Algebraic Homotopy Theory, Groups, and K-Theory

John Frederick Jardine

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The University of British Columbia

Abstract

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Let $\mathbf{M}_k$ be the category of algebras over a unique factorization domain k, and let ind $-\mathbf{Aff}_k$ denote the category of pro-representable functions from $\mathbf{M}_k$ to the category $\mathbf{E}$ of sets. It is shown that ind $-\mathbf{Aff}_k$ is a closed model category in such a way that its associated homotopy category Ho(ind $-\mathbf{Aff}_k$) is equivalent to the homotopy category Ho($\mathbf{S}$) which comes from the category $\mathbf{S}$ of simplicial sets. The equivalence is induced by functors $S_k : \text{ind} -\mathbf{Aff}_k \rightarrow \mathbf{S}$ and $R_k : \mathbf{S} \rightarrow \text{ind} -\mathbf{Aff}_k$.

In an effort to determine what is measured by the homotopy groups $\pi_i(X) = \pi_i(S_k X)$ of X in ind $-\mathbf{Aff}_k$ in the case where k is an algebraically closed field, some homotopy groups of affine reduced algebraic groups G over k are computed. It is shown that, if G is connected, then $\pi_0(G) = *$ if and only if the group $G(k)$ of k-rational points of G is generated by unipotents. A fibration theory is developed for homomorphisms of algebraic groups which are surjective on rational points which allows the computation of the homotopy groups of the universal covering groups of the simple algebraic subgroups of the associated semi-simple group $G/R(G)$, where $R(G)$ is the solvable radical of G.

The homotopy groups of simple Chevalley groups over almost all fields k are studied. It is shown that the homotopy groups of the special linear groups Sl_n and of the symplectic groups Sp_{2m} converge, respectively, to the K-theory and L-theory of the underlying field k. It is shown that there are isomorphisms $\pi_1(S1_n) \cong H_2(Sl_n(k);\mathbb{Z}) \cong K_2(k)$ for $n \geq 3$ and almost all fields k, and $\pi_1(Sp_{2m}) \cong H_2(Sp_{2m}(k);\mathbb{Z}) \cong {}_{-1}L(k)$ for $m \geq 1$ and almost all fields k of characteristic $\neq 2$, where $\mathbb{Z}$ denotes the ring of integers. It is also shown that $\pi_1(Sp_{2m}) \cong H_2(Sp_{2m}(k);\mathbb{Z}) \cong K_2(k)$ if k is algebraically closed of arbitrary characteristic. A spectral sequence for the homology of the classifying space of a simplicial group is used for all of these calculations.

Thesis Supervisor: Dr. Roy R. Douglas

Contents

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Introduction

In an article [24] which appeared in Topology in 1977, Kan and Miller showed that, if k is a unique factorization domain, then the homotopy type of a finite simplicial set K can be recovered from its k-algebra A^0K of Sullivan-de Rham O-forms. A^0K is the evaluation at K of a contravariant functor $A^0 : \mathbf{S} \to \mathbf{M}_k$ from the simplicial set category $\mathbf{S}$ to the category $\mathbf{M}_k$ of algebras over k. For a simplicial set X, A^0X is defined by $A^0X = \mathbf{S}(X, k_*)$, the simplicial set homomorphisms from X to the simplicial k-algebra k_*, whose algebra of n-simplicies is given by

$$k_n = k[x_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i - 1)$$

with faces and degeneracies induced by

$$d_i x_j = \begin{cases} x_j & j < i \\ 0 & j = i \\ x_{j-1} & j > i \end{cases}$$

and

$$s_i x_j = \begin{cases} x_j & j < 1 \\ x_i + x_{i+1} & j = i \\ x_{j+1} & j > i \end{cases}$$

respectively. There is a contravariant functor $F^0 : \mathbf{M}_k \to \mathbf{S}$, which is adjoint to A^0 in the sense that there is an isomorphism

$$\mathbf{S}(S, F^0A) \cong \mathbf{M}_k(A, A^0X)$$

which is natural in both variables. For a k-algebra A, F^0A is defined by specifying its set F^0A_n of n-simplices to be the set of k-algebra homomorphisms $\mathbf{M}_k(A, k_n)$. What Kan and Miller proved in [24] is that the canonical map $\eta_K : K \to F^0A^0K$ is a weak homotopy equivalence if K is a finite simplicial set.

Perhaps the most interesting aspect of this result is that it suggests that there is a homotopy theory for algebras over a unique factorization domain k. By this I mean that one would hope to find a closed model structure in the sense of Quillen [37, 39] on the category $\mathbf{M}_k$ in such a way that the adjoint functors F^0 and A^0 would induce an equivalence of its associated homotopy category with the homotopy category $\mathrm{Ho}(\mathbf{S})$ which comes from simplicial sets. It turns out that set theoretic difficulties arising from the construction of products of algebras preclude the existence of such an equivalence of categories. The adjunction homomorphism $\eta_X : X \to F^0A^0X$ is not a weak equivalence in general, even if k is a finite field. A proof of this fact appears in [23].

The problem referred to above can be avoided by passing to the pro-category $\mathrm{pro}-\mathbf{M}_k$, whose objects consist of all small left filtered diagrams in $\mathbf{M}_k$. In [23] it was shown that, if k is a unique factorization domain, then $\mathrm{pro}-\mathbf{M}_k$ is a closed model category. Moreover, there are contravariant functors $\hat{A} : \mathbf{S} \to \mathrm{pro}-\mathbf{M}_k$ and $\hat{F} : \mathrm{pro}-\mathbf{M}_k \to \mathbf{S}$, which are adjoint and induce an equivalence of the associated homotopy categories $\mathrm{Ho}(\mathbf{S})$ and $\mathrm{Ho}(\mathrm{pro}-\mathbf{M}_k)$. $\hat{A}$ and $\hat{F}$ are closely related to A^0 and F^0; there are canonical isomorphisms $\hat{A}K \cong A^0K$ and $\hat{F}A \cong F^0A$ for finite simplicial sets k and k-algebras A respectively, and the natural map $\eta_X : X \to \hat{F}\hat{A}X$ is a weak equivalence for all simplicial sets X in a way which generalizes the result of Kan and Miller. The proofs of these results are reproduced here; they are the subject of Chapter 1.

The category of pro-representable functors from $\mathbf{M}_k$ to the set category $\mathbf{E}$ will be denoted by $\mathrm{ind}-\mathbf{Aff}_k$. The results of Chapter 1 imply that there are covariant functors $R_k : \mathbf{S} \to \mathrm{ind}-\mathbf{Aff}_k$, $S_k : \mathrm{ind}-\mathbf{Aff}_k \to \mathbf{S}$ and a closed model structure on $\mathrm{ind}-\mathbf{Aff}_k$ in such a way that R_k and S_k induce an equivalence on $\mathrm{Ho}(\mathbf{S})$ with $\mathrm{Ho}(\mathrm{ind}-\mathbf{Aff}_k)$, and this for any unique factorization domain k. For $X \in \mathbf{S}$, $R_k(X)$ is the functor which is represented by $\hat{A}X$. In view of the fact that the inclusion $k[x_1, \dots, x_n] \subset k[x_0, \dots, x_n]$ induces an isomorphism $k[x_1, \dots, x_n] \cong k_n$ for each $n \geq 0$, the n-simplices of $S_k(T)$ for $T \in \mathrm{ind}-\mathbf{Aff}_k$ are specified by

$$S_k(T)_n = \mathrm{ind}-\mathbf{Aff}_k(\mathbb{A}^n_k, T),$$

where $\mathbb{A}^n_k$ is the representable functor $\mathbf{M}_k(k[x_1, \dots, x_n],\ )$. We shall also write $\mathbb{A}^n_k = \mathrm{Spec}_k(k[x_1, \dots, x_n])$, in keeping with the usual definition of the contravariant functor $\mathrm{Spec}_k : \mathbf{M}_k \to \mathrm{ind}-\mathbf{Aff_k}$, which assigns to each k-algebra A the representable functor $\mathrm{Spec}_k(A) = \mathbf{M}_k(A,\ )$. Such functors are called affine k-schemes. Applying the functor Spec_k to the simplicial k-algebra k_* gives a cosimplicial k-scheme, which is denoted by $\mathbb{A}_k$. As noted previously, each $\mathbb{A}^n_k$ is isomorphic to the hyperplane $\mathrm{Spec}_k(k[x_0, \dots, x_n]/(\sum_{i=0}^n (x_i - 1))$ in $\mathrm{Spec}_k(k[x_0, \dots, x_n])$, and one can check that the cofaces and codegeneracies of $\mathbb{A}_k$ are defined in the same way as those of the usual cosimplicial space $\boldsymbol{\Delta}$ which is gotten from the standard n-simplices Δ^n in barycentric co-ordinates. For this reason, S_k is called the *algebraic singular functor*, and R_k is called the *algebraic realization functor*, with respect to k.

These constructions appear to yield a natural way of defining homotopy theory in the affine algebraic geometric setting of $\mathrm{ind}-\mathbf{Aff}_k$ over a large class of bases which includes the integers. (There are other constructions of homotopy theories for schemes; see [2], for example.) How natural is it? In particular, what is measured by the homotopy groups of the spaces S_kT arising from $T \in$

ind $-\mathbf{Aff}_k$? The rest of this thesis consists of an attempt to answer this question in the case where T is a reduced affine scheme of finite type over an algebraically closed field k.

Let me hasten to remark, however, that I don not yet know any geometric criteria for an arbitrary affine variety X to be even path-connected in the sense that $\pi_0(S_k X) = *$. Perversely, path-connectedness is not a local property, so that the problem of finding such criteria seems to be very difficult in general. But answers begin to emerge if one assumes that X has more structure. In fact, one can use classical algebraic group theory to show that an algebraic group G over k of arbitrary characteristic, which is connected in the usual sense, is path-connected if and only if its group $G(k)$ of k-rational points is generated by unipotent elements. This is the subject of Chapter 2.

Because $S_k G$ is a simplicial group and hence a Kan complex when G is an algebraic group, G is path-connected if for any rational point $x \in G(k)$ there is a k-scheme morphism $\omega : \mathbb{A}^1_k \to G$ with $\omega(0) = e$ and $\omega(1) = x$, where 0 and 1 lie in $\mathbb{A}^1_k = k$. If the characteristic of the underlying field k is 0 and x is a unipotent element of the group $Gl_n(k)$ of rational points of the general linear group Gl_n, then a "path" $\omega : \mathbb{A}^1_k \to Gl_n$ from e to x is defined for $s \in \mathbb{A}^1_k(k)$ by $\omega(k)(s) = exp(s \log(x))$. This observation was the starting point for the results of Chapter 2. No such result may be used, however, in the positive characteristic case. There, one must use a well-known result which says that the underlying k-scheme of a connected unipotent group U over k is isomorphic to an affine space $\mathbb{A}^n_k$, where n is the dimension of U.

The most efficient proof of this result uses the theory of fpqc torseurs in the sense of Demazure and Gabriel [13]. These are defined for the fpqc topology analogously to the notion of principal fibration. For example, any homomorphism $\pi : G \to H$ of algebraic groups over k which is surjective on rational points is a K-torseur over H, where K is the group-scheme kernel of π. The pullback $p : X \times_H G \to S$ of π over any k-scheme morphism $f : X \to H$ is a K-torseur over X, and f lifts to G if and only if the induced K-torseur p is trivial in the sense that $X \times_H G$ is equivariantly isomorphic to $X \times K$ over X, where K is the group-scheme kernel of π.

The utility of this idea lies in the fact that such an algebraic group homomorphism π is a fibration in the sense of Chapter 1 if and only if every k-scheme morphism $\mathbb{A}^n_k \to H$ lifts to G, and this for all $n \geq 0$. Chapter 3 is devoted to identifying several different fibrations of algebraic groups. The long exact sequences which result allow one to represent the homotopy group of a path-connected group G over k of arbitrary characteristic in terms of the homotopy groups of Chevalley groups over k which are of universal type.

The special linear group Sl_{n+1} is universal of type A_n for all $n \geq 1$. It is easy to see, using the determinant map, that Sl_{n+1} is the path component of the identity in Gl_{n+1}, and that $\pi_0(Gl_{n+1}) = k^*$, the group of non-zero elements of k. It follows that the inductive algebraic group Sl is the path component of the identity $e \in Gl(k)$ in the inductive group Gl, and $\pi_0(Gl) = k^*$. But the homotopy groups of Gl are defined to be the homotopy groups of the simplicial group $Gl(k_*)$ which is gotten by applying the general linear group functor to the simplicial k-algebra k_*, and it has been known for some time [17] that there is an isomorphism

$$\pi_i(Gl(k_*)) \cong K_{i+1(k)} \text{ for } i \geq 0, \tag{0.0.1}$$

where the groups $K_{i+1}(k)$, $i \geq 0$, are Quillen's algebraic K-groups of the underlying field k.

A new proof of the isomorphism (0.0.1) is given in Chapter 4. The main tools which are used for this proof consist of Quillen's homotopy property for the K-theory of a regular Noetherian ring, and a spectral sequence for the homology of the classifying space of a simplicial group. This spectral sequence is also used, together with certain results of Matsumoto, to show that there are isomorphisms

$$\pi_1(Sl_{n+1}) \cong H_2(Sl_{n+1}(k); \mathbb{Z}) \cong K_2(k) \text{ for } n \geq 1, \text{ and} \tag{0.0.2}$$

$$\pi_1(Sp_{2m}) \cong H_2(Sp_{2m}(k); \mathbb{Z}) \cong K_2(k) \text{ for } m \geq 1, \tag{0.0.3}$$

where $Sp_{2m}(k)$ is the group of k-rational points of the symplectic group Sp_{2m} and $\mathbb{Z}$ denotes the integers. Such isomorphisms also hold in some cases where k is not algebraically closed. In particular, one has

$$\pi_1(Sl_{n+1}) \cong H_2(Sl_{n+1}(k); \mathbb{Z}) \cong K_2(k) \text{ for } n \geq 2, \tag{0.0.4}$$

for Sl_{n+1} and $\pi_1(Sp_{2m})$ defined over almost all fields k, and

$$\pi_1(Sp_{2m}) \cong H_2(Sp_{2m}(k); \mathbb{Z}) \cong {}_{-1}L_2(k) \text{ for } m \geq 1, \tag{0.0.5}$$

for Sp_{2m} and $\pi_1(Sp_{2m})$ defined over almost all fields k of characteristic not equal to 2. ${}_{-1}L_2(k)$ is one of the Karoubi L-groups [25] of the field k; it coincides with $K_2(k)$ if k is algebraically closed. Jan Strooker has recently informed me that the isomorphism (0.0.4) can also be obtained from a result of Krusemeyer [27] by destabilizing a result of Rector [43] which relates the Karubi-Villamayor K-groups of the field k to the groups $\pi_i(Gl(k_*))$.

It is also shown in Chapter 4 that certain results of Bass and Tate [4] on K_2 of a number field may be used to show that K_2 of an algebraically closed field k vanishes when the Kroenecker $\delta(k)$ of k satisfies $\delta(k) \leq 1$, and is a non-trivial uniquely divisible abelian group otherwise. Some consequences for the homotopy groups of arbitrary Chevalley groups of type A_n and C_m over k are also discussed.

It seems that much more can be said in the way of computing homotopy groups of Chevalley groups. For example, I conjecture that, if G_Φ is a Chevalley group over an algebraically closed field k which is universal for an indecomposable root system Φ, then there is an isomorphism $\pi_1(G_\Phi) \cong H_2(G_\Phi(k); \mathbb{Z})$ which generalizes the isomorphisms (0.0.2) and (0.0.3) above. Beyond this, one would like to know to what extent the homotopy groups of an arbitrary Chevalley group G_Φ of universal type over k coincide with the K-theory of k when k is assumed to be algebraically closed. One also hopes for periodicity results for the groups $\pi_1(Sl_{n+1})$ which would reflect the apparent periodicity in Quillen's calculation of the K-theory of the algebraic closure of a finite field.

Chapter 1

Algebraic homotopy theory

1.1 Preliminary Lemmas

As noted in the introduction, this chapter contains a reproduction of those results of [23] which are relevant to the present work. We begin by establishing some terminology. This exposition is not self-contained however; if necessary, the reader should consult the Appendix of [2] for more details.

Let k be a unique factorization domain, and let $\mathbf{M}_k$ denote the category of k-algebras. Recall that a *pro-object* in $\mathbf{M}_k$ (ie. an object in the category pro $-\mathbf{M}_k$) is a contravariant functor $T : \mathbf{I} \to \mathbf{M}_k$, where $\mathbf{I}$ is a small filtered category (see [2, p. 154]). If $S : \mathbf{J} \to \mathbf{M_k}$ is another pro-object then a *pro-map* $\phi : T \to S$ is defined to be an element of the set

$$\varprojlim_j \varinjlim_i \mathbf{M}_k(T_i, S_j)$$

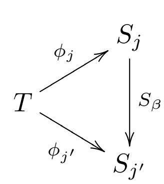
where the limits are taken over $j \in \mathbf{J}$ and $i \in \mathbf{I}$, and $\mathbf{M}_k(T_i, S_j)$ is the set of k-algebra morphisms from T_i to S_j. $\mathbf{M}_k$ is a full subcategory of pro $-\mathbf{M}_k$ in such a way that, for $A \in \mathbf{M}_k$,

$$\text{pro} -\mathbf{M}_k(T, A) = \varinjlim_i \mathbf{M}_k(T_i, A).$$

This is a set of equivalence classes of homomorphisms $\theta : T_i \to A$; the class that θ represents is denoted by $[\theta]$. Thus, a pro-map $\phi : T \to S$ can be thought of as a collection of simpler kinds of pro-maps

$$\phi_j : T \longrightarrow S_j, \quad j \in \mathbf{J},$$

such that for each $\beta : j$‘ $\to j$ in $\mathbf{J}$, the diagram



commutes in the sense that $\phi_{j'} = [S_\beta \cdot \tau]$, where τ represents ϕ_j. This information is summarized by saying that a collection of maps

$$\phi_j : T_{i(j)} \longrightarrow S_j, \quad j \in \mathbf{J},$$

represents the pro-map ϕ if (ambiguously) $[\phi_j] = \phi_j$ for every $j \in \mathbf{j}$. With this terminology, it is convenient to say, given another pro-object $U : \mathbf{K} \to \mathbf{M}_k$ and a collection of maps

$$\psi_k : S_{j(k)} \longrightarrow U_k, \quad k \in \mathbf{K},$$

representing the pro-map $\psi : S \to U$, that the collection

$$T_i(j(k)) \xrightarrow{\phi_{j(k)}} S_{j(k)} \xrightarrow{\psi_k} U_k, \ k \in \mathbf{K},$$

represents the composite $\psi \cdot \phi : T \to U$ in pro $-\mathbf{M}_k$.

