A rigidity theorem for L-theory

J.F. Jardine

Mathematics Department
University of Western Ontario
London, Ontario N6A 5B7
Canada

Preface (1994)

This paper was written in October, 1983, but never published. I am making it available on the internet in this form as a response to the requests for copies of the preprint received in the intervening years. As long as these servers are running, a dvi file of this paper will be available by gopher at gopher.math.uwo.ca and on the world wide web at http://www.math.uwo.ca.

Only very minor changes have been made to the body of the paper for this version. The references have been updated to reflect publication of cited papers that were still preprints at the time when the original manuscript was written. In particular, Karoubi gives proofs for Theorem 1 and (implicitly) Theorem 8 in [9].

Introduction (1983)

Let k be an algebraically closed field, and choose a prime number ℓ which is not equal to the characteristic of k. What is coming to be known as the Gabber-Gillet-Thomason rigidity theorem, [2], [3], asserts that, if x is a rational point on a smooth k-variety X, then the residue map induces an isomorphism

$$K_*(O_x^{sh};\mathbb{Z}/\ell) \overset{\cong}{\to} K_*(k;\mathbb{Z}/\ell),$$

where O_x^{sh} is the strict henselization of the local ring at x. The purpose of this note is to show that this theorem may be promoted to a similar statement for Karoubi L-theory, away from characteristic 2. A computation of the groups ${}_\epsilon L_*(k;\mathbb{Z}/\ell)$ follows, along the lines of the computation of $K_*(k;\mathbb{Z}/\ell)$ given in [5]. The method is to prove the isomorphism conjecture for the groups ${}_\epsilon O(k)$. This is not a generalization of Suslin's arguments [15], [16].

Typeset with $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX.

1. Complements.

Suppose, for the rest of this paper, that the characteristic of the algebraically closed field k is not 2, and choose ϵ in k such that $\epsilon^2 = 1$. The groups ${}_\epsilon O(k)$ are defined as in [6].

The groups ${}_\epsilon L_i(A;\mathbb{Z}/\ell)$ are defined, for a k-algebra A with involution, by

$$ {}_\epsilon L_i(A;\mathbb{Z}/\ell) = [Y^i, E_\epsilon O(A)^+], i \geq 2, $$

where Y^i is the cofibre of the self-map of S^{i-1} which is multiplication by 1. In addition, I define

$$ \begin{aligned} {}_\epsilon L_1(A;\mathbb{Z}/\ell) &= [Y^2, B_\epsilon O(SA)^+], \text{ and} \\ K_1(A;\mathbb{Z}/\ell) &= [Y^2, BG\ell(SA)^+]. \end{aligned} $$

The groups ${}_\epsilon L_i(A;\mathbb{Z}/\ell)$ are abelian, and there are exact sequences

$$ 0 \to {}_\epsilon L_i(A) \otimes \mathbb{Z}/\ell \to {}_\epsilon L_i(A;\mathbb{Z}/\ell) \to \mathrm{Tor}(\mathbb{Z}/\ell, {}_\epsilon L_{i-1}(A) \to 0 $$

for $i \geq 1$.

The fibre squares

$$ \begin{array}{ccc} U_A & \longrightarrow & C(M_2(A)) \\ \downarrow & & \downarrow \\ S(A \times A^0) & \longrightarrow & S(M_2A)) \end{array} \quad \text{and} \quad \begin{array}{ccc} V_A & \longrightarrow & C(A \times A^0) \\ \downarrow & & \downarrow \\ S(A) & \longrightarrow & S(A \times A^0) \end{array} $$

of [8] give rise to long exact sequences

$$ \begin{aligned} &\ldots \to {}_\epsilon U_2(A;\mathbb{Z}/\ell) \to K_2(A;\mathbb{Z}/\ell) \to {}_\epsilon L_2(A;\mathbb{Z}/\ell) \\ &\stackrel{\partial}{\to} {}_\epsilon U_1(A;\mathbb{Z}/\ell) \to K_1(A;\mathbb{Z}/\ell) \to {}_\epsilon L_1(A;\mathbb{Z}/\ell), \end{aligned} \tag{1} $$

$$ \begin{aligned} &\ldots \to {}_\epsilon V_2(A;\mathbb{Z}/\ell) \to {}_\epsilon L_2(A;\mathbb{Z}/\ell) \to K_2(A;\mathbb{Z}/\ell) \\ &\stackrel{\partial}{\to} {}_\epsilon V_1(A;\mathbb{Z}/\ell) \to {}_\epsilon L_1(A;\mathbb{Z}/\ell) \to K_1(A;\mathbb{Z}/\ell). \end{aligned} \tag{2} $$

Here, ${}_\epsilon U_i(A;\mathbb{Z}/\ell) = {}_\epsilon L_{i+1}(U_A;\mathbb{Z}/\ell)$ and ${}_\epsilon V_i(A;\mathbb{Z}/\ell) = {}_\epsilon L_{i+1}(A;\mathbb{Z}/\ell)$ by definition.

