Pro-equivalences of diagrams

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Introduction

The paper [8] constructs model structures for pro-objects in simplicial presheaves. These constructions are based on the standard methods of local homotopy theory [9], and generalize known results for pro-objects in ordinary categories of spaces [2], [5]. The homotopy theory of pro-objects, as displayed in all of these papers, is based on (and generalizes) standard features of étale homotopy theory [1], [3], and is an artifact of the étale topology, and ultimately of Galois theory.

One can ask if there are intrinsic invariants which arise from different Grothendieck topologies, which would generalize and extend existing pro-homotopy theoretic techniques. The theory of cocycle categories [7], [9] suggests this: it involves diagrams of weak equivalences which extend classical diagrams of hypercovers, with techniques that are present in all topologies. These diagrams have very little structure, and must be considered in the context of arbitrary small diagrams of simplicial presheaves, with a suitably defined notions of "pro-map" and "pro-equivalence".

Suppose that $X : I \to s\mathbf{Pre}$ and $Y : J \to s\mathbf{Pre}$ are small diagrams of simplicial presheaves.

If I and J are left filtered and X and Y are pro-objects, the most efficient way to define a pro-map $\phi : X \to Y$ (following Grothendieck) is to say that ϕ is a natural transformation

$$\varinjlim_{j \in J} \hom(Y_j, \ ) \to \varinjlim_{i \in I} \hom(X_i, \ )$$

of the corresponding pro-representable functors. Then, as in [8], the map ϕ is a pro-equivalence if the induced map

$$\varinjlim_{j \in J} \mathbf{hom}(Y_j, Z) \to \varinjlim_{i \in I} \mathbf{hom}(X_i, Z)$$

of colimits of function spaces is a weak equivalence for all fibrant objects Z. The displayed colimits are filtered, and filtered colimits are already homotopy colimits, so this makes perfect sense from a homotopy theoretic point of view.

In more general situations, in which the categories I and J are no longer filtered, the colimits above must be replaced with homotopy colimits. Then, for example, the homotopy colimit for the functor $j \mapsto \hom(Y_j, Z)$ is the nerve of the slice category Y/Z whose objects are the morphisms $Y_j \to Z$, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} Y_j & & \\ {\scriptstyle\alpha_*}\downarrow & \searrow & \\ & & Z \\ & \nearrow & \\ Y_{j'} & & \end{array}$$

in which $\alpha : j \to j'$ is a morphism of J. In this way, the diagram $Y : J \to s\mathbf{Pre}$ homotopy pro-represents a functor $B(Y/?) : s\mathbf{Pre} \to s\mathbf{Set}$, with

$$Z \mapsto B(Y/Z).$$

A pro-map $X \to Y$ for arbitrary index categories is then naturally defined as a natural transformation

$$B(Y/?) \to B(X/?)$$

of homotopy pro-representable functors. One can show that a pro-map, so defined, consists of a functor $\alpha : J \to I$ and a natural transformation $\theta : X \cdot \alpha \to Y$, and is therefore a morphism of the Grothendieck construction associated to the list of diagram categories $s\mathbf{Pre}^I$ and the functors between them which are defined by restriction along functors $J \to I$.

Each such pro-map $(\alpha, \theta) : X \to Y$ induces a commutative diagram

$$\begin{array}{ccccc} \underrightarrow{\text{holim}}_{\,j\in J}\ \mathbf{hom}(Y_j, Z) & \xrightarrow{\theta^*} & \underrightarrow{\text{holim}}_{\,j\in J}\ \mathbf{hom}(X_{\alpha(j)}, Z) & \xrightarrow{\alpha_*} & \underrightarrow{\text{holim}}_{\,i\in I}\ \mathbf{hom}(X_j, Z) \\ \downarrow & & \downarrow & & \downarrow \\ BJ & \xrightarrow[1]{} & BJ & \xrightarrow[\alpha]{} & BI \end{array}$$

and I say that the map (α, θ) is a *pro-equivalence* if the simplicial set map $\alpha : BJ \to BI$ is a weak equivalence, and the top composite

$$\underrightarrow{\text{holim}}_{\,j\in J}\ \mathbf{hom}(Y_j, Z) \to \underrightarrow{\text{holim}}_{\,i\in I}\ \mathbf{hom}(X_j, Z)$$

is a weak equivalence of simplicial sets for each fibrant object Z.

The requirement that the map $\alpha : BJ \to BI$ be a weak equivalence is not an issue if I and J are filtered categories, because the spaces BJ and BI are contractible in that case. The definition of pro-map between arbitrary small diagrams, while motivated by the classical Grothendieck description as a transformation of pro-representable functors, is more rigid, because it involves a comparison of functors that are represented by homotopy colimits.