Now we describe a contravariant functor $\hat{A} : \mathbf{S} \to \text{pro} -\mathbf{M}_k$, where $\mathbf{S}$ denotes the category of simplicial sets. Recall that the (contravariant) 0-forms functor $A^0 : \mathbf{S} \to \mathbf{M}_k$ is defined, for $X \in \mathbf{S}$, by $A^0 X = \mathbf{S}(X, k_*)$, where k_* is the simplicial k-algebra of the Introduction. Now for $X \in \mathbf{S}$ let $\mathbf{Fin}$(X) be the small filtered category which has all finite subcomplexes K of X as objects and all inclusions between them as morphisms. Then a contravariant functor $\hat{A}X : \mathbf{Fin}(X) \to \mathbf{M}_k$ is defined on morphisms $i : K \to L$ by

$$\hat{A}X(i) = A^0(i) : A^0 L \longrightarrow A^0 K.$$

If $f : X \to Y$ is a simplicial map and K is a finite simplicial complex of X then $f(K)$ is a finite subcomplex of Y. Let $f|_K : K \to f(K)$ be the restriction of f to K. Then it is easy to see that the collection

$$A^0(f|_k) : A^0(f(K)) \to A^0 K, \ K \in \mathbf{Fin}(X),$$

represents a pro-map $\hat{A}f : \hat{A}Y \to \hat{A}X$, and that the assignment $f \to \hat{A}f$ determines a contravariant functor $\hat{A} : \mathbf{S} \to \text{pro} -\mathbf{M}_k$. It is worth noting that $\hat{A}$ is the right Kan extension of the restriction of the composition

$$\mathbf{S} \xrightarrow{A^0} \mathbf{M}_k \subset \text{pro} -\mathbf{M}_k$$

to the full subcategory $\mathbf{Fin}$ of finite simplicial sets, along the inclusion of $\mathbf{Fin}$ in $\mathbf{S}$.

Recall that the contravariant functor $F^0 : \mathbf{M}_k \to \mathbf{S}$ is defined for $A \in \mathbf{M}_k$ by requiring the n-simplices $F^0 \mathbb{A}_n$ of $F^0 A$ to be the set of homomorphisms $\mathbf{M}_k(A, k_n)$. Similarly, a contravariant functor $\hat{F} : \text{pro} -\mathbf{M}_k \to \mathbf{S}$ is defined for $T \in \text{pro} -\mathbf{M}_k$ by specifying that

$$\hat{F}T_n = \text{pro} -\mathbf{M}_k(T, k_n).$$

Obviously $\hat{F}A = F^0 A$ for $A \in \mathbf{M}_k$, and it is straightforward to show

Lemma 1.1.1. *There is a natural map $\zeta : A^0X \to \hat{A}X$ for $X \in \mathbf{S}$, with $\zeta_K = A^0(i) : A^0X \to A^0K$ for $K \in \mathbf{Fin}(X)$, where i is the inclusion of K in X. ζ is an isomorphism if X is finite; in this case ζ^{-1} is represented by 1_{A^0X}.*

There is a natural map

$$\psi : \text{pro}-\mathbf{M}_k(T, \hat{A}X) \longrightarrow \mathbf{S}(X, \hat{F}T),$$

such that, for $g : T \to \hat{A}X$ and $x \in S_n,\ n \geq 0$, $\psi g(x)$ is the composition

$$T \xrightarrow{g} \hat{A}X \xrightarrow{\hat{A}i_x} A\Delta_n \xrightarrow{\zeta^{-1}} A^0\Delta^n \xrightarrow{j} k_n,$$

where $i_x : \Delta^n \to X$ is the classifying map for x and j is the isomorphism $A^0\delta^n = \mathbf{S}(\Delta^n, k_*) \to k_n$ defined by $j(f) = f(i_n)$, where $i_n \in \Delta^n_n$ is the standard n-simples. In fact, we have

Proposition 1.1.2. *ψ is a natural bijection, so that $\hat{A}$ and $\hat{F}$ are adjoint on the right.*

Proof. By definition, if $X \in \mathbf{S}$ and $T : \mathbf{I} \to \mathbf{M}_k$ is a pro-object, then

$$\text{pro}-\mathbf{M}_k(T, \hat{A}X) = \varprojlim_K \varinjlim_i \mathbf{M}_k(T_i, A^0K),$$

where the limits are taken over $K \in \mathbf{Fin}(X)$ and $i \in \mathbf{I}$. But there is a natural isomorphism

$$\varprojlim_K \varinjlim_i \mathbf{M}_k(T_i, A^0K) \xrightarrow{\phi} \cong \varprojlim_K \varinjlim_i \mathbf{S}(K, F^0T_i),$$

in view of the fact that A^0 and F^0 are adjoint on the right, and a natural map

$$\varprojlim_K \varinjlim_i \mathbf{S}(K, F^0T_i) \xrightarrow{c} \mathbf{S}(X, \hat{F}T)$$

which is gotten by taking colimits. The map c is an isomorphism, since

$$\varinjlim_i \mathbf{S}(K, F^0T_i) \cong \mathbf{S}(K, \hat{F}T)$$

for each finite $K \in \mathbf{S}$. It is an exercise to show that ψ is the composition of the isomorphisms c and ϕ. □

The key point in all that follows is

Lemma 1.1.3. *Consider the situation*

$$\begin{array}{l} A \\ \cup \\ I \twoheadrightarrow B \end{array}$$

where A and B are k-algebras, B is a unique factorization domain, I is a non-zero ideal of A, and h is a non-zero multiplicative k-module homomorphism. Then there is a unique k-algebra homomorphism $h_* : A \to B$ extending h.

Proof. Choose $u \in I$ such that $h(u) \neq 0$. Then for all $x \in A$ there is a $\beta_x \in B$ such that

$$h(xu) = \beta_x h(u)$$

In effect, if $h(xu) = 0$ then obviously $\beta_x = 0$, and if $h(xu) \neq 0$ then the equation

$$h(xu)^k = h(x^k u)h(u)^{k-1}, \quad k \geq 2,$$

give the result, for if p is a prime and $p^m | h(xu)$, $p^{m+1} | h(xu)$ and $p^r | h(u)$, then $(k-1)r \leq km$ for all $k \geq 2$. Moreover, since B is an integral domain, β_x is unique. Observe that if there is another $v \in I$ such that $h(v) \neq 0$, with corresponding identity

$$h(xv) = \gamma_x h(v),$$

then

$$\beta_x h(u)h(v) = h(xuv) = \gamma_x h(v)h(u),$$

and so $\beta_x = \gamma_x$. Now define $h_* : A \to B$ by $h_*(x) = \beta_x$. □

Some technical lemmas for pro $-\mathbf{M}_k$ will now be listed.

Lemma 1.1.4. *Let* $\mathbf{I}$ *be a small filtered category and consider the pullback diagram*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ W & \xrightarrow[\gamma]{} & Z \end{array}$$

in the category $\mathbf{M}_{\mathbf{k}}^{\mathbf{I}}$ *of contravariant functors* $\mathbf{I} \to \mathbf{M}_{\mathbf{k}}$*, where* $\delta_i : Y_i \to Z_i$ *is surjective for every* $i \in \mathbf{I}$*. Let* $B \in \mathbf{M}_k$ *be a unique factorization domain. Then the diagram*

$$\begin{array}{ccc} \text{pro} - \mathbf{M}_k(Z, B) & \xrightarrow{\gamma^*} & \text{pro} - \mathbf{M}_k(W, B) \\ {\scriptstyle\delta^*}\downarrow & & \downarrow{\scriptstyle\beta^*} \\ \text{pro} - \mathbf{M}_k(Y, B) & \xrightarrow[\alpha^*]{} & \text{pro} - \mathbf{M}_k(X, B) \end{array}$$

is a pushout of sets.

Proof. Suppose that there is a commutative diagram of sets

$$\begin{array}{ccc} \text{pro} - \mathbf{M}_k(X, B) & \xrightarrow{\gamma^*} & \text{pro} - \mathbf{M}_k(W, B) \\ {\scriptstyle\delta^*}\downarrow & & \downarrow{\scriptstyle g} \\ \text{pro} - \mathbf{M}_k(Y, B) & \xrightarrow[f]{\qquad\qquad} & E \end{array} .$$

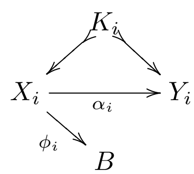
Take a pro-map $\phi : X \to B$ and let

$$K_i = ker\{\delta_i : Y_i \twoheadrightarrow Z_i\} = ker\{\beta_i : X_i \twoheadrightarrow W_i\}.$$

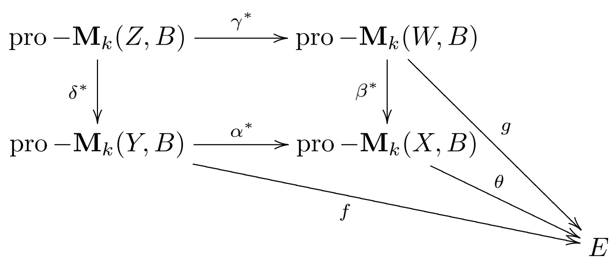
Then:

1. If there is a representative $\phi_i : X_i \to B$ of ϕ such that $\phi(K_i) = 0$, then there is a unique pro-map $\psi : W \to B$ such that $\psi B = \phi$.
2. If there is no representative ϕ_i of ϕ which kills K_i, then there is a unique pro-map $\eta : Y \to B$ such that $\eta\alpha = \phi$.

The uniqueness of ψ in (1)follows from the fact that β is an epi of pro $-\mathbf{M}_k$; this will be proven in Lemma 1.1.7. To see (2), consider the diagram



By Lemma 1.1.3, there is a unique map $\eta_i : Y_i \to B$ such that $\eta_i\alpha_i = \phi_i$. Lemma 1.1.3 also guarantees that $\eta = [\eta_i]$ is independent of the choice of representative ϕ_i of ϕ, and that η is the unique pro-map such that $\eta\alpha = \phi$. Now, in order to get a commutative diagram



we are obliged to define

$$\theta(\phi) = \begin{cases} g(\psi) \text{ if } \phi \text{ satisfies (1), and} \\ f(\eta) \text{ if } \phi \text{ satisfies (2).} \end{cases}$$

A straightforward case check shows that $\theta\alpha^* = f$ and $\theta\beta^* = g$, and the Lemma is proved. $\square$

Corollary 1.1.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ W & \xrightarrow[\gamma]{} & Z \end{array}$$

be a pullback diagram of k*-algebras in which* δ *is surjective and let the* k*-algebra* B *be a unique factorization domain. Then the following diagram is a pushout of sets:*

$$\begin{array}{ccc} \mathbf{M}_k(Z,B) & \xrightarrow{\gamma^*} & \mathbf{M}_k(W,B) \\ {\scriptstyle\delta^*}\downarrow & & \downarrow{\scriptstyle\beta^*} \\ \mathbf{M}_k(Y,B) & \xrightarrow[\alpha^*]{} & \mathbf{M}_k(X,B) \end{array}$$

Proposition 1.1.6 (Kan, Miller [24]). *The natural map $\eta_K : K \to F^0A^0K$ is a weak equivalence if K is a finite simplicial set.*

Proof. In effect, each finite complex K is built up via a finite sequence of pushouts

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & L \\ {\scriptstyle i}\downarrow & & \downarrow \\ \Delta^n & \longrightarrow & L' \end{array}$$

where L and L' are finite, and $\partial\Delta^n$ is the subcomplex of Δ^n which is generated by the (n-1)-simplices $d_i i_n$, $0 \leq i \leq n$. By Corollary 1.1.5, the induced diagram

$$\begin{array}{ccc} F^0A^0(\partial\Delta^n) & \longrightarrow & F^0A^0L \\ {\scriptstyle F^0A^0i}\downarrow & & \downarrow \\ F^0A^0(\delta^n) & \longrightarrow & F^0A^0L' \end{array}$$

is also a pushout of simplicial sets. Moreover, F^0A^0i is an inclusion of simplicial sets, hence a cofibration. Now, $\eta_{\Delta^n} : \Delta^n \longrightarrow F^0A^0\Delta^n$ is a weak equivalence by the Extension Lemma of [5, p. 3]. Thus, if we assume that $\eta_{\partial\Delta^n}$ and η_L are weak homotopy equivalences, then the Brown Glueing Lemma of [19, p.80] implies that $\eta_{L'}$ is a weak homotopy equivalence as well. An induction on dimension finishes the proof. □

The missing link in the proof of Lemma 1.1.4 was

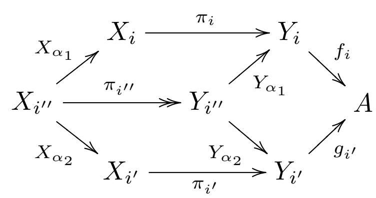
Lemma 1.1.7. *(i) Take $X, Y : \mathbf{I} \to \mathbf{M}_k$ in* pro $-\mathbf{M}_k$ *and let $\pi : X \to Y$ be a natural transformation such that $\pi_i : X_i \to Y_i$ is surjective for every $i \in \mathbf{I}$. Then π represents an epimorphism of* pro $-\mathbf{M}_k$.

(ii) In every pullback diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow[\beta]{} & Y \end{array}$$

in pro $-\mathbf{M}_k$ *with π as above, p is epi.*

Proof. (i) Take $f, g : Y \to A$, $A \in \mathbf{M}_k$, with$f\pi = g\pi$, and let $f_i : Y_i \to A$ and $g_i : Y_{i'} \to A$ represent f and g respectively. Then there is a commutative diagram



so that $f_i Y_{\alpha_1} = g_{i'} Y_{\alpha_2}$ and $f = g$. The general case follows easily.

(ii) Suppose that T is the pro-object $T : \mathbf{J} \to \mathbf{M_k}$. By Section 3 of the Appendix of [2], there is a small filtered category $\mathbf{M}_\beta$, cofinal functors $\phi : \mathbf{M}_\beta \to \mathbf{J}$ and $\psi : \mathbf{M}_\beta \to \mathbf{I}$, a natural transformation $\beta_* : T\phi \to Y\psi$ and a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\beta} & Y \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ T\phi & \xrightarrow[\beta_*]{} & Y\psi \end{array}$$

in pro $-\mathbf{M}_k$, where the vertical arrows are canonical isomorphisms which are represented by identity maps. Moreover, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ X\psi & \xrightarrow[\pi\psi]{} & Y\psi \end{array}$$

commutes in pro $-\mathbf{M}_k$, where $\pi\psi$ is represented by the obvious natural transformation. Thus, by [2, A.4.4], our pullback is isomorphic to the pullback

$$\begin{array}{ccc} S' & \xrightarrow{\alpha'} & X\psi \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi\psi} \\ T\phi & \xrightarrow[\beta_*]{} & Y\psi \end{array}$$

of $\mathbf{M}_{\mathbf{k}}^{\mathbf{M}_\beta} \cdot \pi\psi_m : X\psi_m \to Y\psi_m$ is surjective for every $m \in \mathbf{M}_\beta$, so that p' is an epi of pro $-\mathbf{M}_k$ by (i). □

It is necessary at this point to recall briefly the construction of filtered inverse limits in pro $-\mathbf{M}_k$ from [2, A.4.4]. Let $\mathbf{J}$ be a small filtered category, and consider a contravariant functor $T : \mathbf{J} \to$ pro $-\mathbf{M}_k$. In particular, we have $T(j) : \mathbf{I}(j) \to \mathbf{M}_k$ for every $j \in \mathbf{J}$. Let $\mathbf{K}$ be a category having as objects all pairs (j, i) with $j \in \mathbf{J}$ and $i \in \mathbf{I}(j)$, and such that a morphism $(\alpha, \phi) : (j, i) \to (j', i')$ consists of an arrow $\alpha : j \to j'$ of $\mathbf{J}$, together with a k-algebra map $\phi : T(j')_{i'} \to T(j)_i$ representing $T(\alpha)_i$. $\mathbf{K}$ is small and filtered. Let $L : \mathbf{K} \to \mathbf{M_k}$ be the pro-object which is defined on morphisms by $L(\alpha, \phi) = \phi$. The pro-maps $\pi_j : L \to T(j)$ with $(\pi_j)_i$ represented by $1_{T(j)_i}$ form a limiting cone in pro $-\mathbf{M}_k$.

As an example, take $A_i \in \mathbf{M}_k$, where the i ranges over some index set X. The product of the A_i in pro $-\mathbf{M}_k$, denoted by $\hat{\prod}_{i \in X} A_i$, is the functor $P : \mathbf{Fin}(X) \to \mathbf{M}_k$, where $\mathbf{Fin}(X)$ denotes the finite subsets of X considered as a small filtered category, and with

$$P(K) = \prod_{i \in K} A_i, \ \mathit{for}\ K \in \mathbf{Fin}(X),$$

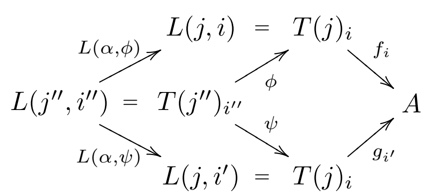
this finite product being taken in $\mathbf{M}_k$.

Lemma 1.1.8. *For $T : \mathbf{J} \to \text{pro}-\mathbf{M}_k$ as above, if $T(\alpha) : T(j') \to T(j)$ is an epi of* pro $-\mathbf{M}_k$ *for every $\alpha : j \to j'$ in $\mathbf{J}$, then the maps*

$$\pi_j : L \longrightarrow T(j), \quad j \in \mathbf{J},$$

are also epimorphisms of pro $-\mathbf{M}_k$.