The statement of Karoubi's fundamental theorem that I require asserts that there is a natural homotopy equivalence

$$ {}_\epsilon\mathcal{L}(V_A) \cong \Omega_{-\epsilon}\mathcal{L}(U_A) $$

of L-theory spaces, in the notation of [8]. This equivalence induces isomorphisms

$$ {}_\epsilon V_i(A;\mathbb{Z}/\ell) \cong {}_{-\epsilon}U_{i+1}(A;\mathbb{Z}/\ell) \tag{3} $$

for $i \geq 1$.

2. The main results.

The following rigidity theorem for L-theory is proved first:

THEOREM 1. *Let k be an algebraically closed field of characteristic not equal to 2. Let ℓ be a prime which is distinct from the characteristic of k. Let x be a rational point of a smooth k-scheme X. Then the residue map induces an isomorphism*

$$\pi_* : {}_{\epsilon}L_i(O_x^{sh};\mathbb{Z}/\ell) \overset{\cong}{\to} {}_{\epsilon}L_i(k;\mathbb{Z}/\ell) \quad \text{for } i \geq 1,\ \epsilon = \pm 1,$$

where O_x^{sh} is the strict henselization of the local ring at x.

The main reduction step in the proof of the theorem is

LEMMA 2. *If π_* is an isomorphism for $i = 1$ and 2 and all ϵ, then π_* is an isomorphism for $i \geq 1$ and all ϵ.*

PROOF: For notational convenience, ${}_{\epsilon}L_i(A)$ and $K_i(A)$ should be understood to have $\mathbb{Z}/\ell$ coefficients.

The residue map $\pi : O_x^{sh} \to k$ induces an epimorphism in all invariants, since O_x^{sh} is a k-algebra. Suppose that

$$\pi_* : {}_{\epsilon}L_i(O_x^{sh}) \to {}_{\epsilon}L_i(k)$$

is an isomoprhism for all ϵ, and consider the diagram

$$\begin{array}{ccccccccc}
{}_{\epsilon}L_{i+1}(O_x^{sh}) & \longrightarrow & K_{i+1}(O_x^{sh}) & \longrightarrow & {}_{\epsilon}V_i(O_x^{sh}) & \longrightarrow & {}_{\epsilon}L_i(O_x^{sh}) & \longrightarrow & K_i(O_x^{sh}) \\
\downarrow & & \downarrow\cong & & \downarrow\pi_* & & \downarrow\cong & & \downarrow\cong \\
{}_{\epsilon}L_{i+1}(k) & \longrightarrow & K_{i+1}(k) & \longrightarrow & {}_{\epsilon}V_i(k) & \longrightarrow & {}_{\epsilon}L_i(k) & \longrightarrow & K_i(k)
\end{array}$$

The induced maps in K-theory are isomorphisms by the Gabber-Gillet-Thomason rigidity theorem. Chasing the diagram shows that

$$\pi_* : {}_{\epsilon}V_i(O_x^{sh}) \to {}_{\epsilon}V_i(k)$$

is monic, and hence an isomorphism. But then

$$\pi_* : {}_{-\epsilon}U_{i+1}(O_x^{sh}) \to {}_{-\epsilon}U_{i+1}(k)$$

is an isomorphism by (3), and this for all ϵ. Chasing the diagram

$$\begin{array}{ccccccccc}
{}_{\epsilon}U_{i+2}(O_x^{sh}) & \longrightarrow & K_{i+2}(O_x^{sh}) & \longrightarrow & {}_{\epsilon}L_{i+2}(O_x^{sh}) & \longrightarrow & {}_{\epsilon}U_{i+1}(O_x^{sh}) & \longrightarrow & K_{i+1}(O_x^{sh}) \\
\downarrow & & \downarrow\cong & & \downarrow\pi_* & & \downarrow\cong & & \downarrow\cong \\
{}_{\epsilon}U_{i+2}(k) & \longrightarrow & K_{i+2}(k) & \longrightarrow & {}_{\epsilon}L_{i+2}(k) & \longrightarrow & {}_{\epsilon}U_{i+1}(k) & \longrightarrow & K_{i+1}(k)
\end{array}$$

shows that

$$\pi_* : {}_\epsilon L_{i+2}(O_x^{sh}) \to {}_\epsilon L_{i+2}(k)$$

is an isomorphism. ∎

Lemma 3. *The map*

$$\pi_* : {}_\epsilon L_0(O_x^{sh}) \to {}_\epsilon L_0(k)$$

(no coefficients) is an isomorphism for $\epsilon = \pm 1$.

