}(Sm|_S)_{Nis}) \simeq \mathrm{Ho}_{\mathcal{A}}(s\mathbf{Spt}_T(Sm|_S)_{Nis})$$

of the associated homotopy categories.

References

[1] W.G. Dwyer, D.M. Kan and C.R. Stover, *An E_2 model category structure for pointed simplicial spaces*, J. Pure App. Algebra **90** (1993), 137–152.

[2] W.G. Dwyer, D.M. Kan and C.R. Stover, *The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space X*, J. Pure App. Algebra **103** (1995), 167–188.

[3] P.G. Goerss and M.J. Hopkins, *Resolutions in model categories*, Preprint.

[4] P.G. Goerss and J.F. Jardine, *Localization theories for simplicial presheaves*, Can. J. Math. **50(5)** (1998), 1048–1089.

[5] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, **174**, Birkhäuser, Basel-Boston-Berlin (1999).

[6] M. Hovey, *Model Categories*, Math. Surveys and Monographs **63**, AMS, Providence (1999).

[7] M. Hovey, B. Shipley and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.

[8] J.F. Jardine, *Simplicial presheaves*, J. Pure Applied Algebra **47** (1987), 35–87.

[9] J.F. Jardine, *Stable homotopy of simplicial presheaves*, Can. J. Math. **39** (1987), 733–747.

[10] J.F. Jardine, *Generalized Etale Cohomology Theories*, Progress in Math. **146**, Birkhäuser, Basel-Boston-Berlin (1997).

[11] J.F. Jardine, *Presheaves of symmetric spectra*, J. Pure App. Alg. **150** (2000) 137–154.

[12] J.F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552.

[13] F. Morel and V. Voevodsky, $\mathbb{A}^1$ *homotopy theory of schemes*, Publ. Math. IHES **90** (1999), 45–143.