Proof. Take $f, g : T(j) \to \mathbf{M}_k$, with $A \in \mathbf{M}_k$, such that $f\pi_j = g\pi_j$, and let $f_i : T(j)_i \to A$ and $g_{i'} : T(j)_{i'} \to A$ represent f and g respectively. By using the filtered condition on $\mathbf{J}$ if necessary we can assume that there is a commutative diagram in pro $-\mathbf{M}_k$ of the form



But ϕ and ψ both represent $T\alpha$, so $fT\alpha = gT\alpha$ and $f = g$. The general case again follows easily. □

Corollary 1.1.9. *For T with L as in Lemma 1.1.8 and $A \in \mathbf{M}_k$,*

$$\text{pro}-\mathbf{M}_k(L, A) = \text{pro}-\mathbf{M}_k(\varprojlim_j T_j, A) = \varinjlim_j \text{pro}-\mathbf{M}_k(T_j, A).$$

Proof. Every map $l \to A$ is represented by a map $T(j)_i \to A$ of $\mathbf{M}_k$ □

Corollary 1.1.10. *Let $b \in \mathbf{M}_k$ be an integral domain. Then*

$$\text{pro}-\mathbf{M}_k(\hat{\prod}_{i\in I} A_i, B) \cong \coprod_{i\in I} \mathbf{M}_k(A_i, B).$$

This isomorphism is natural in B.

1.2 The main results

The reader will recall [37, 39] that specifying a closed model structure for a category $\mathbf{C}$ requires the definition of three classes of maps, called fibrations, cofibrations, and weak equivalences respectively, such that the following axioms are satisfied.

CM1 $\mathbf{C}$ is closed under finite direct and inverse limits.

CM2 Given $U \xrightarrow{f} T \xrightarrow{g} S$ in $\mathbf{C}$, if any two of g, f and $g \cdot f$ are weak equivalences, then so is the third.

CM3 If f is a retract of g in the category of arrows of $\mathbf{C}$ and g is a cofibration, fibration, or weak equivalence, then so is f.

CM4 Given any solid arrow diagram in **C** of the form

$$\begin{array}{ccc} U & \longrightarrow & T \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ V & \longrightarrow & S \end{array}$$

where i is a cofibration and p is a fibration, then a dotted arrow exists making the diagram commutative if either i or p is a weak equivalence.

CM5 Any map f may be factored as

1. $f = pi$ where i is a cofibration and a weak equivalence and p is a fibration, and
2. $f = qj$ where j is a cofibration and q is a fibration and a weak equivalence.

It should be pointed out that the notions of trivial fibration and cofibration, and right and left lifting property have their customary meanings here (see [37]). Now say that, in pro $-\mathbf{M}_k$:

1. $f : T \to S$ is a *cofibration* if f has the left lifting property with respect to all maps of the form

$$A^0(i) : A^0(\Delta^n) \longrightarrow A^0(\Lambda^n_r), \quad n \geq 1, \quad 0 \leq r \leq n,$$

where Λ^n_r is the subcomplex of Δ^n which is generated by all of the faces of i_n except $d_r i_n$, and $i : \Lambda^n_r \subset \Delta^n$ is the inclusion.

2. $f : T \to S$ is a *weak equivalence* provided that $\hat{F}f$ is a weak equivalence of **S**, and

3. $f : T \to S$ is a *fibration* if f has the right lifting property with respect to all cofibrations which are also weak equivalences.

Then we have

Theorem 1.2.1. *With these definitions,* pro $-\mathbf{M}_k$ *is a closed model category if* k *is a unique factorization domain.*

Proof. **CM1** comes from [2, A.4.2]. **CM2** and **CM3** are trivial. One shows first of all that the factorization axiom **CM5**(2) holds. Take $f : T \to S$ and consider the set of all diagrams D of the form

$$\begin{array}{ccc} T & \xrightarrow{\alpha_D} & A^0(\Delta^{n_D}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A^0 i} \\ S & \xrightarrow[\beta_D]{} & A^0(\Lambda^{n_D}_{r_D}) \end{array}$$

Form the pullback

$$\begin{array}{ccc} S_1 & \xrightarrow{pr} & \hat{\prod}_D A^0(\Delta^{n_D}) \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \hat{\Pi} A^0 i} \\ S_0 = S & \xrightarrow[(\beta_D)]{} & \hat{\prod}_D A^0(\Lambda^{n_D}_{r_D}) \end{array}$$

where $\hat{\Pi} A^0 i$ is the obvious natural transformation. All of the maps occurring in $\hat{\Pi} A^0 i$ are surjective and so σ_1 is epi by Lemma 1.1.7. Lemma 1.1.8, together with Proposition 1.1.6, ensures that $\hat{F}\hat{\Pi} A^0 i)$ is a trivial cofibration of $\mathbf{S}$. Thus, by Lemma 1.1.4, so is $\hat{F}\sigma_1$. Let $f_1 : T \to S_1$ be the unique map such that

$$\begin{array}{ccccc} T & & & & \\ & \searrow^{f_1} & & \searrow^{(\alpha_D)} & \\ {\scriptstyle f_0 = f}\downarrow & & S_1 & \xrightarrow[pr]{} & \hat{\prod}_D A^0(\Delta^{n_D}) \\ & & {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \hat{\Pi} A^0 i} \\ S_0 & = & S & \xrightarrow[(\beta_D)]{} & \hat{\prod}_D A^0(\Lambda^{n_D}_{r_D}) \end{array}$$

commutes, and now iterate the construction to produce a tower of maps

$$\ldots \to S_3 \xrightarrow{\sigma_3} S_2 \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = S,$$

together with a cone

$$f_i : T \to S_i, \quad i \geq 0.$$

Let $S_\infty = \varprojlim_i S_i$, with limiting cone

$$\pi_i : S_\infty \to S_i, \quad i \geq 0.$$

Then f has a factorization $f = \pi_0 f_\infty$, where f_∞ is the unique map induced by the f_i, $i \geq 0$. f_∞ is a cofibration, since any $\beta : S_\infty \to A^0(\Lambda^n_r)$ factors through some S_m, according to the construction of filtered inverse limits in pro $-\mathbf{M}_k$. π_0 has the right lifting property with respect to all cofibrations, so in particular π_0 is a fibration. The factorization **CM5**(1) is obtained similarly, by using the fact that $f : T \to S$ is a trivial cofibration of pro $-\mathbf{M}_k$ if and only if f has the left lifting property with respect to all maps

$$A^0 i : A^0(\Delta^n) \to A^0(\partial\Delta^n), \quad n \geq 0,$$

induced by the inclusion $i : \partial\Delta^n \subset \Delta^n$, where $\partial\Delta^0 = \emptyset$by convention. The non-trivial part of **CM4** is a standard consequence of the construction which is used for the proof of **CM5**(2). □

Theorem 1.2.2. *$\hat{F}$ and $\hat{A}$ induce an equivalence of* $\mathrm{Ho}(\mathbf{S})$ *with* $\mathrm{Ho}(\text{pro} -\mathbf{M}_k)$.

Proof. We begin by showing that the unit of the adjunction,

$$\hat{\eta}_X \to \hat{F}\hat{A}X,$$

is a weak equivalence for arbitrary $X \in \mathbf{S}$. First of all, one uses the naturality of the map ζ of Lemma 1.1.1 together with the fact that $\hat{\eta}_X(x)$, for $x \in X_n$, is the composition

$$\hat{A}X \xrightarrow{\hat{A}i_x} \hat{A}\Delta^n \xrightarrow{\zeta^{-1}} A^0\Delta^n \xrightarrow{j} k_n,$$

to show that there is for every simplicial set X, a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{\eta}_X} & \hat{F}\hat{A}X \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \hat{F}f} \\ F^0A^0X & = & \hat{F}A^0X \end{array}$$

Thus, by Lemma 1.1.1 and Proposition 1.1.6, $\hat{\eta}_X$ is a weak equivalence for finite $X \in \mathbf{S}$. For the general case, X can be regarded as a filtered colimit

$$X = lim_{\to}K, \quad K \in \mathbf{Fin}(X)$$

in $\mathbf{S}$, while by Corollary 1.1.9,

$$\hat{F}\hat{A}X = lim_{\to}\hat{F}\hat{A}K, \quad K \in \mathbf{Fin}(X).$$

Thus, $\hat{\eta}_X$ is a weak equivalence, since it is a filtered colimit of weak equivalences $\hat{\eta}_K : K \to \hat{F}\hat{A}K$, $K \in \mathbf{Fin}(X)$. Now, the counit of the adjunction,

$$\hat{\epsilon}_T : T \to \hat{A}\hat{F}T,$$

is a weak equivalence of pro $-\mathbf{M}_k$ by the triangle identity

$$\hat{F}\hat{\epsilon}_T \cdot \hat{\eta}_{\hat{F}T} = 1_{\hat{F}T}.$$

$\hat{A}$ preserves weak equivalences since $\hat{\eta}$ is a natural weak equivalence, and $\hat{F}$ preserves weak equivalences by definition. The theorem follows easily. □

This section closes with the following observation:

Proposition 1.2.3. *Suppose that* $\mathbf{C}$ *is a closed model category and that the functor* $F : \mathbf{C} \to \mathbf{D}$ *is a (resp. contravariant) equivalence of categories with quasi-inverse* $G : \mathbf{D} \to \mathbf{C}$*. Say that an arrow* f *of* $\mathbf{D}$ *is a fibration (resp. cofibration), cofibration(resp. fibration) or weak equivalence if* $G(f)$ *is, respectively, a fibration, cofibration, or weak equivalence of* $\mathbf{C}$*. Then, with these definitions,* $\mathbf{D}$ *is a closed model category in such a way that the functors* F *and* G *induce and equivalence of* $\mathrm{Ho}(\mathbf{C})$ *with* $\mathrm{Ho}(\mathbf{D})$*.*

What is meant by saying that$F : \mathbf{C} \to \mathbf{D}$ is an equivalence of categories with quasi-inverse $G : \mathbf{D} \to \mathbf{C}$ is that there are natural isomorphisms $1_{\mathbf{C}} \cong GF$ and $1_{\mathbf{D}} \cong FG$.

Chapter 2

Algebraic groups

Chapter Introduction

The category ind $-\mathbf{Aff}_k$ of inductive affine schemes over a unique factorization domain k is contravariantly equivalent to the category pro $-\mathbf{M}_k$ of pro-objects in the category of k-algebras. In view of the results of Chapter 1, ind $-\mathbf{Aff}_k$ has a closed model structure in such a way that Ho(ind $-\mathbf{Aff}_k$) is equivalent to the ordinary homotopy category Ho($\mathbf{S}$) which is associated to the category $\mathbf{S}$ of simplicial sets. In other words, what we have here is an affine algebraic geometric setting for homotopy theory. In particular, one may associate to each affine k-scheme X its homotopy groups $\pi_i(X)$, $i \geq 0$, in a seemingly very natural way, just by passage to the simplicial category. One could reasonably ask what these invariants measure in the classical case where k is an algebraically closed field.

The first problem that one encounters in this programme is that of finding naturally occurring algebraic geometric objects whose homotopy groups can be computed, at least partially. It seems that the category of affine reduced schemes of finite type over k is not quite adequate for this investigation. Even the notion of path-connectedness is difficult to understand there. It is a difficult problem to see how to "draw lines" on an arbitrary affine (even smooth) variety X in the sense of finding enough homomorphisms $\mathbb{A}^1_k \to X$. It can be seen, for example, that, while the affine line $\mathbb{A}^1_k$ is contractible in the sense that its associated simplicial set is contractible, the simplicial set which is associated to the open subset $\mathbb{A}^1_k - \{0\} = \mathrm{Spec}(k[x, x^{-1}])$ is a discrete simplicial group on the group of units k^* of the underlying field k.

There is, however, at least one classical geometric setting in which drawing lines is very natural. Let $Gl_n(k)$ be the group of rational points of the general linear group and suppose that the characteristic of the algebraically closed field k is 0. It is well-known (see [22, p.96]) that,if an element x of $Gl_n(k)$ is unipotent, then there is a homomorphism $\phi : \alpha_k \to Gl_n$ of algebraic groups from the additive group α_k, which is defined on $\alpha_{\mathbf{k}}(k) = k$ by $\phi(a) = \exp(a \cdot \log(x))$ for

$a \in k$. Moreover, ϕ defines an isomorphism of α_k onto the unique smallest closed algebraic subgroup of Gl_n which contains x in its group of rational points. The underlying variety of α_k is the affine line $\mathbb{A}^1_k$, so that this result implies that, if G is an algebraic group over k and $x \in G(k)$ is unipotent, then there is a scheme homomorphism $\omega : \mathbb{A}^1_k \to G$ with $\omega(0) = e$ and $\omega(1) = x$ in $G(k)$, where e denotes the identity of $G(k)$. We call such an ω a *path* from e to x. An algebraic group G is said to be *path-connected* if there is a path from e to any x of $G(k)$. This observation in characteristic 0 was the starting point for the classification of path-connected algebraic groups over k of arbitrary characteristic which is given in the second section of this Chapter.

It turns out that the theory of algebraic groups is rich enough that one can begin to calculate their homotopy groups. In succeeding Chapters we will set up a fibration theory for homomorphisms of algebraic groups and start to compute the fundamental group. For this, it will be necessary to work in several categories which are described in the first section.

2.1 Preliminaries

Let k be an algebraically closed field and let $\mathbf{Sch}_k$ be the category of schemes over k. $\mathbf{F}_k$ will denote the full subcategory of $\mathbf{Sch}_k$ whose objects are reduced and of finite type over k. A group-scheme is a group-object of $\mathbf{Sch}_k$, while an algebraic group over k will be a group-scheme which is affine and lies in $\mathbf{F}_k$. Where it is convenient, we shall follow the standard practice of identifying objects of $\mathbf{F}_k$ and algebraic groups in particular with their sets of k-rational points, together with the induced topology and sheaf of rings. The language and basic results of classical algebraic group theory will be assumed; the reference is [22].

As before, let $\mathbf{M}_k$ be the category of algebras (or modèles; see [13]) over k. Associated to each k-scheme X is a covariant functor, which is also denoted by X, from $\mathbf{M}_k$ to the category $\mathbf{E}$ of sets. X is defined by

$$X(A) = \mathbf{Sch}_k(\mathrm{Spec}(A), X)$$

for $A \in \mathbf{M}_k$. We call $X(A)$ the set of A-points of X. As is well known, this association determines a fully faithful functor from $\mathbf{Sch}_k$ to the category $\mathbf{M}_k\mathbf{E}$ of functors from $\mathbf{M}_k$ to $\mathbf{E}$, so that $\mathbf{Sch}_k$ is equivalent to its image in $\mathbf{M}_k\mathbf{E}$. Thus, a k-scheme X can be identified with the functor that it "represents"; affine k-schemes are associated with honestly representable functors in this way.

$\mathbf{M}_k\mathbf{E}$ also contains the full subcategory of pro-representable functors. These are all of the functors of the form

$$\mathrm{Sp}_k(T)(A) = \varinjlim_{i \in I} \mathbf{M}_k(T_i, A), \quad for\ A \in \mathbf{M}_k$$

where $T : \mathbf{I} \to \mathbf{M}_k$ is a pro-algebra as in Chapter 1. The contravariant functor which is gotten from $T \mapsto \mathrm{Sp}_k(T)$ for $T \in \text{pro}-\mathbf{M}_k$ is fully faithful, so that pro $-\mathbf{M}_k$ is contravariantly equivalent with the category of pro-representable functors in $\mathbf{M}_k\mathbf{E}$, which will be denoted by ind $-\mathbf{Aff}_k$ in view of the equivalence with the usual category of inductive affine k-schemes. If we take for each $X \in$ ind $-\mathbf{Aff}_k$ a fixed representing object $\Gamma X \in$ pro $-\mathbf{M}_k$, then a functor $\Gamma :$ ind $-\mathbf{Aff}_k \to$ pro $-\mathbf{M}_k$ is defined, which is the inverse of Sp_k up to natural isomorphism.

If K is a finite simplicial set, we shall often write K for the simplicial set itself, for the affine k-scheme $\mathrm{Spec}(A^0K)$, and for the representable functor $\mathbf{M}_k(A^0K, \ )$ in $\mathbf{M}_k\mathbf{E}$. In this notation, Δ^n is the affine n-space $\mathbb{A}^n_k$ for $n \geq 0$ in either $\mathbf{M}_k\mathbf{E}$ or the scheme category, since $A^0\Delta^n \cong k[x_1, \ldots, x_n]$ in $\mathbf{M}_k$. The cosimplicial k-scheme that one gets by applying Spec to the simplicial algebra k_* will be denoted by $\mathbb{A}_k$.

If $f : X \to Y$ is a morphism of ind $-\mathbf{Aff}_k$, we may now say the following:

1) f is a *fibration* if in every solid arrow diagram in ind $-\mathbf{Aff}_k$ of the form

$$\begin{array}{ccc} \Lambda^n_r & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \mathbb{A}^n_k = \Delta^n & \longrightarrow & Y \end{array}$$

there is a dotted arrow making the diagram commute,

2) f is a *weak equivalence* if Γf is a weak equivalence of pro $-\mathbf{M}_k$ (equivalently, if $\hat{F}\Gamma f$ is a weak equivalence of simplicial sets), and

3) f is a *cofibration* if f has the left lifting property with respect to every map which is both a fibration and a weak equivalence.

In view of the results of §2 of Chapter 1, we have

Theorem 2.1.1. *With these definitions, the category* ind $-\mathbf{Aff}_k$ *is a closed model category in such a way that the functors* Γ *and* Sp_k *induce a contravariant equivalence of* $\mathrm{Ho}(\text{pro} -\mathbf{M}_k)$ *with* $\mathrm{Ho}(\text{ind} -\mathbf{Aff}_k)$.

Corollary 2.1.2. *The covariant functors* $\hat{F} \cdot \Gamma :$ ind $-\mathbf{Aff}_k \to \mathbf{S}$ *and* $\mathrm{Sp}_k \cdot \hat{A} :$ $\mathbf{S} \to$ ind $-\mathbf{Aff}_k$ *induce an equivalence of* $\mathrm{Ho}(\text{ind} -\mathbf{Aff}_k)$ *with* $\mathrm{Ho}(\mathbf{S})$.