Proof: Suppose that M is a non-singular $n \times n$ symmetric matrix with coefficients in O_x^{sh}, and let $\overline{M}$ be its image in $Gl_n(k)$. By pulling back from k^n, one can find a column vector v in $(O_x^{sh})^n$ such that t_{vMv} is a unit of O_x^{sh}. Since char$(k) \neq 2$, and O_x^{sh} is henselian, we may assume that $t_{vMv} = 1$. It follows by induction on n that $(O_x^{sh})^n$ has an orthonormal basis for the symmetric form which is defined by M. This implies that ${}_1L_0(O_x^{sh}) \cong {}_1L_0(k) \cong \mathbb{Z}$.

A form which is defined by a non-singular anti-symmetric matrix over O_x^{sh} is always hyperbolic, since O_x^{sh} is local. This implies that there are isomorphisms ${}_{-1}L_0(O_x^{sh}) \cong {}_{-1}L_0(k) \cong \mathbb{Z}$. ∎

Lemma 4. *The maps*

$$\pi_* : {}_\epsilon L_1(O_x^{sh}) \to {}_\epsilon L_1(k)$$

are isomorphisms for $\epsilon = \pm 1$.

Proof: The groups $SO(O_x^{sh})$ and $Sp(O_x^{sh})$ are generated by elementary transformations [10] (see also the proof of Lemma 5), and hence are perfect. It follows immediately that ${}_{-1}L_1(O_x^{sh}) = {}_{-1}L_1(k) = 0$. Also, $SO(O_x^{sh})$ is the commutator subgroup of ${}_1O(O_x^{sh})$, and so there are isomorphisms ${}_1L_1(O_x^{sh}) \cong {}_1L_1(k) \cong \mathbb{Z}/2$. ∎

Lemma 5. *The maps*

$$\pi_* : {}_\epsilon L_2(O_x^{sh}) \otimes \mathbb{Z}/\ell \to {}_\epsilon L_2(k) \otimes \mathbb{Z}/\ell$$

are isomorphisms for $\epsilon = \pm 1$ *if char*$(k) \neq 2$.

Proof: Let G_r be the universal group of type D_r, where $r \geq 3$. The covering map $\gamma : G_r \to SP_{2r}$ determines the central extension

$$1 \to \mathbb{Z}/2 \to G_r(O_x^{sh}) \xrightarrow{\gamma_*} SO_{2r}(O_x^{sh}) \to 1.$$

In effect, the obstructions to the surjectivity of γ_* lie in $H^1_{fl}(Sp(O_x^{sh}); \boldsymbol{\mu}_2) = 0$ (see [4]), and $\boldsymbol{\mu}_2(O_x^{sh}) = \mathbb{Z}/2$ since O_x^{sh} is an integral domain by the smoothness assumption on X.

$G_r(O_x^{sh})$ is generated by elementary transformations [10], and so $H_2(G_r(O_x^{sh}); \mathbb{Z})$ is Stein's group $L(D_r; O_x^{sh})$ (see [13], [14]). Stein shows that this group is generated by symbols $\{u, v\}_\alpha$, $\alpha \in B_r$, $u \in (O_x^{sh})^*$. The relation

$$\{u^2, vw\}_\alpha = \{u^2, v\}_\alpha \cdot \{u^2, w\}_\alpha$$

holds in $L(D_r; O_x^{sh})$. Every element in $(O_x^{sh})^*$ has a square root and an ℓ^{th} root by the assumptions on k and the henselian property. It follows that

$$\{u, v\}_\alpha = \{u, w\}_\alpha^\ell$$

where $w^\ell = v$, and hence that $L(D_r; O_x^{sh})$ is ℓ-divisible. A similar argument shows that

$$H_2(Sp_{2r}(O_x^{sh}); \mathbb{Z}) \cong L(C_r; O_x^{sh})$$

is ℓ-divisible if $r \geq 3$.