If I is a fixed small category, then a natural transformation $f : X \to Y$ of I-diagrams of simplicial presheaves is a pro-map in the sense described above, and it is a pro-equivalence if and only if it induces weak equivalences

$$f^* : \underrightarrow{\text{holim}}_{\,i\in I}\ \mathbf{hom}(Y_i, Z) \to \underrightarrow{\text{holim}}_{\,i\in I}\ \mathbf{hom}(X_i, Z)$$

for all injective fibrant objects Z. Observe that if I is left filtered, then the homotopy colimits can be replaced by filtered colimits, and then the transformation f would be a pro-equivalence of pro-objects.

One wants to show that the general description of pro-equivalence that is displayed above is part of a homotopy theoretic structure for small diagrams, but this has so far not been realized.

The purpose of the present paper is more modest, to show that the category of I-diagrams of simplicial presheaves, with ordinary cofibrations and pro-equivalences, has a homotopy theoretic structure in that it satisfies the axioms for a left proper closed simplicial model category that is cofibrantly generated. This is the main result of this paper, and appears as Theorem 12 below.

This has not previously been done for classical pro-objects, and that theory is worked out as a test case in Section 1. The corresponding model structure is given by Proposition 5.

The main steps in the arguments for Proposition 5 and Theorem 12 are bounded monomorphism statements (in the style of [9]), which appear as Lemma 1 and Lemma 11, respectively. Lemma 1 is a special case of Lemma 11, but it appears in the first section with a much easier argument that makes heavy use of standard filtered colimit techniques.

The proof of Lemma 11 is more interesting, in that it involves restrictions of I-diagrams to finite diagrams defined on order complexes of finite simplicial complexes, in conjunction with subdivision arguments that appeal to the stalklike structure of Kan's Ex^{∞} construction.

Contents

1 The left filtered case

Suppose that I is a small category, and let $\mathcal{C}$ be a small Grothendieck site.

In all of this paper, we assume that α is a regular cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$ and $\alpha > |I|$, and define the injective fibrant model functor $X \mapsto L(X)$ for simplicial presheaves X by formally inverting the α-bounded trivial cofibrations for the injective model structure on the simplicial presheaf category $s\mathbf{Pre}$.

In this case, we know that there is a regular cardinal $\lambda > \alpha$, such that if X is a simplicial presheaf such that $|X| < \lambda$ then the fibrant model $L(X)$ satisfies

$|L(X)| < \lambda$. Further, the fibrant model construction satisfies

$$L(X) = \varinjlim_{Y \in B_\lambda(X)} L(Y),$$

where $B_\lambda(X)$ denotes the family of λ-bounded subobjects of X. The functor $X \mapsto L(X)$ also preserves monomorphisms and intersections. See Lemma 7.16 of [9].

It follows, for example, that, given a general lifting problem of simplicial presheaves

$$\begin{array}{ccc} A & \xrightarrow{f} & Z \\ {\scriptstyle i}\downarrow & \nearrow & \\ B & & \end{array} \tag{1}$$

with i a cofibration, A λ-bounded and Z injective fibrant, we can replace Z by the λ-bounded object $L(A)$. In effect, we find a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & L(A) \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Z \end{array}$$

of the map f, where the map j is the fibrant model. It follows that, if we can solve the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{j} & L(A) \\ {\scriptstyle i}\downarrow & \nearrow & \\ B & & \end{array}$$

then we can solve the lifting problem (1) for all injective fibrant objects Z.

Suppose, for the rest of this section, that the index category I is left filtered.

Suppose given a diagram of cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

where A is λ-bounded and i is a pro-equivalence.

The map

$$\varinjlim_i \mathbf{hom}(Y_i, Z) \to \varinjlim_i \mathbf{hom}(X_i, Z)$$

is a trivial fibration of simplicial sets for all injective fibrant objects Z. This is equivalent to the assertion that any lifting problem

$$\begin{array}{ccc} (\partial\Delta^n \times Y_i) \cup (\Delta^n \times X_i) & \xrightarrow{f} & Z \\ \downarrow & \nearrow & \\ \Delta^n \times Y_i & & \end{array} \tag{2}$$

can be solved after refining along i. Solution after refinement means that there is a morphism $\alpha : j \to i$ of I with a commutative diagram

$$\begin{array}{ccccc} (\partial\Delta^n \times Y_j) \cup (\Delta^n \times X_j) & \xrightarrow{\alpha_*} & (\partial\Delta^n \times Y_i) \cup (\Delta^n \times X_i) & \xrightarrow{f} & Z \\ \downarrow & & & \overset{\theta}{\nearrow} & \\ \Delta^n \times Y_j & & & & \end{array}$$