One may think of $\hat{F} \cdot \Gamma$ as an analogue of the ordinary singular functor. We shall write $S_k = \hat{F} \cdot \Gamma$ and call it the *algebraic singular functor*. Similarly, $R_k = \mathrm{Sp}_k \cdot \hat{A}$ is called the *algebraic realization functor*. Observe that, for $X \in$ ind $-\mathbf{Aff}_k$, $S_k(X)$ is isomorphic to the simplicial set $X(k_*)$, which comes from applying the functor X to the simplicial algebra k_*. The homotopy group $\pi_i(X, x)$ of X based at $x \in X(k)$ is defined to be the i^{th} homotopy group $\pi_i(|X(k_*)|, x)$ of the ordinary geometric realization in the topological category of the simplicial set $X(k_*)$. Observe that a morphism $X \to Y$ of ind $-\mathbf{Aff}_k$ is a weak equivalence if and only if it induces an isomorphism of the homotopy groups so defined.

The rest of this thesis consists of computations of some of the homotopy groups of affine group-schemes G of finite type over k. In fact, it suffices to consider only algebraic groups in the sense described above, since the reduced part G_{red} of G is a closed subgroup-scheme of G and we have the elementary

Proposition 2.1.3. *The closed immersion $G_{red} \subset G$ induces an isomorphism $G_{red}(k_*) \cong G(k_*)$.*

2.2 Path-components of algebraic groups

One of the principal reasons that something may be said about the set $\pi_0(G)$ of path-components of an algebraic group G has to do with the fact that the simplicial group $G(k_*)$ is a Kan complex. It follows from the combinatorial definition of the homotopy groups of a Kan complex (see [33]) that $G(k_*)$ is connected as a simplicial set if and only if G satisfies the property that for every $x \in G(k)$ there is a k-scheme morphism $\omega : \mathbb{A}^1_k \to G$ with $\omega(0) = e$ and $\omega(1) = x$ in $G(k)$, where e denotes the identity of $G(k)$. In effect, one identifies $G(k[x_1])$ with the k-scheme homomorphisms $\mathbb{A}^1_k \to G$, and identifies the face maps d_0 and d_1 of $G(k_*)$ with the functions from $\mathbf{Sch}_k(\mathbb{A}^1_k, G)$ to $\mathbf{Sch}_k(A^0_k, G) \cong G(k)$ which are induced by the choice of rational points 1 and 0 respectively in $\mathbb{A}^1_k \cong k$. An algebraic group which satisfies this property is said to be *path-connected*. The morphism ω is called a *path* from e to x.

When one says that G is *connected*, on the other hand, one means, as usual, that G is irreducible as a k-scheme. It is not the case that all connected algebraic groups are path-connected. Let μ_k denote the multiplicative group $\text{Spec}(k[t, t^{-1}])$. The inclusion $k \subset k[x_1, \ldots, x_n]$ induces an isomorphism

$$\mu_k(k) = k^* \cong k[x_1, \ldots, x_n]^* = \mu_k(k[x_1, \ldots, x_n]) \quad for\ n \geq 1.$$

It follows that the simplicial group $\mu_k(k_*$ is discrete on the group k^* of units of k, so that μ_k, while it is connected, is not path-connected. Similar remarks hold for any torus. It is the case, however, that we have

Proposition 2.2.1. *All path-connected algebraic groups over k are connected.*

Proof. Use the fact that the continuous image of an irreducible topological space is irreducible. □

In fact, we shall prove

Theorem 2.2.2. *A connected algebraic group G over k is path-connected if and only if the group $G(k)$ of rational points of G is generated by unipotent elements.*

We begin the proof of this theorem with a construction. Let

$$G^e(k) = \{x \in G(k) | \text{ there is } \omega : \mathbb{A}^1_k \to G \text{ with } \omega(0) = e \text{ and } \omega(1) = x \text{ in } G(k)\}$$

Then we have

Lemma 2.2.3. *$G^e(k)$ is a closed, connected, and normal subgroup of $G(k)$.*

Proof. $G^e(k)$ is a normal subgroup of $G(k)$ by multiplication and conjugation of paths. Let Y_ω be the image in $G(k)$ of $\mathbb{A}^1_k(k)$ under a path ω. Then $Y_\omega \subset G^e(k)$. The subgroup H of $G(k)$ generated by all such Y_ω is closed and connected [22, p. 55]. Clearly $H = G^e(k)$. □

It follows that $G^e(k)$ is the group of rational points of a closed algebraic subgroup G^e of G. Using the fact that all of the affine spaces $\mathbb{A}^n_k$ are reduced one can show that the simplicial group $G^e(k_*)$ is the connected component of the vertex e in $G(k_*)$. For this reason, we call G^e the *path-component of the identity* in G.

Lemma 2.2.4. *Let G be a connected algebraic group and let $\langle G_u\rangle(k)$ be the subgroup of $G(k)$ which is generated by the unipotent elements of G. Then $\langle G_u\rangle(k)$ is a closed, connected, normal subgroup $G(k)$.*

Proof. Let B be a Borel subgroup of G with unipotent subgroup B_u. By the Density Theorem [22, p. 139],

$$G(k) = \bigcup_{x\in G(k)} xB(k)x^{-1}$$

Thus, every unipotent element of $G(k)$ is contained in the unipotent subgroup $B_u(k)$ of some Borel subgroup B of G. All such B_u are connected closed algebraic subgroups of G [22, p. 122], so that the subgroup H of $G(k)$ generated by all of the $B_u(k)$ is closed and connected as well. But $H = \langle G_u\rangle(k)$, and $\langle G_u\rangle(k)$ is normal in $G(k)$. □

Again, $\langle G_u\rangle(k)$ is the group of rational points of a connected algebraic subgroup $\langle G_u\rangle$ of G. The connectedness assumption on G was used in Lemma 2.2.4 in that the Density Theorem was invoked. Assume that G is connected from now on. Now we have

Lemma 2.2.5. *G^e is a closed algebraic subgroup of $\langle G_u\rangle$.*

Proof. It suffices to show that $G^e(k) \subset \langle G_u\rangle(k)$. Consider the exact sequence of groups

$$e \to \langle G_u\rangle(k) \to G(k) \xrightarrow{P} (G/\langle G_u\rangle)(k) \to e.$$

$G/\langle G_u\rangle$ is a connected algebraic group, and every element of $(G/\langle G_u\rangle)(k)$ is semi-simple by the naturality of the Jordan-Chevalley decomposition. Thus, $G/\langle G_u\rangle$ is a torus (see [22, p. 137]). Every torus is discrete in a homotopical sense, so that, if $x \in G^e(k)$, then $p(x) = e$, whence $x \in \langle G_u\rangle(k)$. □

It is well known that any connected unipotent algebraic group U over k is isomorphic as a k-scheme with $\mathbb{A}^n_k$, where n is the dimension of U. The most efficient proof of this fact uses the theory of fpqc-torseurs; this will be sketched in the next Chapter. But it is easy to see that there is an isomorphism of simplicial sets

$$\mathbb{A}_k^m(k_*) = k_* \times \ldots \times k_* \text{ (}n\text{ copies)},$$

so that $\mathbb{A}_k^m(k_*)$ is contractible by the Extension Lemma of [5]. It follows, in particular, that any connected unipotent algebraic group over k is path-connected. This proves

Lemma 2.2.6. *$\langle G_u \rangle$ is a closed subgroup of G^e.*

The proof of Theorem 2.2.2 is complete as well.

Examples of path-connected algebraic groups abound in nature, in view of Theorem 2.2.2. Any semi-simple group, for example, is generated by unipotent elements. Thus, all of the classical simple algebraic groups are path-connected. It is also easy to see that the special linear group Sl_n is the path-component of the identity for the general linear group Gl_n for all $n \geq 1$.

It follows from the proof of Lemma 2.2.5 that the group $\pi_0(G)$ of path-components of a connected algebraic group G is the group of rational points of a torus $T(G)$. By appealing to a fundamental result of the next Chapter, we may establish the rank of $T(G)$ as follows:

Proposition 2.2.7. *Let G be a connected algebraic group. Then the rank of $T(G)$ is the rank of a maximal torus of the solvable radical $R(G)$.*

Proof. Let T be a maximal torus of $R(G)$. Let $R_u(G)$ be the unipotent radical of G and write $H = G/R_u(G)$. Then $R(H)$ is a torus, and $rk(R(H)) = rk(T)$. Moreover, a result of the next Chapter implies that, since $R_u(G)$ is a connected unipotent algebraic group, the sequence

$$e \rightarrow R_u(G)(k_*) \rightarrow G(k_*) \xrightarrow{p} H(k_*) \rightarrow e$$

is an exact sequence of simplicial groups, where p is induced by the canonical projection. It follows that p induces an isomorphism $P_*\pi_0(G) \cong \pi_0(H)$, and that the tori $T(G)$ and $T(H)$ have the same rank. Since $H/[H,H]$ is a torus and the commutator subgroup $[H,H]$ of H is semi-simple, we have $\langle H_u \rangle = [H,H]$, and there is an exact sequence of groups

$$e \rightarrow F \rightarrow R(H)(k) \rightarrow \pi_0(H) = (H/[H,H])(k) \rightarrow e,$$

where F is a finite abelian group. The conclusion follows □

Remark 2.2.8. *In the proof above, the canonical projection $G \rightarrow H$ actually induces an isomorphism $TG \cong T(H)$ of tori, by a smoothness argument.*

An easy consequence of Proposition 2.2.7 is the following criterion for path-connectedness:

Corollary 2.2.9. *A connected algebraic group G is path-connected if and only if its solvable radical $R(G)$ is unipotent.*

Some of the consequences of this Corollary are:

Corollary 2.2.10. *Let H be a closed, connected,normal algebraic subgroup of a path-connected algebraic group G. Then H is path-connected.*

Proof. Use $R(H) \subset R(G)$. □

Corollary 2.2.11. *Let L be an algebraically closed field containing k, and suppose that G is a connected algebraic group which is defined over k. Let $G \otimes_k L$ denote the base extension of G over L. Then $G \otimes_k L$ is path-connected over L in the obvious sense if and only if G is path-connected over k.*

Proof. $G \otimes_k L$ is connected over L, and $R(G) \otimes_k L = R(G \otimes_k L)$. Then $R(G)$ is unipotent if and only if $R(G \otimes_k L)$ is unipotent. □

Chapter 3

Fibrations of algebraic groups

Chapter Introduction

It was shown in Chapter 2 that a connected algebraic group G, which is defined over an algebraically closed field k, is path-connected if and only if its group $G(k)$ of rational points is generated by unipotent elements. In this Chapter, we begin to answer the question of what is measured by the higher homotopy groups of an algebraic group; the main purpose here is to give a description of the higher homotopy groups of a path-connected algebraic group in terms of the homotopy groups of Chevalley groups of universal type. This is done by developing a theory of fibrations for algebraic groups.

A homomorphism $f : G \to H$ of algebraic groups is said to be a *fibration* if it induces a fibration $S_k f : S_k G \to S_k H$ in the simplicial set category. This means precisely that f is a fibration of ind $-\mathbf{Aff}_k$ in the sense described in the last Chapter. Now, $S_k f$ is a homomorphism of simplicial groups, and it is well known that every surjective simplicial group homomorphism is a fibration of simplicial sets. Thus, for $S_k f$ to be a fibration, it suffices that every k-scheme homomorphism $\mathbb{A}^n_k \to H$ can be lifted to G, for every $n \geq 0$. Part of this condition (the 0-simplices part) says that f is surjective on rational points. Such a homomorphism of algebraic groups will be said to be *surjective* henceforth. One could ask if every surjective homomorphism of algebraic groups is a fibration. It will be seen in §2 that this is the case if the characteristic of k is 0; it is not true in positive characteristics.

The situation in arbitrary characteristic is far from unmanageable, however. Starting with a path-connected algebraic group G, one divides off by its solvable radical $R(G)$ to get the associated semi-simple group $S = G/R(G)$, together with a surjective homomorphism.

$$G \to G/R(G) = S \tag{3.0.1}$$

which is just the canonical map. Consider now the finite collection of minimal closed normal connected subgroups of positive dimension $S_1, \ldots, S_n$ of S. These subgroups centralize each other and generate S, so that a surjective algebraic group homomorphism

$$m : S_1 \times \cdots \times S_n \to S \tag{3.0.2}$$

is defined using the multiplication map of S. It will be shown, also in §2 that the homomorphisms (3.1) and (3.2) are fibrations. It will follow from an application of the long exact sequence for a fibration that $\pi_1(G) \cong \bigoplus_{j=1}^{n} \pi_i(S_j)$ for $i > 1$, and that there is a short exact sequence of abelian groups

$$0 \to \bigoplus_{j=1}^{n} \pi_1(S_j) \to \pi_1(G) \to \Gamma \to 0,$$

where Γ is a finite abelian group.

By the Classification Theorem [10] for simple algebraic groups, each of the groups S_j is isomorphic to a Chevalley group G_{ρ_j} over k which comes from a faithful representation ρ_j of a simple Lie algebra L_j over the complex numbers $\mathbb{C}$. If ϕ_j denotes the root system of L_j and G_{ϕ_j} is the Chevalley group over k which is universal of type ϕ_j, then there is a surjective comparison homomorphism

$$\lambda_{\phi_j} : G_{\phi_j} \to G_{\phi_j} \tag{3.0.3}$$

for $j = 1, \ldots, n$. These homomorphisms are shown to be fibrations in §3, after a review of definitions and notation. The main tools are the "big cell" and a result (Corollary 3.2.4) of §2.

The technical point that is exploited throughout this Chapter is that the obstruction to lifting a map $\mathbb{A}_k^n \to H$ over a surjective algebraic group homomorphism $f : G \to H$ lies in the pointed set $\tilde{H}^1(\mathbb{A}_k^n; K)$ of fpqc-torseurs over $\mathbb{A}_k^n$ with coefficients in the 0 group-scheme kernel K of f. The first section of this Chapter contains a quick introduction to this theory, together with proofs of results about unipotent algebraic groups that were used in Chapter 2. The approach to torseur theory that is taken here follows that of the book of Demazure and Gabriel [13].

3.1 Sheaves and torseurs

We shall work in the Demazure-Gabriel version [13] of the category $\widetilde{\mathbf{M}_k\mathbf{E}}$ of fpqc-sheaves over the algebraically closed field k. The category of schemes over k is a full and faithful subcategory of $\widetilde{\mathbf{M}_k\mathbf{E}}$. Let G be a sheaf of groups over k and take $S \in \widetilde{\mathbf{M}_k\mathbf{E}}$. Then we recall that a right *G-torseur* over S consists of:

1. a map $p : X \to S$ in $\widetilde{\mathbf{M}_k\mathbf{E}}$, and
2. a right G-action $m : X \times G \to X$ on X,

such that $p(xg) = p(x)$ for all $x \in X(A)$, $g \in G(A)$, and all $A \in \mathbf{M}_k$, and such that there is a sheaf epi $T \to S$ for which the pullback $T \times_S X \to T$ is equivariantly isomorphic to the projection $pr_T : T \times G \to T$. Recall further that any G-torseur which is equivariantly isomorphic to the torseur $pr_S : S \times G \to S$ is said to be *trivial*, and that a G-torseur $p : X \to S$ is trivial if and only if p has a section. The class of isomorphism classes of G-torseurs over S is denoted by $\tilde{H}^1(S; G)$. The class of trivial torseurs provides $\tilde{H}^1(S; G)$ with a base point. Every surjective homomorphism $\pi : G \to G'$ of algebraic groups over k is a K-torseur over G', where K is the group-scheme kernel of π, and every quotient map $G \to G/H$, for H a closed normal algebraic subgroup of G, is an H-torseur over G/H.

A key point (see [13, p. 361] is:

Proposition 3.1.1. *Let $p : X \to S$ be a G-torseur over S, and let $g : T \to S$ be a k-sheaf morphism. Then g lifts to X if and only if the G-torseur $T \times_S X \to T$, which is induced by pullback over g, is trivial.*

Suppose that $p : X \to S$ is a G-torseur over S, and that $f : G \to H$ is a homomorphism of sheaves of groups over k. Define a left G-action on $X \times H$ via $g \cdot (x, h) = (x \cdot g^{-1}, f(g) \cdot h)$, and let the k-sheaf $X \vee^G H$ be the coequalizer in $\widetilde{\mathbf{M}_k\mathbf{E}}$ of the pair of maps

$$G \times (X \times H) \underset{pr_{X \times H}}{\overset{m}{\rightrightarrows}} X \times H,$$

where m is the G-action and $pr_{X \times H}$ is the obvious projection. H acts on $X \vee^G H$ on the right and there is a unique map $X \vee^G H \to S$ factoring through the composition

$$S \times H \xrightarrow{pr_X} X \xrightarrow{p} S,$$

which makes $X \vee^G H$ into an H-torseur over S (see [13, p.368]). This operation defines a natural pointed set map $f_* : \tilde{H}^1(S; G) \to \tilde{H}^1(S; H)$ and makes $\tilde{H}^1(S;\ )$ a functor on the category of sheaves of groups over k.

Now consider an exact sequence

$$e \to K \xrightarrow{i} G \xrightarrow{\pi} H \to e \tag{3.1.1}$$

of sheaves of groups over k. Recall that, if G and H are algebraic groups, then exactness of (3.4) means precisely that π is surjective (on rational points) and that K is the group-scheme kernel of π. For $S \in \widetilde{\mathbf{M}_k\mathbf{E}}$ and G a sheaf of groups over k, we define $\tilde{H}^0(S; G) = \widetilde{\mathbf{M}_k\mathbf{E}}(S; G)$. Then it is well known that $\tilde{H}^1(S;\ )$ is the first right derived functor of $\tilde{H}^0(S;\ )$ in the sense that we have

Proposition 3.1.2. *For any $S \in \widetilde{\mathbf{M}_k\mathbf{E}}$ and any exact sequence (3.4) as above, there is a 6-term sequence of pointed sets*

$$\begin{aligned} * \to &\tilde{H}^0(S; K) \xrightarrow{i_*} \tilde{H}^0(S; G) \xrightarrow{\pi_*} \tilde{H}^0(S; H) \\ &\xrightarrow{\delta} \tilde{H}^1(S; K) \xrightarrow{i_*} \tilde{H}^1(S; G) \xrightarrow{\pi_*} \tilde{H}^1(S; H), \end{aligned}$$

which is exact in the sense that kernel = image everywhere. The coboundary map δ is defined by pullback.