The comparison of central extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & G_r(O_x^{sh}) & \longrightarrow & SO_{2r}(O_x^{sh}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & G_r(k) & \longrightarrow & SO_{2r}(k) & \longrightarrow & 1
\end{array}$$

induces a comparison of Hochschild-Serre spectral sequences which determines a diagram of exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(G_r(O_x^{sh}); \mathbb{Z}) & \longrightarrow & H_2(SO_{2r}(O_x^{sh}); \mathbb{Z}) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2(G_r(k); \mathbb{Z}) & \longrightarrow & H_2(SO_{2r}(k); \mathbb{Z}) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}$$

Thus, tensoring with $\mathbb{Z}/\ell$ gives an isomorphism

$$H_2(SO_{2r}(O_x^{sh}); \mathbb{Z}) \otimes \mathbb{Z}/\ell \cong H_2(SO_{2r}(k); \mathbb{Z}) \otimes \mathbb{Z}/\ell.$$

The ℓ-divisibility of $H_2(Sp(O_x^{sh}); \mathbb{Z})$ and $H_2(Sp(k); \mathbb{Z})$ give the result for ${}_{-1}L_2$. ∎

Consider the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_\epsilon L_i(O_x^{sh}) \otimes \mathbb{Z}/\ell & \longrightarrow & {}_\epsilon L_i(O_x^{sh}; \mathbb{Z}/\ell) & \longrightarrow & Tor(\mathbb{Z}/\ell, {}_\epsilon L_{i-1}(O_x^{sh})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \pi_* & & \downarrow & & \\
0 & \longrightarrow & {}_\epsilon L_i(k) \otimes \mathbb{Z}/\ell & \longrightarrow & {}_\epsilon L_i(k; \mathbb{Z}/\ell) & \longrightarrow & Tor(\mathbb{Z}/\ell, {}_\epsilon L_{i-1}(k)) & \longrightarrow & 0
\end{array}$$

for $i = 1$ and 2. Lemmas 3 and 4 imply that π_* is an isomorphism when $i = 1$. Lemmas 4 and 5 imply that π_* is an isomorphism when $i = 2$. Theorem 1 is proved, by Lemma 2.

Let $(Sm|_k)_{et}$ denote the site of smooth k-schemes with the etale topology, as in [5], and let the algebraic group ${}_\epsilon O_{n,n}$ represent a sheaf of groups on this site. The sheaf of groups ${}_\epsilon O$ is defined by

$${}_\epsilon O = \varinjlim_n {}_\epsilon O_{n,n}$$

where the filtered colimit is taken in the sheaf category on $(Sm|_k)et$. Consider the canonical map

$$\tilde{\epsilon} : \Gamma^*_\epsilon O(k) \to {}_\epsilon O,$$

where $\Gamma^*_\epsilon O(k)$ is the constant sheaf on the discrete group of rational points ${}_\epsilon O(k)$. $\tilde{\epsilon}$ determines a map of homology sheaves

$$\tilde{\epsilon}_* : H_*(B\Gamma^*_\epsilon O(k); \mathbb{Z}/\ell) \to H_*(B_\epsilon O; \mathbb{Z}/\ell),$$

and Theorem 1 implies (see also [16])

Corollary 6. *The comparison map*

$$\tilde{\epsilon}_* : H_*(B\Gamma^*_\epsilon O(k); \mathbb{Z}/\ell) \to H_*(B_\epsilon O; \mathbb{Z}/\ell),$$

is an isomorphism of sheaves on $(Sm|_k)_{et}$.

The methods of [5] then yield

Theorem 7. *The induced comparsion map*

$$\tilde{\epsilon}^* : H^*_{et}(B_\epsilon O; \mathbb{Z}/\ell) \to H^*(B_\epsilon O(k); Z/\ell)$$

is an isomorphism.

$H^*(B_\epsilon O(k); \mathbb{Z}/\ell)$ is the cohomology of the discrete group ${}_\epsilon O(k)$. The assertion of Theorem 7 is a special case of the generalized isomorphism conjecture.

When k is the field $\mathbb{C}$ of complex numbers, one knows that there is an isomorphism

$$H^*_{\text{top}}(B_\epsilon O(\mathbb{C}); \mathbb{Z}/\ell) \cong H^*_{et}(B_\epsilon O; \mathbb{Z}/\ell),$$

and so $H^i(B_\epsilon O(\mathbb{C}); \mathbb{Z}/\ell)$ and $H^i_{\text{top}}(B_\epsilon O(\mathbb{C}); \mathbb{Z}/\ell)$ are finite dimensional $\mathbb{Z}/\ell$-vector spaces of the same dimension, for $i \geq 0$ [12]. There is a commutative diagram

$$\begin{array}{ccc}
H^i_{\text{top}}(B_\epsilon O_{n,n}(\mathbb{C}); \mathbb{Z}/\ell) & \xrightarrow{\eta} & H^i(B_\epsilon O_{n,n}(\mathbb{C}); \mathbb{Z}/\ell) \\
\uparrow & & \uparrow \\
H^i_{\text{top}}(B_\epsilon O(\mathbb{C}); \mathbb{Z}/\ell) & \longrightarrow & H^i(B_\epsilon O(\mathbb{C}); \mathbb{Z}/\ell)
\end{array}$$

in which η is monic [11]. It follows that the map

$$H^i_{\rm top}(B_\epsilon O(\mathbb{C});\mathbb{Z}/\ell) \to H^i(B_\epsilon O(\mathbb{C});\mathbb{Z}/\ell)$$

is a monomorphism, and hence an isomorphism.