The basic technical device in the sections that follow will be to show, for group-scheme kernels K of certain surjective algebraic group homomorphisms, that $\tilde{H}^1(\mathbb{A}_k^m; K) = *$ for all $m \geq 1$, and then invoke Proposition 3.1.1. This will involve Proposition 3.1.2 and proceed from basic facts.

Proposition 3.1.3. *[13, p. 397] Suppose that X is a scheme which is of finite type over k. Then $\tilde{H}^1(X; Gl_n)$ is in one to one correspondence with the set of quasi-coherent X-modules which are locally free of rank n.*

If X is affine, then the class of X-modules of this last Proposition coincides with the class of projective A-modules of rank n, where $X = \mathrm{Spec}(A)$. Thus, if X is one of the affine spaces $\mathbb{A}_k^m$, then the Serre "conjecture" for k [42] can be restated as

Proposition 3.1.4. $\tilde{H}^1(\mathbb{A}_k^m; Gl_n) = *$ *for all* $m, n \geq 1$.

In particular, since Gl_1 is the multiplicative group $\mu_{\mathbf{k}}$, and any torus is a finite product of copies of $\mu_{\mathbf{k}}$, we have

Corollary 3.1.5. $\tilde{H}^1(\mathbb{A}_k^m; T) = *$ *for any torus T and all $m \geq 1$.*

Corollary 1.5 may also be inferred from the vanishing of the Picard group $\mathrm{Pic}(\mathbb{A}_k^m)$ of $\mathbb{A}_k^m$(see [13, p.371]).

Another basic fact is

Proposition 3.1.6. *[13, p.383] Suppose that X is an affine k-scheme. Then* $\tilde{H}^1(X; \alpha_k) = *$.

We may now prove

Proposition 3.1.7. *Let U be a connected unipotent algebraic group and suppose that X is an affine k-scheme. Then $\tilde{H}^1(X; U) = *$.*

Proof. In effect, U has a connected closed normal algebraic subgroup which is isomorphic to α_k [22, p.115, 131], and U/α_k is a connected unipotent algebraic group of lower dimension. Proceed by induction on the dimension of U, using the exact sequence of Proposition 3.1.2. □

It follows from Proposition 3.1.7 that taking the quotient of a connected algebraic group G by its unipotent radical $R_u(G)$ gives a fibration $G \to G/R_u(G)$. This fact was used in the proof of Proposition 2.2.7 of Chapter 2. We also have the following well-known result:

Proposition 3.1.8. *The underlying k-scheme of a connected unipotent algebraic group U is isomorphic to $\mathbb{A}_k^n$, where n is the dimension of U.*

Proof. As in the proof of Proposition 3.1.7, U has a closed connected normal algebraic subgroup isomorphic to α_k. Then the quotient map $U \to U/\alpha_k$ is a right α_k-torseur over the affine k-scheme U/α_k, so that $U = \alpha_k \times (U/\alpha_k)$ by Proposition 3.1.6. The proof is finished by induction on the dimension of U, as before. □

It follows that every connected unipotent group U is weakly equivalent to a point Spec(k) in the sense of Chapter 1. This completes the proofs of Proposition 2.2.7, Lemma 2.2.6, and Theorem 2.2.2 of Chapter 2.

3.2 Dévissage

Let G be a connected algebraic group over k, with solvable radical $R(G)$. The first task will be to show that the canonical homomorphism $G \to G/R(G)$ is a fibration.

Proposition 3.2.1. *Let S be a connected solvable algebraic group. Then it follows that $\tilde{H}^1(\mathbb{A}^m_k; S) = *$ for all $m \geq 1$.*

Proof. Let U be the set of unipotent elements of S and let T be a maximal torus. Then U is a closed connected normal subgroup of S and there is a split exact sequence of algebraic groups of the form

$$e \to U \to S \to T \to e.$$

But $\tilde{H}^1(\mathbb{A}^m_k; S) = *$ by Proposition 3.1.7 and $\tilde{H}^1(\mathbb{A}^m_k; T) = *$ by Corollary 3.1.5. An application of the exact sequence of Proposition 3.1.2 finishes the proof. □

Corollary 3.2.2. *Let G be a connected algebraic group with solvable radical $R(G)$. Then the canonical map $G \to G/R(G)$ is a fibration of algebraic groups.*

Now let Γ be a finitely generated abelian group, and consider the functor $D(\Gamma) : \mathbf{M}_k \to \mathbf{E}$, which is defined at $A \in \mathbf{M}_k$ by

$$D(\Gamma)(A) = \mathbf{Ab}(\Gamma, A^*),$$

where **Ab** denotes the category of abelian groups. $D(\Gamma)$ is an affine group-scheme over k which is represented by the group algebra $k[\Gamma]$ of Γ over k, where the comultiplication $\Delta : k[\Gamma] \to k[\Gamma] \otimes_k k[\Gamma]$, the counit $\epsilon : k[\Gamma] \to k$, and the involution $\sigma : k[\Gamma] \to k[\Gamma]$ are specified respectively by $\Delta(x) = x^{\otimes} x$, $\epsilon(x) = e$, and $\sigma(x) = x^{-1}$, for $x \in \Gamma$, and together give the Hopf algebra structure for $k[\Gamma]$. Here, Γ is written multiplicatively with identity e. Such an affine group-scheme $D(\Gamma)$ over k is said to be *multiplicative* (see [13, p.144]). Any subgroup-scheme of a torus is multiplicative; in fact, the multiplicative group-schemes over k as defined above are precisely the subgroup-schemes of tori. Perhaps the most useful general result of this Chapter is

Theorem 3.2.3. *Let* Γ *be a finitely generated abelian group and let* $D(\Gamma)$ *be its associated multiplicative group-scheme over* k. *Then, for all* $m \geq 1$, $\tilde{H}^1(\mathbb{A}^m_k; D(\Gamma)) = *$

Proof. One may show that applying D to an exact sequence of abelian groups

$$e \to \mathbb{Z}^r \xrightarrow{i} \mathbb{Z}^s \xrightarrow{P} \Gamma \to e$$

gives an exact sequence

$$e \to D(\Gamma) \xrightarrow{p^*} D(\mathbb{Z}^s) \xrightarrow{i^*} S(\mathbb{Z}^r) \to e$$

of group-schemes over k [13, p. 473]. Take $m \geq 1$ and consider the 6-term exact sequence of Proposition 3.1.2

$$* \to \tilde{H}^0(\mathbb{A}^m_k; D(\Gamma)) \to \tilde{H}^0(\mathbb{A}^m_k; D(\mathbb{Z}^s)) \xrightarrow{i^*} \tilde{H}^0(\mathbb{A}^m_k; D(\mathbb{Z}^r))$$

$$\xrightarrow{\delta} \tilde{H}^1(\mathbb{A}^m_k; D(\Gamma)) \to \tilde{H}^1(\mathbb{A}^m_k; D(\mathbb{Z}^s)) \to \tilde{H}^1(\mathbb{A}^m_k; D(\mathbb{Z}^r)).$$

$D(\mathbb{Z}^s)$ is a torus of rank s, so that $\tilde{H}^1(\mathbb{A}^m_k; D(\mathbb{Z}^s)) = *$ by Corollary 3.1.5. On the other hand, there is a commutative diagram

$$\begin{array}{ccc}
\tilde{H}^0(\mathbb{A}^m_k; S(z^s)) & \xrightarrow{\quad i^* \quad} & \tilde{H}^0(\mathbb{A}^m_k; D(\mathbb{Z}^r)) \\
\downarrow \cong & & \downarrow \cong \\
D(\mathbb{Z}^s)(k[x_1, \ldots, x_m]) & & D(\mathbb{Z}^r)(k[x_1, \ldots, x_m]) \\
\uparrow j_* & & \uparrow j_* \\
D(\mathbb{Z}^s)(k) & \xrightarrow{\quad i^*(k) \quad} & D(\mathbb{Z}^r)(k)
\end{array}$$

where j denotes the inclusion $k \subset k[x_1, \ldots, x_m]$. But the vertical maps j_* are isomorphisms since $k^* = k[x_1, \ldots, x_m]^*$, and $i^*(k)$ is surjective because i^* is faithfully flat. The theorem follows. □

Corollary 3.2.4. *Any surjective algebraic group homomorphism with multiplicative group-scheme kernel is a fibration.*

The argument of Theorem 3.2.3 breaks down if the field k is not assumed to be algebraically closed. If $\mathbb{A}^m_Q$ is the affine m-space over the rational numbers Q and $D_Q(\mathbb{Z}/2\mathbb{Z})$ is the group-scheme over Q corresponding to $D(\mathbb{Z}/2\mathbb{Z})$ above, then one may show that there is an isomorphism $\tilde{H}^1(\mathbb{A}^m_Q; D_Q(\mathbb{Z}/2\mathbb{Z})) \cong Q^*/(Q^*)^2$, and this for all $m \geq 0$.

We need some more notation at this point. Let $X : \mathbf{M}_k \to \mathbf{E}$ be a functor. Take $A \in \mathbf{M}_k$ and let $\mathbf{M}_A$ denote the category of A-algebras. Then a functor $X_A : \mathbf{M}_A \to \mathbf{E}$ is defined at $B \in \mathbf{M}_A$ by $X_A(B) = X(B)$, since $B \in \mathbf{M}_k$ as well. If Y is another element of $\mathbf{M}_k\mathbf{E}$, then a functor $\mathbf{hom}(X, Y) : \mathbf{M}_k \to \mathbf{E}$

is defined at $A \in \mathbf{M}_k$ by $\mathbf{hom}(X,Y)(A) = \mathbf{M_A E}(X_A, Y_A)$. $\mathbf{hom}(X,Y)$ is the canonical function complex for $\mathbf{M}_k\mathbf{E}$ in that there is a natural isomorphism

$$\mathbf{M}_k\mathbf{E}(X \times S, Y) \cong \mathbf{M}_k\mathbf{E}(S, \mathbf{hom}(X,Y)),$$

which is natural in X, Y, and S. The functors $\mathbf{End}(X)$ and $\mathbf{Aut}(X)$ which are associated to $X \in \mathbf{M}_k\mathbf{E}$ have corresponding obvious definitions (see [13, p.160]).

Now suppose that T is a maximal torus of a semi-simple algebraic group G, and identify these algebraic groups with their associated group-valued functors on $\mathbf{M}_k$. T acts on G by conjugation, and the above natural isomorphism associates to this action a natural transformation $\rho : T \to \mathbf{Aut}(G)$, which is defined for $t \in T(A)$, $g \in G(B)$, and $B \in \mathbf{M}_A$ by $\rho_A(t)_B(g) = t_B g t_B^{-1}$. Here, following [13], t_B denotes the image in $T(B)$ of t under the map $T(A) \to T(B)$ which is induced by the structure map of the A-algebra B. A functor $G^T : \mathbf{M}_k \to \mathbf{E}$ is defined at $A \in \mathbf{M}_k$ by

$$G^T(A) = \{\ x \in G(A)|\ t x_B t^{-1} = x_B \text{ for all } B \in \mathbf{M}_A \text{ and } t \in T(B)\}$$

G^T is a subgroup-scheme of G [13, p. 165]. Its group $G^T(k)$ of k-rational points can be identified with the centralizer $C_{G(k)}(T(k))$ of $T(k)$ in $G(k)$. But $T(k) = C_{G(k)}(T(k))$ since G is reductive [22, p. 159]. Moreover, G^T is smooth over k, hence reduced, by the "Smoothness of Centralizers Theorem" of [13, p.240]. It follows that the subgroup-schemes G^t and T of G coincide. This is the essential point in the proof of the following Theorem:

Theorem 3.2.5. *Let $S_1, \ldots, S_n$ be the finite collection of minimal closed connected normal subgroups of a semi-simple algebraic group G. Then the algebraic group homomorphism $m : S_1 \times \cdots \times S_n \to G$, which is defined by $(s_1, \ldots, s_n) \mapsto s_1 \cdot \ldots \cdot s_n$, is a fibration.*

Proof. That m is a homomorphism of algebraic groups comes from the fact that the subgroups $_i(k)$ of $G(k)$ centralize each other [22, p. 167]. It follows that the subgroups $S_i(A)$ of $G(A)$ centralize each other, and hence that the group $K(A)$ of A-points of the group-scheme kernel K of m is central in $(S_1 \times \cdots \times S_n)(A)$, for all $A \in \mathbf{M}_k$. Let T be a maximal torus of the semi-simple algebraic group $S_1 \times \cdots \times S_n$. Then $K(A)$ is centralized by $T(A)$ for every $A \in \mathbf{M}_k$, hence K is a closed subgroup-scheme of T by the above remarks. Thus, K is multiplicative and the Theorem is proved by applying Corollary 3.2.4. □

Since the kernel K in the proof of Theorem 3.2.5 is multiplicative and K(k) is finite [22, p.167], it follows that K has the form $D(\Gamma)$, where Γ is a finite abelian group. Write

$$\tilde{\Gamma} = \begin{cases} \Gamma & \text{if } \mathrm{char}(k) = 0, \text{ and} \\ \Gamma/\Gamma^{(p)} & \text{if } \mathrm{char}(k) = p, \end{cases}$$

where $\Gamma^{(p)}$ denotes the p-primary part of Γ. Then the simplicial group $D(\Gamma)(k_*)$ is discrete with group of vertices $\tilde{\Gamma}$, so that the long exact sequence for a fibration gives

Corollary 3.2.6. *If* $G, S_1, \ldots, S_n$ *are as in Theorem 3.2.5, then there is a short exact sequence of abelian groups*

$$0 \to \bigoplus_{i=1}^{n} \pi_1(S_i) \to \pi_1(G) \to \tilde{\Gamma} \to 0,$$

and isomorphisms $\bigoplus_{i=1}^{n} \pi_j(S_i) \cong \pi_j(G)$ *which are induced by* m *for all* $j > 1$.

If the characteristic of the underlying field k is 0, then every surjective homomorphism of algebraic groups over k is smooth, and hence has reduced group-scheme kernel. This is what allows us to prove

Theorem 3.2.7. *Suppose that* $\mathrm{char}(k) = 0$. *Then every surjective homomorphism* $f : G \to H$ *of algebraic groups over* k *is a fibration.*

Proof. Let K be the group-scheme kernel of f. As noted above, K is reduced and normal in G. Moreover, H can be identified with the quotient G/K as a sheaf of groups over k. Assume for the present that the Theorem has been proven in the case where both G and K are connected. Now suppose only that G is connected, and consider the diagram

$$\begin{array}{ccc} & & G \\ & \swarrow & \downarrow \pi \\ G/K^0 & \xrightarrow{\pi'} & G/K \end{array} \tag{3.2.1}$$

in which the morphisms are canonical projections and hence surjective. The kernel of π' is K/K^0, where K^0 denotes the connected component of the identity in K. K/K^0 is a closed normal finite subgroup of G/K^0. The connectedness of G/K^0 implies that K/K^0 is central in G/K^0, hence is abelian. Every finite abelian algebraic group over k is multiplicative if $\mathrm{char}(k) = 0$ [13, p. 517], so that π' is a fibration by Corollary 3.2.4. It follows from the assumption that π is a fibration as well.

For the case in which G and K are arbitrary, recall that the canonical map $\pi : G \to G/K$ is a finite union of copies of a surjective homomorphism

$$G^0 \cdot K \to (G^0 \cdot K)/K,$$

and that there is an isomorphism

$$\phi : (G^0 \cdot K)/K \xrightarrow{\cong} G^0/(G^0 \cap K)$$

of sheaves of groups over k such that the following diagram commutes:

$$\begin{array}{ccc} G^0 \cdot K & \longrightarrow & (G^0 \cdot K)/K \\ \cup & & \phi \downarrow \cong \\ G^0 & \longrightarrow & G^0/(G^0 \cap K) \end{array}$$

By a translation argument, we may assume that any map $\mathbb{A}_k^n \to G/K$ factors through $(G^0 \cdot K)/K$ and hence lifts to $G^0 \subset G^0 \cdot K \subset G$, whence $\pi : G \to G/K$ is a fibration.

It remains to prove the Theorem in the case where G and K are connected. Observe, first of all, that, if $R(G)$ is the solvable radical of G, then there is an isomorphism of algebraic groups

$$(G/K)/(R(G)/(K \cap R(G))) \cong (G/R(G))/(K/(K \cap R(G))).$$

These two quotients can be identified with a connected algebraic group H in such a way that there is a commutative diagram

$$\begin{array}{ccccccc}
K \cap R(G) & \subset & K & \longrightarrow & K/(K \cap R(G)) & \xleftarrow[m']{} & S_1 \times \cdots \times S_r \\
\cap & & \cap & & \downarrow j & & \downarrow \\
R(G) & \subset & G & \xrightarrow{p'} & G/R(G) & \xleftarrow[m]{} & S_1 \times \cdots \times S_n \\
\downarrow p & & \downarrow \pi & & \downarrow & & \downarrow pr \\
R(G)/(K \cap R(G)) & \xrightarrow{i} & G/K & \longrightarrow & H & \xleftarrow[m'']{} & S_{r+1} \times \cdots \times S_n
\end{array}$$

Here, $S_1, \ldots, S_n$ are the simple normal algebraic subgroups of the semi-simple algebraic group $G/R(G)$, and $S_1, \ldots, S_r$ are those simple normal algebraic subgroups which are contained n the closed connected normal subgroup

$$K/(K \cap R(G))$$

of $G/R(G)$. pr is the obvious projection map; it splits, so it is a fibration. The maps m and m' are surjective with finite central kernel [22, p. 167], so that m'' is as well. It follows from Corollary 3.2.4 that m'' is a fibration. p' is a fibration by Proposition 3.2.1. $K \cap R(G)$ is a solvable algebraic group, so that an argument similar to that which was used for the diagram (3.4) together with Proposition 3.2.1 shows that p is a fibration as well. To put it another way, we have seen that the homomorphisms pr, m'', p', and p are surjective on the level of $k[x_1, \ldots, x_n]$-points for all $n \geq 1$. A diagram chase using the exactness of the sequences in sight now shows that π as well is surjective on $k[x_1, \ldots, x_n]$-points for all $n \geq 1$. □

Theorem 3.2.7 does not hold if the characteristic of the underlying field k is non-zero. The homomorphism $F_p : \alpha_{\mathbf{k}} \to \alpha_{\mathbf{k}}$, which is defined over k of characteristic p by $F_p(x) = x^p$ for $x \in \alpha_k(A) = A$, $A \in \mathbf{M}_k$, is not even surjective on the level of $k[x_1]$-points, since x_1 has no p^{th} root in $k[x_1]$.

3.3 Chevalley groups

In previous sections, we have seen that the problem of computing $\pi_i(G)$ for a connected algebraic group G over k can be effectively reduced to the case where

G is a simple algebraic group, via the fibration theory. All simple algebraic groups may be constructed from representations of simple Lie algebras over the complex numbers $\mathbb{C}$, in a systematic way which is due to Chevalley [9] and Steinberg [45]. Groups which are produced in this way are called Chevalley groups over k. There is a nice class of naturally occurring surjective homomorphisms between Chevalley groups of a fixed type, which I call comparison maps. The purpose of this section is to show that these comparison maps are fibrations. Among other things, this will allow us to compute π_1 of a Chevalley group in terms of its "geometric fundamental group" and π_1 of the Chevalley group which is universal of the same type.

We begin with a quick review of notation. For more detail the reader should consult [21] and especially the notes of Steinberg [45].

The data for a Chevalley group consists of the following:

(1) a simple Lie algebra L over $\mathbb{C}$, together with a choice of maximal toral subalgebra H,

(2) the root system Φ of L relative to H, with a choice of a set of simple roots $\mathbf{\Delta} = \{\alpha_1, \dots, \alpha_r\} \subset \Phi$, where the rank of Φ is $r = \dim_{\mathbb{C}}(H)$,

(3) a Chevalley basis $\mathbf{B} = \{h_i,\ i = 1, \dots, r;\ x_\alpha,\ \alpha \in \Phi\}$ for L (see [21, p. 146]),

(4) a faithful representation $\rho : L \to g1(V)$, and

(5) an admissible lattice M for ρ

The subgroup $G_\rho(k)$ of $Gl(M \otimes_{\mathbb{Z}} k)$ which is generated by all of the $x_\alpha(t)$, $\alpha \in \Phi$, $t \in k$, where $x_\alpha(t)$ is the specialization of of $\exp(T\rho(x_\alpha)) \in Gl(M \otimes_{\mathbb{Z}} \mathbb{Z}[T])$ to $Gl(M \otimes_{\mathbb{Z}} k)$ at t, is what is known as a Chevalley group over k relative to the above data.

Since $x_\alpha(t+u) = x_\alpha(t) \cdot x_\alpha(u)$ for $t, u \in k$, $\alpha \in \Phi$, $G_\rho(k)$ is the group of k-rational points of an algebraic group G_ρ which is defined over the prime subfield of k, and moreover the assignment $t \mapsto x_\alpha(t)$ determines a homomorphism $s_\alpha : \alpha_k \to G_\rho$ of algebraic groups over k. We will say that the $x_\alpha(t),\ \alpha \in \Phi,\ t \in k$, are *elementary matrices* of $G_\rho(k)$.

Within $G_\rho(k)$, one has the following subgroups:

(1) $U(k)$, which is generated by the $x_\alpha(t)$, $t \in k$, where α is positive,

(2) $U^-(k)$, generated by the $x_\alpha(t)$, $t \in k$, where α is negative,

(3) $N(k)$, generated by the $w_\alpha(t)$, where $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, for $t \in k^*$, $\alpha \in \Phi$, and

(4) $H(k)$, which is generated by the $h_\alpha(t)$, where $h_\alpha(t) = w_\alpha(t)w_\alpha(-1)$, for $t \in k^*$, $\alpha \in \Phi$.

The groups $U(k)$, $U^-(k)$, and $H(k)$ are groups of k-rational points of algebraic groups U, U^-, and H respectively. H is a maximal torus of G_ρ, and U and U^- are maximal unipotent algebraic subgroups of G_ρ which are normalized by H. $B = U \cdot H$ and $B^- = U^- \cdot H$ are Borel subgroups of G_ρ. Also, $N(k)$ is the normalizer of $H(k)$ in $G_\rho(k)$, and an isomorphism from the Weyl group W of Φ to the quotient $N(k)/H(k)$ is defined on a simple reflection σ_α of W by $\sigma_\alpha \mapsto w_\alpha(1)$.

The weight lattice Γ_ρ of ρ can be identified with the character group $X(H)$ of H. G_ρ is said to be *universal* of type Φ if Γ_ρ is the entire abstract weight lattice Γ_1. If π is another faithful representation of L and $\Gamma_\pi \subset \Gamma_\rho$, then there is a unique homomorphism $\lambda_{\rho,\pi} : G_\rho \to G_\pi$, which is surjective on rational points, such that $\lambda_{\rho,\pi}(x_\alpha(t)) = x_\alpha(t)$. $\lambda_{\rho,\pi}$ is called a *comparison map*, at least for our purposes. One sees that every G_ρ has a covering by a Chevalley group of universal type, which is unique up to isomorphism and will be denoted by G_ϕ. Other comparison-type maps $i_* : G_{\rho\cdot i} \to G_\rho$ arise from the restrictions of the representation ρ to the Lie subalgebras L' of L which are generated by connected subdiagrams $\mathbf{\Delta}'$ of $\mathbf{\Delta}$. Again $i_*(x_\alpha(t)) = x_\alpha(t)$, $G_{\rho\cdot i}$ is of universal type if G_ρ is, and i_* is a closed immersion in general.

The special linear groups $Sl_{n+1}(k)$, $n \geq 1$, and the symplectic groups $Sp_{2m}(k)$, $m \geq 1$, are the groups of rational points of Chevalley groups over k which are universal of type A_n, $n \geq 1$, and of type C_m, $m \geq 1$, respectively. The elementary matrices of $Sl_{n+1}(k)$ are the matrices $X_{i,j}(t) = I + te_{i,j}$, $t \in k$, $1 \leq i \neq j \leq n+1$, where $e_{i,j}$ is the matrix which is 1 in the (i,j)-position and 0 elsewhere. The elementary matrices of $Sp_{2m}(k)$ are matrices of the types:

I. $I + t \cdot (e_{i,j} - e_{m+j,m+i}), \quad 1 \leq i \neq j \leq m,$

II. $I + t \cdot e_{i,m+i} \quad 1 \leq i \leq m,$

III. $I + t \cdot (e_{i,m+j} + e_{j,m+i}), \quad 1 \leq i \neq j \leq m,$

IV. $I + t \cdot e_{m+i,i}, \quad 1 \leq i \leq m$, and

V. $I + t \cdot (e_{m+i,j} + e_{m+j,i}), \quad 1 \leq i \neq j \leq m.$

The canonical inclusions $Sl_n(k) \subset Sl_{n+1}(k)$ and $Sp_{2m}(k) \subset Sp_{2m+2}(k)$ are instances of the maps i_*. In any given irreducible root system there are at most two root lengths. The root systems C_m are characterized by the fact that any two long roots are orthogonal. For this reason, a Chevalley group G is said to be *symplectic* if $\mathbf{\Delta}$ is one of the root systems C_m up to isomorphism, and *non-symplectic* otherwise. Observe that $Sl_2 = Sp_2$ is symplectic, and the Sl_n are non-symplectic for $n \geq 3$.

The *geometric fundamental group* $\Pi(G_\rho)$ of G_ρ is defined to be the quotient Γ_1/Γ_ρ. $\Pi(G_\rho)$ is a finite abelian group. If G_ρ is of type A_n, $n \geq 1$, then $\Pi(G_\rho)$ is a quotient of $\mathbb{Z}/(n+1)\mathbb{Z}$. If G_ρ is of type C_m, $m \geq 1$, then $\Pi(G_\rho)$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$. It follows that the groups of type C_m which can occur are Sp_{2m} and the adjoint group PSp_{2m}.

Suppose that ρ and π are representations of L with $\Gamma_\pi \subset \Gamma_\rho$. We are now in a position to show that the resulting comparison map $\lambda_{\rho,\pi} : G_\rho \to G_\pi$ is a fibration of algebraic groups. We have the subgroups $U(k)$, $U^-(k)$, $H(k)$, $N(k)$, and $B(k)$ of $G_\rho(k)$. The corresponding subgroups of $G_\pi(k)$ are denoted by $U_\pi(k)$, $U_\pi^-(k)$, $H_\pi(k)$, $N_\pi(k)$, and $B_\pi(k)$.

Lemma 3.3.1. *$\lambda_{\rho,\pi}^{-1}(U_\pi^-(k) \cdot B_\pi(k)) = U^-(k) \cdot B(k)$ in $G_\rho(k)$.*

Proof. We use the Bruhat decomposition. Explicitly,

$$G_\rho(k) = \bigcup_{w \in W} B(k)wB(k),$$

with $B(k)w_1B(k) = B(k)w_2B(k)$ if and only if $w_1 = w_2$ (the w in $B(k)wB(k)$ is really a coset representative in $N(k)$ for $w \in W$ via the isomorphism $W \to N(k)/H(k)$). Observe that $\lambda_{\rho,\pi}$ preserves all of the subgroups mentioned above, and induces an isomorphism $N(k)/H(k) \to N_\pi(k)/H_\pi(k)$ which commutes with the respective isomorphisms with W, so that $\lambda_{\rho,\pi}$ preserves the Bruhat decomposition as well. Now choose $w_0 \cdot G(k) = G(k)$ and $w_0 \cdot W = W$, we see that

$$G_\rho(k) = \bigcup_{w \in W} U^-(k)wB(k),$$

with

$$U^-(k)w_1B(k) = U^-(k)wB(k)$$

if and only if $w_1 = w_2$, and that $\lambda_{\rho,\pi}$ preserves this decomposition. Thus, if $x \in U^-(k)wB(k)$ and $\lambda_{\rho,\pi}(x) \in U_\pi^-(k)B_\pi(k)$, then $w = e$ and $x \in U^-(k)B(k)$, and the Lemma is proved. □

$U^-B = U^-HU$ is the big cell of G_ρ. It is an affine open subvariety of G_ρ, and the multiplication map $m : U^- \times H \times U \to U^-HU$ is an isomorphism of k-schemes. It follows from Lemma 3.3.1 that the commutative square

$$\begin{array}{ccc} U^- \times H \times U & \xrightarrow{m} & G_\rho \\ \downarrow & & \downarrow{\scriptstyle \lambda_{\rho,\pi}} \\ U_\pi^- \times H_\pi \times U_\pi & \xrightarrow{m} & G_\pi, \end{array}$$

in which the map on the left is induced by the restriction of $\lambda_{\rho,\pi}$ to each factor, is a pullback in the category of schemes over k.

For the structure of U (respectively U_π), choose an ordering of the positive roots of ϕ which is consistent with addition, and suppose that n is the number of such roots. Then the map $\mathbb{A}_k^n \to U$ defined by

$$(t_1, \dots, t_n) \mapsto x_{\alpha_1}(t_1) \cdot \dots \cdot x_{\alpha_n}(t_n),$$

where the α_i are arranged in the indicated order, is an isomorphism of k-schemes [45, p.63]. A similar statement holds for U^- (respectively U_π^-). It follows that the maps $\lambda_{\rho,\pi}|_{U^-}$ and $\lambda_{\rho,\pi}|_U$ are isomorphisms of group-schemes over k.

Theorem 3.3.2. *$\lambda_{\rho,\pi}$ is a fibration of algebraic groups, with group-scheme kernel $K_{\rho,\pi}$ given at $A \in \mathbf{M}_k$ by $K_{\rho,\pi}(A) = \mathbf{Ab}(\Gamma_\rho/\Gamma_\pi, A^*)$*

Proof. From the discussion above, we see that $K_{\rho,\pi}(A) \subset H(A)$ for every $A \in \mathbf{M}_k$, and hence that $K_{\rho,\pi}$ is a closed subgroup-scheme of H. But then $K_{\rho,\pi}$ is multiplicative, and so $\lambda_{\rho,\pi}$ is a fibration by Corollary 3.2.4. Steinberg shows [45, p. 43] that every $h \in H(k)$ can be written $h = h_{\alpha_1}(t_1)\cdot \ldots \cdot h_{\alpha_n}(t_n)$, where $\mathbf{\Delta} = \{\alpha_1, \ldots, \alpha_n\}$. It is an exercise to show that, for $\mu \in \Gamma_\rho$, the assignment

$$\hat{\mu}(h_{\alpha_1}(t_1) \cdot \ldots \cdot h_{\alpha_n}(t_n)) = t_1^{\mu(h_{\alpha_1})} \cdot \ldots \cdot t_n^{\mu(h_{\alpha_n})},$$

where the h_{α_i} are co-roots corresponding to the α_i in the maximal toral subalgebra of L, defines a character $\hat{\mu}$ in $X(H)$ and an isomorphism $\eta_\rho : \Gamma_\rho \to X(H)$. It follows that the diagram

$$\begin{array}{ccc} \Gamma_\pi & \xrightarrow{\eta_\pi} & X(H_\pi) \\ \cap & & \big\downarrow \lambda^*_{\rho,\pi} \\ \Gamma_\rho & \xrightarrow{\eta_\rho} & X(H) \end{array}$$

commutes. But then $\lambda_{\rho,\pi}|_H(A) : H(A) \to H_\pi(A)$ is the map

$$\mathbf{Ab}(\Gamma_\rho, A^*) \to \mathbf{Ab}(\Gamma_\pi, A^*)$$

for every $A \in \mathbf{M}_k$, whence the Theorem. □

Let $\Pi(G_\rho = \Gamma_1/\Gamma_\rho$ be the geometric fundamental group of G_ρ as before and define

$$\tilde{\Pi}(G_\rho) = \begin{cases} \Pi(G_\rho) & \text{if } \mathrm{char}(k) = 0, \text{ and} \\ \Pi(G_\rho)/\Pi(G_\rho)^{(p)} & \text{if } \mathrm{char}(k) = p, \end{cases}$$

where $\Pi(G_\rho)^{(p)}$ is the p-primary component of the finite group $\Pi(G_\rho)$. Then we have

Corollary 3.3.3. *There is a short exact sequence of abelian groups*

$$0 \to \pi_1(G_\phi) \xrightarrow{\lambda_*} \pi_1(G_\rho) \to \tilde{\Pi}(G_\rho) \to 0,$$

where λ_* is the homomorphism which is induced by the universal covering map $\lambda : G_\phi \to G_\rho$.

Proof. $\tilde{\Pi}(G_\rho)$ is the group of rational points of the group-scheme kernel of λ. □

Chapter 4

Algebraic K-theory

Chapter Introduction

Perhaps a historical remark is in order at this point. We have been using a simplicial algebra k_* which is defined over an algebraically closed field k in order to define homotopical invariants of affine k-schemes. Obviously the construction of k_* generalizes to produce complexes A_* for arbitrary rings A. My interest in such complexes originally came from the fact that $\mathbb{Q}_*$ may be used to define the $\mathbb{Q}$-algebra $A^0(X)$ of Sullivan-de Rham 0-forms of a simplicial set X, just by setting $A^0(X) = \mathbf{S}(X, \mathbb{Q}_*)$. This construction first appeared explicitly in a paper of Swan [46], and later in an AMS Memoir of Bousfield and Guggenheim [5].

But the complexes A_* have a history in algebraic K-theory as well. Rector showed [43] in 1971 that the homotopy groups of the simplicial group $Gl(A_*)$ coincide with the Karoubi-Villamayor K-theory of the ring A, up to a dimension shift. Using this result, Gersten showed [17, p. 28] that there is an isomorphism

$$K_{i+1}(A) \cong \pi_i(Gl(A_*)), \quad \text{for } i \geq 0, \tag{4.0.1}$$

if A is a regular Noetherian ring, where the K-theory is that of Quillen. Anderson gives a proof of this result in [1] which uses some results of Quillen's on the group completion of a simplicial monoid.

For us, the interest in this isomorphism lies in the fact that the homotopy groups $\pi_i(Gl(k_*))$ are the homotopy groups of the inductive affine algebraic group Gl, as defined in Chapter 2. It seems worthwhile, therefore, to include a very explicit proof of the isomorphism (4.1) here. This is done in the first section of this Chapter after a few preliminaries about Quillen K-theory are mentioned. The proof that is given here differs from that of Gersten or Anderson in that the bar construction $BGl(A_*)$ or the simplicial group $Gl(A_*)$ is explicitly identified as $BGl(A)^+$ when A is assumed to be Noetherian and regular.

These assumptions on the ring A are made in order that we may use the theorem of Quillen which asserts that the canonical inclusion $A \subset A[x_1]$ induces an isomorphism $K_i(A) \cong K_i(A[x_1])$ for all $i \geq 1$ if A is Noetherian and regular.

This isomorphism may be thought of as good stable behaviour, in a way that will become clear in the second section, where this result is used together with certain theorems of Matsumoto to show that there are isomorphisms

$$\pi_1(Sl_n) \cong K_2(k) \quad \text{for } n \geq 2\text{, and} \tag{4.0.2}$$

$$\pi_1(Sp_{2m}) \cong K_2(k) \quad \text{for } m \geq 1, \tag{4.0.3}$$

where Sl_n and Sp_{2m} are the obvious group-schemes defined over an algebraically closed field k which correspond to special linear and symplectic groups respectively. No assumption is made on the characteristic of k here. It should be pointed out, however, that the lower bound for the first isomorphism and the lack of a condition on the characteristic of k for the second depends very much on the fact that k is algebraically closed. Something can be said about the case where k is not algebraically closed; this will be sketched at the end of the section.

The third and final section of this Chapter contains a vanishing theorem for $K_2(k)$ of an algebraically closed field k, which depends on an analysis of K_2 of a global field that was given by Bass and Tate [4]. Explicitly, $K_2(k) = 0$ if the Kroenecker dimension of k is strictly less than 2, but is a non-trivial uniquely divisible abelian group otherwise. A calculation of the fundamental group of an arbitrary Chevalley group of type A_n or C_m in terms of $K_2(k)$ and the geometric fundamental group is also given.

In closing, let me say that Jan Strooker has kindly informed me that the isomorphism (4.2) could have been inferred from a result of Krusemeyer [27, p. 23], by showing that the result of Rector which was cited above can be destabilized. Krusemeyer has a notion of path-connectedness of an algebraic group which coincides with that given in Chapter 2. He also cites results which are crucial for the proof of the isomorphism (4.3).