Also, $H^*(B_\epsilon O;\mathbb{Z}/\ell)$ is invariant of the underlying algebraically closed field, so that an inclusion $k \subset L$ of algebraically closed fields induces an isomorphism

$$H^*(B_\epsilon O(L);\mathbb{Z}/\ell) \to H^*(B_\epsilon O(k);\mathbb{Z}/\ell).$$

Putting this together with the Brauer lift techniques of [1] gives a complete calculation of the groups ${}_\epsilon L_i(k;\mathbb{Z}/\ell)$.

THEOREM 8. *Suppose that k is an algebraically closed field such that char$(k) \neq 2$, and let ℓ be an odd prime such that $\ell \neq$ char(k). Then, for $i \geq 1$, the groups ${}_\epsilon L_i(k;\mathbb{Z}/2)$ and ${}_\epsilon L_i(k;\mathbb{Z}/\ell)$ are as follows:*

i(*mod* 8)	${}_1L_i(\ ,\mathbb{Z}/2)$	${}_{-1}L_i(\ ,\mathbb{Z}/2)$	${}_1L_i(\ ,\mathbb{Z}/\ell)$	${}_{-1}L_i(\ ,\mathbb{Z}/\ell)$
0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/\ell$	$\mathbb{Z}/\ell$
1	$\mathbb{Z}/2$	0	0	0
2	$\mathbb{Z}/4$	0	0	0
3	$\mathbb{Z}/2$	0	0	0
4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/\ell$	$\mathbb{Z}/\ell$
5	0	$\mathbb{Z}/2$	0	0
6	0	$\mathbb{Z}/4$	0	0
7	0	$\mathbb{Z}/2$	0	0

Recall that the Moore space Y^2 is a copy of $\mathbb{R}P^2$, and $\widetilde{KO}(\mathbb{R}P^2) \cong \mathbb{Z}/4$: the generator is the tangent bundle.

BIBLIOGRAPHY

[1]. Z. Fiedorowicz and S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Springer Lecture Notes in Math. **674** (1978).

[2]. O. Gabber, *K-theory of Henselian local rings and Henselian pairs*, Contemporary Math. **126** (1992), 59–70.

[3]. H. Gillet and R. Thomason, *The K-theory of strict local hensel rings and a theorem of Suslin*, J. Pure Appl. Algebra **34** (1984), 241–254.

[4]. J.F. Jardine, *Fibrations of algebraic groups*, Journal of Algebra **89** (1984), 335–348.
[5]. J.F. Jardine, *Simplicial objects in a Grothendieck topos*, Contemporary Math. **55,I** (1986), 193–239.
[6]. M. Karoubi, *Periodicité de la K-théorie hermitienne*, Springer Lecture Notes in Math. **343** (1973), 412–463.
[7]. M. Karoubi, *Théorie de Quillen et homologie du groupe orthogonal*, Ann. of Math. **(2) 112** (1980), 206–258.
[8]. M. Karoubi, *Le théoreme fondamental de la K-théorie hermitienne*, Ann. of Math. **(2) 112** (1980), 259–282.
[9]. M. Karoubi, *Relations between algebraic K-theory and hermitian K-theory*, J. Pure Applied Algebra **34** (1984), 259–263.
[10]. H. Matsumoto, *Subgroups of finite index in certain algebraic groups*, Proc. AMS Symp. Pure Math. **IX** (1966), 99–103.
[11]. J. Milnor, *On the homology of Lie groups made discrete*, Comment. Math. Helv. **58** (1983), 72–85.
[12]. D. Quillen, *The Adams conjecture*, Topology **10** (1971), 67–90.
[13]. M. Stein, *Generators, relations, and coverings of Chevalley groups over commutative rings*, Amer. J. Math. **93** (1971), 965–1004.
[14]. M. Stein, *Surjective stability in dimension* 0 *for K_2 and related functors*, Trans. AMS **178** (1973), 165–191.
[15]. A.A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. **73** (1983), 241–245.
[16]. A.A. Suslin, *On the K-theory of local fields*, J. Pure Applied Algebra **34** (1984), 301–318.