4.1 Quillen K-theory

As mentioned in the Introduction, this section begins with a brief review of what, for us, are the basics of Quillen's K-theory of a unitary ring. Details of these constructions can best by found in Loday's thesis [28].

One starts with a unitary ring A and considers the classifying space $BGl(A)$ of the discrete group $Gl(A)$ as the realization of the nerve of the appropriate one-object groupoid. The subgroup of elementary matrices $E(A)$ is a perfect normal subgroup of $Gl(A) \cong \pi_1(BGl(A))$, by the Whitehead Lemma. By adding 2-cells and 3-cells to $BGl(A)$ one forms a complex $BGl(A)^+$ in such a way that the inclusion $i : BGl(A) \subset BGl(A)^+$ induces an isomorphism in homology with any (twisted) coefficients, and induces a map i_* on the level of the fundamental group which fits into a commutative diagram

$$\begin{array}{ccc}
\pi_1(BGl(A)) & \xrightarrow{i_*} & \pi_1(BGl(A)^+) \\
\cong \downarrow & & \downarrow \cong \\
Gl(A) & \longrightarrow & Gl(A)/E(A)
\end{array}$$

where the bottom arrow is the canonical surjection.

An obstruction-theoretic argument shows that $i : BGl(A) \to BGl(A)^+$ is universal in the pointed homotopy category for maps $f : BGl(A) \to X$ satisfying $f_*(E(A)) = e$ in $\pi_1(X)$, in the sense that there is a map f^+, which is unique up to pointed homotopy, such that the diagram

$$\begin{array}{ccc}
BGlA & \longrightarrow & BGl(A)^+ \\
& {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\
& X &
\end{array}$$

homotopy commutes. It is an easy consequence that the assignment

$$A \mapsto \pi_i(BGl(A)^+)$$

determines a functor from unitary rings to abelian groups for all $i \geq 1$. We write $K_i(A) = \pi_i(BGl(A)^+)$, for $i \geq 1$; these are the Quillen K-groups of the ring A. Observe that $K_1(A) \cong Gl(A)/E(A)$. One may also show that there is an isomorphism $K_2(A) \cong H_2(E(A); \mathbb{Z})$.

It is also important to know that the group homomorphism $\oplus : Gl(A) \times Gl(A) \to Gl(A)$, which is defined by

$$(M \oplus N)_{i,j} = \begin{cases} M_{k,r} & \text{if } (i,j) = (2k-1, 2r-1), \\ N_{k,r} & \text{if } (i,j) = (2k, 2r), \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

for matrices M and N, induces the structure of an H-space on $BGl(A)^+$. It follows from this and the universal property of the $+$ construction that, if a CW complex X is an H-space and there is a map $BGl(A) \to X$ which induces an isomorphism of integral homology, then X has the homotopy type of $BGl(A)^+$.

We conclude this survey of preliminaries by noting the following result of Quillen:

Theorem 4.1.1. *The inclusion $A \subset A[x_1]$ induces isomorphisms $K_i(A) \cong K_i(A[x_1])$ for all $i \geq 1$ if A is a Noetherian regular ring.*

The proof of Theorem 4.1.1 is started in [41] and finished in [18].

Now we prove

Theorem 4.1.2. *There are isomorphisms $\pi_i(Gl(A_*)) \cong K_{i+1}(A)$ for all $i \geq 0$ if A is a regular Noetherian ring.*

The proof of this Theorem requires a certain amount of simplicial group theory. Let G be a simplicial group. Applying the classifying space fibration functor $E(\ ) \to B(\ )$ to each of the groups G_p of p-simplices of G yields a homomorphism $E(G) \to B(G)$ of bisimplicial sets, where $E(G)_{p,q} = E(G_p)_q$ and $B(G)_{p,q} = B(G_p)_q$. The diagonal simplicial set $dE(G)$ of $E(G)$ has p-simplices

$$dE(G)_p = \{(g_p, \ldots, g_0) |\ g_i \in G_p\},$$

and faces and degeneracies defined by

$$d_i(g_p, \ldots, g_0) = (d_i g_p, d_i g_{p-1}, \ldots, d_i g_{p-i}, d_i g_{p-i-1}, \ldots, d_i g_0), \text{ and}$$

$$s_i(g_p, \ldots, g_0) = (s_i g_p, \ldots, s_i g_{p-i}, e, s_i g_{p-i-1}, \ldots, s_i g_0)$$

(see, for example, [12, p. 161]). The actions of G_p on $E(G_p)_p$ yield an action of G on $dE(G)$ in such a way that the map $dE(G) \to dB(G)$ is a principal Kan fibration. This fibration is classified by a G-equivariant morphism $dE(G) \to E(G)$, which is defined on p-simplices by

$$(g_p, \ldots, g_0) \mapsto (g_p, d_0 g_p, \ldots, d_0^p g_0) \in E(G_p).$$

Since the realization of a Kan fibration is a Serre fibration [38], we obtain a morphism of Serre fibrations

$$\begin{array}{ccc} dEG & \longrightarrow & E(G) \\ \downarrow & & \downarrow \\ dB(G) & \longrightarrow & B(G), \end{array}$$

with common fibre G. A Lemma of Quillen [41, p. 86] asserts that the *space* $dE(G)$ is the realization of the simplicial space whose p^{th} object is the realization of $E(G_p)$. Finally, since the spaces $E(G_p)$ are all contractible, a Theorem of May [34, p. 107] implies that the space $dE(G)$ is contractible as well. It follows that $dB(G)$ has the homotopy type of $B(G)$; in particular, these two spaces have isomorphic integral homology. The homology spectral sequence of a double complex [30, p. 340], together with a generalized Eilenberg-Zilber Theorem of Dold and Puppe [15, p. 213], gives

Lemma 4.1.3. *Associated to any simplicial group G, there is a spectral sequence, whose E^{-1}-term is given by $E^1_{p,q} = H_q(G_p; \mathbb{Z})$, and which converges to $H_*(B(G); \mathbb{Z})$. This construction is functorial in G.*

Lemma 4.1.4. *For an arbitrary ring A, the direct sum homomorphism*

$$\oplus : Gl(A_*) \times Gl(A_*) \to Gl(A_*)$$

induces an H-space structure on $BGl(A_*)$.

Proof. The proof is achieved by mimicking the proof given by Loday [28, p. 321] for the corresponding result about $BGl(A)^+$. Let $u : N \to N$ be an injective function, where N denotes the set of natural numbers. u induces a homomorphism $u : Gl(A_*) \to Gl(A_*)$ via

$$u(M)_{i,j} = \begin{cases} M_{k,r} & \text{if } (i,j) = (u(k)), u(r)), \text{ and} \\ \delta_{i,j} & \text{otherwise,} \end{cases}$$

for matrices M. u preserves elementary matrices, and hence induces

$$u : E(A_*) \to E(A_*).$$

Think of strings of the form (e,..., e) as a base point e for $BGl(A_*)$, and observe that the composition

$$BGl(A_*) \xrightarrow{(id,e)} BGl(A_*) \times BGl(A_*) \xrightarrow{\oplus} BGl(A_*)$$

is the map $u : BGl(A_*) \to BGl(A_*)$ which is induced by $u(i) = 2i - 1$. A similar statement holds on the right. The Lemma will be proved, then, once we have shown that all such maps u are pointed-homotopic to the identity on $BGl(A_*)$. This is done by showing that u is a homotopy equivalence, for then a Lemma of Loday [28, p. 322], which asserts that the monoid of in jective self-maps of N has trivial Grothendieck group, finishes the job. But $BE(A_*)$ is simply-connected since $E(A_*)$ is path-connected, so it suffices to show that $u : BE(A_*) \to BE(A_*)$ is an isomorphism in integral homology, for then u induces a homotopy equivalence on $E(A_*)$, hence on $Gl(A_*)$, and hence on $BGl(A_*)$. Finally to see that u is an integral homology isomorphism, we use a result of Loday [28, p. 321] which says that $u : BE(A) \to BE(A)$ is an $H_*(\ ;\mathbb{Z})$ isomorphism for arbitrary A, and then compare the spectral sequences of Lemma 4.1.3 □

What remains for the proof of Theorem 4.1.2 is

Lemma 4.1.5. *The canonical map $j : BGl(A) \to BGl(A_*)$ induces an $H_*(\ ;\mathbb{Z})$ isomorphism if A is a regular Noetherian ring.*

Proof. The map j comes from thinking of A as a discrete simplicial ring, and then taking the map which is induced by $j : Gl(A) \to Gl(A_*)$, which is in turn induced on the level of n-simplices by the inclusion $A \subset A[x_1, \ldots, x_n]$. The Lemma is proved by comparing the spectral sequences of Lemma 4.1.3 for $Gl(A)$ and $Gl(A_*)$ via j, using the observation that Theorem 4.1.1 implies that the induced map $BGl(A) \to BGl(A[x_1, \ldots, x_n])$ is an isomorphism on $H_*(\ ;\mathbb{Z})$. □

It follows from Lemma 4.1.4 and Lemma 4.1.5 that $BGl(A_*)$ is pointed homotopy equivalent to $BGl(A)^+$ if A is a regular Noetherian ring. This finishes the proof of Theorem 4.1.2, for then

$$K_{i+1}(A) \cong \pi_{i+1}(BGl(A_*)) \cong \pi_i(Gl(A_*))$$

if $i \geq 0$.

In the context of Chapters 2 and 3 of inductive affine k-schemes over an algebraically closed field k, Theorem 4.1.2 implies that the homotopy groups of the inductive algebraic group Gl coincide with the K-theory of k, up to a dimension shift. A well known result of Quillen [40, p. 40], when paraphrased, asserts that $K_{2i-1}(k) = k^*$ and$K_{2i}(k) = 0$ for all $i \geq 1$ if k is the algebraic closure of a finite field. Not much is known, however, about the higher K-theory of more general algebraically closed fields k, beyond a certain amount of number-theoretic information about $K_2(k)$. The theory of K_2 of a field is based on a Theorem of Matsumoto [32], that is now described, which gives a presentation for the group $H_2(G_\Phi(k);\mathbb{Z})$, where G_Φ is a Chevalley group which is defined over k and is of universal type for an indecomposable root system Φ.

The following relations are satisfied in $G_\Phi(k)$:

R1 $x_\alpha(t)\cdot x_\alpha(u) = x_\alpha(t+u)$ for $\alpha \in \Phi$, and $t, u \in k$,

R2 $[x_\alpha(t), x_\alpha(u)] = \prod x_{i\alpha+j\beta}(C_{i,j}t^iu^j)$, if $\alpha+\beta \neq 0$, where the product is taken over all roots $i\alpha + j\beta$, $i, j \in \mathbb{N}$, arranged in some fixed order, and where the $C_{i,j}$ are integers which depend only on the ordering and on α and β. Moreover, $C_{1,1} = N_{\alpha,\beta}$, which is the integer such that $[x_\alpha, x_\beta] = N_{\alpha,\beta}x_{\alpha+\beta}$ in the simple Lie algebra L which gives rise to G_Φ(see §3 of Chapter 3).

R3 $w_\alpha(t)\cdot x_\alpha(u)\cdot w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$ for $t \in k^*$ and $u \in k$ (recall that $w_\alpha(t) = x_\alpha(t)\cdot x_{-\alpha}(-t^{-1})\cdot x_\alpha(t)$), and

M $h_\alpha(t\cdot s) = h_\alpha(t)\cdot h_\alpha(s)$ for $t, s \in k^*$ (recall that $h_\alpha(t) = w_\alpha(t)\cdot w_\alpha(-1)$).

Steinberg shows [45, p. 78] that, since k is large enough (ie. infinite), the symbols $x_\alpha(t)$, $\alpha \in \Phi$, $t \in k$, with the relations **R1**, **R2**, **R3** and **M** together give a presentation for $G_\Phi(k)$.

One sees that $G_\Phi(k)$ is perfect by using the relation (see [45, p. 52])

$$[h_\alpha(t), x_\alpha(u)] = x_\alpha((t^2-1)\cdot u),$$

valid for all $\alpha \in \Phi$, $t \in k^*$, and $u \in k$. The Steinberg group $St_\Phi(k)$ is defined to be the group which is generated by the symbols $x_\alpha(u)$, $u \in k$, $\alpha \in \Phi$, subject to the relations **R1** and **R2** if $\mathrm{rank}(\Phi) > 1$, or **R1** and **R3** if $\mathrm{rank}(\Phi) = 1$, where the elements $w_\alpha(t)$ and $h_\alpha(t)$ are defined analogously to the above. If $\mathrm{rank}(\Phi) > 1$, then the relation **R3** is a consequence of **R1** and **R2**. Steinberg has shown that the canonical homomorphism $\pi : St_\Phi(k) \to G_\Phi(k)$ is a universal central extension of $G_\Phi(k)$. It follows that $Ker(\pi)$ is isomorphic to the group $H_2(G_\Phi(k);\mathbb{Z})$.

$H_2(G_\Phi(k);\mathbb{Z})$ measures the failure of the relations **R1**, **R2**, and **R3** to imply that h_α is multiplicative in $St_\Phi(k)$. More precisely, let α be a long root of Φ and define elements $c_\alpha(t, s)$ of $Ker(\pi)$ for $t, s \in k^*$ by

$$c_\alpha(t, s) = h_\alpha(t)\cdot h_\alpha(s)\cdot h_\alpha(t\cdot s)^{-1}$$

Matsumoto shows that $c = c_\alpha$ satisfies the following relations [32, p. 26]:

S1 $c(x,y) \cdot c(xy,z) = c(x,yz) \cdot c(y,z)$,

S2 $c(1,1) = 1$, $c(x,y) = c(x^{-1},y^{-1})$, and

S3 $c(x,y) = c(x,(1-x)y)$ if $x \neq 1$,

for all x, y, and z in k^*. Moreover, if G_Φ is non-symplectic, then the existence of a long root β which is not orthogonal to α forces $c = c_\alpha$ to satisfy:

S1^o $c(xy,z) = c(x,z) \cdot c(y,z)$,

S2^o $c(x,yz) = c(x,y) \cdot (x,z)$, and

S3^o $c(x,1-x) = 1$ if $x \neq 1$.

The group $S(k^*)$ of *Steinberg cycles* over k^* is defined to be the abelian group which is generated by the symbols $c(x,y)$, $x,y \in k^*$, and subject to the relations **S1**, **S2**, and **S3**. The group $S^o(k^*)$ of *bilinear Steinberg cycles* over k^* is the abelian group which is generated by $c(x,y)$, $x,y \in k^*$, but subject to the relations **S1**^{o}, **S2**^{o}, and **S3**^{o}. There is a canonical surjective homomorphism $S(k^*) \to S^o(k^*)$. Matsumoto's Theorem is the following [32, p.30]:

Theorem 4.1.6. *Let α be a long root of Φ. Then the homomorphism $S(k^*) \to Ker(\pi)$, which is defined by $c(t,s) \mapsto c_\alpha(t,s)$ for $t,s \in k^*$, is an isomorphism if G_Φ is symplectic, and factors through an isomorphism $S^o(k^*) \to Ker(\pi)$ if G_Φ is non-symplectic.*

The field k does not need to be algebraically closed for Theorem 4.1.6 to hold; the proof given in [32] requires only that k be infinite. For such a field k, define a Steinberg *cocycle* over k^* with coefficients in an abelian group A to be a member of $\mathbf{Ab}(S(k^*),A)$. Bilinear Steinberg cocycles are defined similarly. The Steinberg cocycles and the bilinear Steinberg cocycles with coefficients in A form abelian groups, which are denoted by $S(k^*,A)$, and $S^o(k^*,A)$ respectively. $S^o(k^*,A)$ is a subgroup of $S(k^*,A)$. Matsumoto has shown [32, p.28]

Proposition 4.1.7. *If c is a Steinberg cocycle over k^* with coefficients in A, then the function c_2, which is defined on the set $k^* \times k^*$ by $c_2(x,y) = c(x,y^2)$, is a bilinear Steinberg cocycle over k^* with coefficients in A. Moreover, $c(x,y^2) = c(x^2,y)$ for all x and y in k^*.*

Krusemeyer remarks in [27, p.23] that it is a consequence of this last result that, if an infinite field k is quadratically closed, then every Steinberg cocycle on k^* is bilinear.

Corollary 4.1.8. *The canonical map $S(k^*) \to S^o(k^*)$ is an isomorphism if k is algebraically closed.*

The constructions of Matsumoto's Theorem also respect the comparison maps i_* of §3 of Chapter 3. This yields the following stability result (recall that Sl_2 is symplectic):

Corollary 4.1.9. *If k is an infinite field, then*

1. *the canonical inclusion $i : Sl_n(k) \subset Sl_{n+1}(k)$ induces isomorphisms*

$$H_2(Sl_n(k); \mathbb{Z}) \cong H_2(Sl_{n+1}(k); \mathbb{Z}) \cong S^o(k^*)$$

for $n \geq 3$, and

2. *the following diagram commutes:*

$$\begin{array}{ccc} S(k^*) & \longrightarrow & S^o(k^*) \\ \cong \downarrow & & \downarrow \cong \\ H_2(Sl_2(k); \mathbb{Z}) & \xrightarrow{i_*} & H_2(Sl_3(k); \mathbb{Z}), \end{array}$$

where the vertical maps are the isomorphisms of Theorem 4.1.6.

Putting Corollary 4.1.8 together with Corollary 4.1.9 shows that there are isomorphisms

$$H_2(Sl_n(k); \mathbb{Z}) \cong H_2(Sl_{n+1}(k); \mathbb{Z}) \cong S^o(k^*) \text{ for } n \geq 2,$$

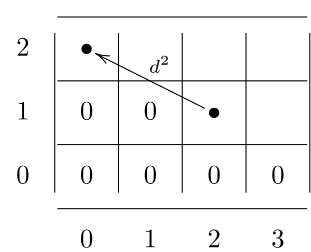
if k is an algebraically closed field.

4.2 The fundamental group

We begin with some generalities about simplicial groups. As in §1, let G be a simplicial group, and recall the spectral sequence of Lemma 4.1.3. Suppose that

(1) the groups G_0 and G_1 are perfect.

Then the E^2-term of the spectral sequence is as follows:



It follows that there is a well-defined composition

$$H_2(G_0; \mathbb{Z}) = E^1_{0,2} \to E^2_{0,2} \to E^\infty_{0,2} = H_2(BG; \mathbb{Z})$$

Observe that the map $E^1_{0,2} \to E^2_{0,2}$ is the cokernel of the map $(d_{0^*} - d_{1^*}) : H_2(G_1; \mathbb{Z}) \to H_2(G_0; \mathbb{Z})$. Thus, the above composition is an isomorphism if the extra conditions

(2) $s_{0^*} : H_2(G_0; \mathbb{Z}) \to H_2(G_1; \mathbb{Z})$ is an isomorphism, and

(3) G_2 is perfect

are satisfied by G. Finally, if we assume that

(4) G is path-connected,

then we may identify $H_2(BG; \mathbb{Z})$ with $\pi_1(G)$, for then BG is simply-connected, and there are isomorphisms

$$H_2(BG; \mathbb{Z}) \cong \pi_2(BG) \cong \pi_1(G).$$

All of these identifications and maps are natural for the simplicial groups G for which they make sense.

Now consider the algebraic group Sl_n, which is defined over an algebraically closed field k, and suppose that $n \geq 3$. $\pi_1(Sl_n)$ is defined, as before, to be the fundamental group of the simplicial group $Sl_n(k_*)$. $Sl_n(k)$ and $Sl_n(k[x_1])$ are perfect, since both are generated by elementary transformations, and the respective elementary transformation groups are perfect by the relation

$$[h_\alpha(t), x_\alpha(u)] = x_\alpha((t^2 - 1)u),$$

which is valid for all $\alpha \in A_{n-1}$, $u \in k[x_1]$, and $t \in k^*$. Observe also that, for all n, the simplicial groups $Sl_n(k_*)$ are path-connected by the results of Chapter 2, hence so is $Sl(k_*)$. By the discussion above, there is a commutative diagram

$$\begin{array}{ccc} H_2(Sl_n(k); \mathbb{Z}) & \twoheadrightarrow & H_2(BSl_n(k_*); \mathbb{Z}) \cong \pi_1(Sl_n(k_*)) \\ \downarrow & & \downarrow \\ H_2(Sl(k); \mathbb{Z}) & \twoheadrightarrow & H_2(BSl(k_*); \mathbb{Z}) \cong \pi_1(Sl(k_*)) \end{array}$$

for which the vertical maps are induced by the canonical inclusion $Sl_n(k_*) \subset Sl(k_*)$, and the horizontal maps come from the respective spectral sequences. Observe that $H_2(Sl_n(k); \mathbb{Z}) \to H_2(Sl(k); \mathbb{Z})$ is an isomorphism for $n \geq 2$ by the remark following Corollary 4.1.9.

The inclusion $k \subset k[x_1, \ldots, x_n]$ induces an isomorphism

$$K_i(k) \cong K_i(k[x_1, \ldots, x_n])$$

for all $i, n \geq 1$, by Theorem 4.1.1. It is an easy consequence of the K_1 isomorphism that $E(k[x_1, \ldots, x_n]) = Sl(k[x_1, \ldots, x_n])$ in $Gl(k[x_1, \ldots, x_n])$ for all $n \geq 1$, so, in particular, $Sl(k[x_1, x_2])$ is perfect. Observe also that the homomorphism $s_{0^*} : H_2(Sl(k); \mathbb{Z}) \to H_2(Sl(k[x_1]); \mathbb{Z})$ which is induced by $k \subset k[x_1]$; this is an isomorphism, again by Theorem 4.1.1. We have shown that the simplicial group $Sl(k_*)$ satisfies the conditions (1)-(4) which are given above. It follows that the bottom horizontal map in the above diagram is an isomorphism, so that the top map is an isomorphism as well. We have proved

Proposition 4.2.1. *There are isomorphisms*

$$\pi_1(Sl_n) \cong H_2(Sl_n(k);\mathbb{Z}) \cong K_2(k)$$

for all $n \geq 2$ if k is algebraically closed.

One of the basic components of the proof of Proposition 4.2.1 was the fact that the groups $Sl_n(k[x_1])$ are perfect for $n \geq 2$. In a similar way, starting from a result of Matsumoto [31, p. 102], which says that the groups $G_\Phi(k[x_1])$ of $k[x_1]$-points of a Chevalley group G_Φ of universal type over k is generated by elementary transformations, once can show that all such groups $G_\Phi(k[x_1])$ are perfect. Thus, there is a surjective homomorphism

$$(*) \qquad H_2(G_\Phi(k);\mathbb{Z}) \to H_2(BG_\Phi(k_*);\mathbb{Z}) \cong \pi_1(G_\Phi(k_*))$$

which arises from the spectral sequence for $H_*(BG_\Phi(k_*);\mathbb{Z})$ just as before. Now we may prove

Theorem 4.2.2. *If k is an algebraically closed field, then there is an isomorphism $H_2(Sp_{2m}(k);\mathbb{Z}) \cong \pi_1(Sp_{2m})$, for every $m \geq 1$.*

Proof. The fixed points of the non-trivial diagram automorphism of the Lie algebra $s1_{2m}(C)$ form a simple Lie algebra of type C_m in such a way that the root for $s1_{2m}(C)$ corresponding to the $(m, m+1)$ entry of the matrices involved restricts uniquely to a long root α on C_m. Moreover, the induced representation of L is of universal type. Thus, the Chevalley group over k which arises from this representation can be identified with Sp_{2m},and gives rise to an imbedding $i : Sp_{2m} \subset Sl_{2m}(k)$ which has the property that the elements of the form $x_\alpha(u)$ and $x_{-\alpha}(u)$ may be identified with $X_{m,m+1}(u)$ and $S_{m+1,m}(u)$ respectively, for all $u \in k$. It follows that there is a commutative diagram

$$\begin{array}{ccc} S(k^*) & \longrightarrow & S^o(k^*) \\ \cong \downarrow & & \downarrow \cong \\ H_2(Spp_{2m}(k);\mathbb{Z}) & \xrightarrow{i_*} & H_2(Sl_{2m}(k);\mathbb{Z}), \end{array}$$

in which the top horizontal map is the canonical one. But this homomorphism is an isomorphism since k is algebraically closed, by Corollary 4.1.8, so that i_* is an isomorphism as well. Comparing spectral sequences as in the proof of Proposition 4.2.1 yields a commutative diagram

$$\begin{array}{ccc} H_2(Sp_{2m}(k);\mathbb{Z}) & \twoheadrightarrow & H_2(BSp_{2m}(k_*);\mathbb{Z}) \cong \pi_1(Sp_{2m}(k_*)) \\ \cong \downarrow i_* & & \downarrow i_* \\ H_2(Sl_{2m}(k);\mathbb{Z}) & \xrightarrow{\cong} & H_2(BSl_{2m}(k_*);\mathbb{Z}) \cong \pi_1(Sl_{2m}(k_*)), \end{array}$$

in which the bottom morphism is an isomorphism by Proposition 4.2.1. It follows that the top homomorphism is injective as well as surjective, and the Theorem is proved. □

An imbedding $Sp_{2m}(k) \subset Sl_{2m}(k)$ which is suitable for the proof of Theorem 4.2.2 may also be obtained by applying a Lemma of Matsumoto [32, p. 37]. One conjectures that there are isomorphism

$$\pi_1(G_\Phi) \cong H_2(G_\Phi(k); \mathbb{Z}) \cong K_2(k),$$

if G_Φ is any Chevalley group of universal type over any algebraically closed field k.

There is some basis for believing that the conjecture is true over more general fields. Steinberg showed [45, p. 72], for example, that $H_2(G_\Phi(k); \mathbb{Z}) \cong 0$ for all finite fields k and indecomposable root systems Φ such that $|k| \geq 4$ and $|k| \neq 9$ if $\mathrm{rank}(\Phi) = 1$ (these are the cases where $St_\Phi(k)$ may not be a universal central extension of $G_\Phi(k)$ [45, p. 78]). The Matsumoto result that was used to construct the homomorphism $(*)$ above is broad enough so that there is such a surjective homomorphism for all fields k outside of the small list just referred to (from now on the expression "almost all fields" will mean every field except those in the list.) This proves

Proposition 4.2.3. *$H_2(G_\Phi(k); \mathbb{Z}) \cong \pi_1(G_\Phi) = 0$ for all finite fields k and indecomposable root systems Φ such that $|k| > 4$ and $|k| \neq 9$ if $\mathrm{rank}(\Phi) = 1$.*

Proposition 4.2.1 is also true much more generally, in view of the remark of the last section which says that Theorem 4.1.6 is valid over all infinite fields. It follows that the argument given for Proposition 4.2.1 generalizes to all infinite fields, provided that we avoid the symplectic group Sl_2. Putting this together with the last Proposition yields

Theorem 4.2.4. *There are isomorphisms*

$$\pi_1(Sl_n) \cong H_2(Sl_n(k); \mathbb{Z}) \cong K_2(k),$$

for all $n \geq 3$, and for any field k such that $|k| > 4$.

Another one of the main devices that is used in the proofs of both Proposition 4.2.1 and this last Theorem is the homotopy property for Quillen K-theory, which is Theorem 4.1.1. A similar property holds for the Karoubi ${}_{-1}L$-theory of a commutative ring A (see [28, 25]), which is defined, for $n \geq 1$, by ${}_{-1}L_n(A) = \pi_n(BSp(A)^+)$, where Sp is the infinite symplectic group, and the space $BSp(A)^+$ is defined by the analogy with $BGl(A)^+$. Explicitly, this homotopy property says that, if A is a regular Noetherian ring which contains $1/2$, then the inclusion $A \subset A[x_1]$ induces an isomorphism ${}_{-1}L_n(A) \cong_{-1} L_n(A[x_1])$ for every $n \geq 1$. Proceeding as in the proof of Theorem 4.2.4 gives

Theorem 4.2.5. *For almost all fields k such that $\mathrm{char}(k) \neq 2$, there are isomorphisms*

$$H_2(Sp_{2m}(k); \mathbb{Z}) \cong \pi_1(Sp_{2m}) \cong S(k^*),$$

for all $m \geq 1$, where $S(k^*)$ is the group of Steinberg cycles over k^*.

The "char(k) $\neq 2$" condition of Theorem 4.2.5 is forced upon us by the "containing 1/2" part of the statement of the homotopy property for Karoubi's ${}_{-1}L$-theory. I don not know if this condition may be removed outside of the algebraically closed case. The same phenomenon appears in the following analogue of Theorem 4.1.2:

Theorem 4.2.6. *Suppose that A is a commutative Noetherian regular ring which contains $1/2$. Then $BSp(A_*)$ is an H-space which has the homotopy type of $BSp(A)^+$, and there are isomorphisms $\pi_i(Sp(A_*)) \cong_{-1} L_{i+1}(A)$ for all $i \geq 0$.*

4.3 A vanishing result

It was established in the previous Section that, if k is an algebraically closed field and Sl_n is defined over k, then $\pi_1(Sl_n) \cong K_2(k)$, and this for all $n \geq 2$. Recall that Sl_{n+1} is the universal Chevalley group over k of type A_n. In the last Chapter, it was shown that the covering map $\lambda : Sl_{n+1} \to G_\rho$, onto a Chevalley group G_ρ over k of the same type, is a fibration, with a group-scheme kernel whose A-points consist of the abelian group homomorphisms $\mathbf{Ab}(\Pi(G_\rho), A^*)$ for all $A \in \mathbf{M}_k$, where $\Pi(G_\rho)$ is the geometric fundamental group of G_ρ. In this case, $\Pi(G_\rho)$ is a quotient of $\mathbb{Z}/(n+1)\mathbb{Z}$. Recall that the long exact sequence of a fibration gives isomorphisms $\pi_i(Sl_{n+1}) \cong \pi_i(G_\rho)$, which are induced by λ for all $i > 1$, and a short exact sequence

$$0 \to \pi_1(Sl_{n+1}) \to \pi_1(G_\rho) \to \tilde{\Pi}(G_\rho) \to 0,$$

where $\tilde{\Pi}(G_\rho)$ is $\Pi(G_\rho)$, mod its p-primary component if char$(k) = p$. A similar result holds in the symplectic case.

In this Section, well-known Theorems on the structure of $K_2(k)$ are applied to the study of $\pi_1(Sl_n)$ and $\pi_1(Sp_{2m})$ in the algebraically closed setting.

A result of Steinberg that was quoted in the last Section implies that $K_2(k) \cong H_2(Sl(k); \mathbb{Z}) = 0$ if k is the algebraic closure of a finite field. It can be shown that $K_2(k)$ is a uniquely divisible group for any algebraically closed field k (see [4, p. 357]). This implies that $K_2(k)$ is injective, so that we have

Proposition 4.3.1. *Let G_ρ be a Chevalley group over the algebraically closed field k which is of type A_n or C_m for all $n, m \geq 1$. Then there is an isomorphism $\pi_1(G_\rho) \cong K_2(k) \oplus \tilde{\Pi}(G_\rho)$.*

One may ask when the groups Sl_n and Sp_{2m} are simply-connected in the sense that they have trivial fundamental group; these groups are simply-connected in a geometric sense, so that one is asking when (ie. for what fields k) these two notions coincide. This is the case, for example, if k is the algebraic closure of a finite field. Milnor has shown, however, that $K_2(\mathbb{C})$ is uncountable [35, p. 107], so that $K_2(k)$ does not vanish in general. In particular, $\pi_1(Sl_{n+1})$ does not coincide with the usual fundamental group of the Lie group $Sl_{n+1}(\mathbb{C})$ when Sl_{n+1} is defined over $\mathbb{C}$. Milnor's method of proof may be used to say more.

The Kroenecker dimension $\delta(k)$ of k is defined by

$$\delta(k) = \begin{cases} \mathrm{Tr.deg}_{\mathbb{Q}}(k) + 1 & \text{if } \mathrm{char}(k) = 0, \text{ and} \\ \mathrm{Tr.deg}_{\mathbb{F}_p}(k) & \text{if } \mathrm{char}(k) = p. \end{cases}$$

Then, following Milnor, we show

Lemma 4.3.2. *$K_2(k)$ is non-trivial if $\delta(k) > 1$.*

Proof. Since $\delta(k) > 1$, k is either an algebraic extension of a field $F(x)$, where F contains $\mathbb{Q}$ or F contains $\mathbb{F}_p(y)$ if $\mathrm{char}(k) = p$. Take the x-adic valuation $v : F(x)^* \to \mathbb{Z}$ and extend it to a valuation of k. The restriction of this valuation to a finite algebraic extension L of $F(x)$ determines a discrete valuation $v : L^* \to \mathbb{Z}$ in such a way that, if L' is a finite algebraic extension of L, then there is a commutative diagram

$$\begin{array}{ccccc} F(x)^* & \subset & L^* & \subset & L'^* \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ \mathbb{Z} & \xrightarrow[\times n]{} & \mathbb{Z} & \xrightarrow[\times m]{} & \mathbb{Z}, \end{array}$$

where the positive integers m and n are ramification indices. Associated to any discrete valuation $v : L^* \to \mathbb{Z}$ is a surjective homomorphism $\partial_v : K_2(L) \to K_1(L(v)) = L(v)^*$, where $L(v)$ is the residue class field of v, which is defined on a representative $x \otimes y$ by $\partial_v(x \otimes y) = (-1)^{v(x)v(y)}[x^{v(y)}/y^{v(x)}]$. There is a commutative diagram

$$\begin{array}{ccccc} K_2(f(x)) & \longrightarrow & K_2(L) & \longrightarrow & K_2(L') \\ \downarrow{\scriptstyle \partial_v} & & \downarrow{\scriptstyle \partial_v} & & \downarrow{\scriptstyle \partial_v} \\ F(x)(v)^* & \xrightarrow[P_n]{} & L(v)^* & \xrightarrow[P_M]{} & L'(v)^*, \end{array}$$

for L and L' as above, where, for example $P_n(x) = x^n$. Observe that $F(x)(v) = F$. Dividing by the roots of unity from the groups $L(v)^*$ gives groups $\overline{L(v)}^*$ and induces injections $\overline{P}_n$, etc. Passing to the direct limit determines maps

$$\begin{array}{ccc} & & K_2(k) \\ & & \downarrow \\ \overline{F}^* & \subset & \overline{L}, \end{array}$$

where $overline L = \lim \overline{L(v)}^*$. Not every element of $\mathbb{Q}$ or $\mathbb{F}_p(y)$ is a root of unity, so that $\overline{F}^* \neq 0$, whence $K_2(k) \neq 0$. □

Thus, $K_2(k) = 0$ if $\delta(k) = 0$, and $K_2(k) \neq 0$ if $\delta(k) > 1$. This leaves the case $\delta(k) = 1$, in which k is either a union of number fields in characteristic 0, or a union of function fields if $\mathrm{char}(k) = p$. K_2 of such fields has been studied extensively (see [35, 4, 3, 16, 14]). Central to that study is the homomorphism $\partial_v : K_2(L) \to K_1(L(v))$, which is associated to a discrete valuation v of L, and which was introduced in the proof of Lemma 4.3.2. If L is a global field in the sense that it is either a number field or a function field, then the residue class field $L(v)$ is finite for any discrete valuation v. Moreover, L inherits the property from either $\mathbb{Q}$ of $\mathbb{F}_p(y)$ that all but finitely many discrete valuations vanish on a fixed element $x \in L$. Thus, by adding up the homomorphisms ∂_v, we may form a homomorphism

$$\partial : K_2(L) \to \bigoplus_v K_1(L(v)),$$

where the sum is taken over all discrete valuations v of L. Putting together well-known results of Bass, Tate, and Garland (see the discussion [4, p.396] and [3, p.243]), one sees that the kernel of ∂ is a finite group, and it follows that $K_2(L)$ is a torsion group. But then $K_2(k)$ is a direct limit of such $K_2(L)$, so that $K_2(k)$ is a torsion group as well as being uniquely divisible. It follows that $K_2(k) = 0$. We have, in effect, shown

Theorem 4.3.3. *Suppose that Sl_n and Sp_{2m} are defined over an algebraically closed field k. Then the groups $\pi_1(Sl_n)$ and $\pi_1(Sp_{2m})$ are trivial if $\delta(k) < 2$, and are non-trivial uniquely divisible abelian groups otherwise.*

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