We replace the map to Z in the picture by the fibrant model

$$(\partial\Delta^n \times Y_i) \cup (\Delta^n \times X_i) \xrightarrow{j} Z(Y)_i.$$

Write $Z(B)_i$ for the fibrant model of the object

$$(\partial\Delta^n \times B_i) \cup (\Delta^n \times (B_i \cap X_i)),$$

where B varies through the λ-bounded subobjects of Y. There is a relation

$$\varinjlim_{B} Z(B)_i = Z(Y)_i$$

where the colimit is indexed over the λ-bounded subobjects B of Y. In effect, every λ-bounded subobject C of

$$(\partial\Delta^n \times Y_i) \cup (\Delta^n \times X_i)$$

is contained in some

$$(\partial\Delta^n \times B_i) \cup (\Delta^n \times (B_i \cap X_i)),$$

with $B \subset Y$ λ-bounded, while $Z(Y_i)$ is a filtered colimit of the objects $Z(C)$.

It follows that the image of the composite map

$$\Delta^n \times A_j \to \Delta^n \times Y_j \xrightarrow{\theta} Z(Y)_i$$

lies in $Z(B)_i$ for some λ-bounded B such that $A \subset B$.

This is the start of an inductive process. There is a λ-bounded subobject B with $A \subset B$ such that every lifting problem

$$\begin{array}{ccc} (\partial\Delta^n \times A_i) \cup (\Delta^n \times (A_i \cap X_i)) & \xrightarrow{j} & Z(A)_i \\ \downarrow & \nearrow & \\ \Delta^n \times A_i & & \end{array}$$

(i.e. for all i) is solved over B after refinement. It follows that there is a λ-bounded C with $A \subset C \subset Y$ such that every lifting problem

$$\begin{array}{ccc} (\partial\Delta^n \times C_i) \cup (\Delta^n \times (C_i \cap X_i)) & \xrightarrow{j} & Z(C)_i \\ \downarrow & \nearrow & \\ \Delta^n \times C_i & & \end{array}$$

is solved after refinement, and this for all i.

We have proved the following bounded monomorphism statement:

Lemma 1. *Suppose that I is a left filtered category. Suppose given a diagram of cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

where A is λ-bounded and i is a pro-equivalence. Then there is an λ-bounded subobject $C \subset Y$ with $A \subset C$ such that the map $C \cap X \to C$ is a pro-equivalence.

Say that a map $p : X \to Y$ is a *pro-fibration* if it has the right lifting property with respect to all cofibrations $A \to B$ which are pro-equivalences.

Lemma 2. *A map $p : X \to Y$ is a pro-fibration and a pro-equivalence if and only if it has the right lifting property with respect to all cofibrations.*

Proof. Suppose that p is a pro-fibration and a pro-equivalence. We show that it has the right lifting property with respect to all λ-bounded cofibrations.

The converse assertion is clear: if p has the right lifting property with respect to all cofibrations, then it is a sectionwise equivalence and hence a pro-equivalence, and it has the right lifting property with respect to a cofibrations which are pro-equivalences.

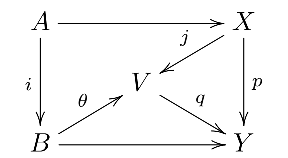
Suppose given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

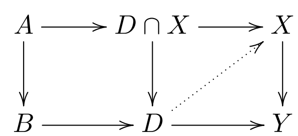
where i is an λ-bounded cofibration and p is a pro-fibration and a pro-equivalence. We show that the indcated lift exists.

This will be true for all λ-bounded cofibrations, and these generate the class of all cofibrations, so it follows that p has the right lifting property with respect to all cofibrations.

Factorize p as $p = q \cdot j$ where q is a trivial injective fibration and j is a cofibration. Then there is a diagram



The cofibration j is a pro-equivalence, and the image $\theta(B)$ is λ-bounded, so there is a subobject $D \subset V$ with $\theta(B) \subset D$, such that the map $D \cap X \to D$ is a pro-equivalence, by Lemma 1. We have found a factorization



of the original diagram, with a pro-equivalence $D \cap X \to D$, and the lifting problem has the indicated solution. $\square$

Corollary 3. *A map $p : X \to Y$ is a pro-fibration and a pro-equivalence if and only if it is a trivial injective fibration.*

The following statements are also clear:

Lemma 4. *1) A map p is a pro-fibration if and only it has the right lifting property with respect to all λ-bounded cofibrations which are pro-equivalences.*

2) The class of maps which are cofibrations and pro-equivalences is closed under pushout.

We can now prove the following:

Proposition 5. *Suppose that the category I is left filtered, and that $\mathcal{C}$ is a Grothendieck site. The category $s\mathbf{Pre}^I$ of I-diagrams in simplicial presheaves on $\mathcal{C}$, together with the classes of cofibrations, pro-equivalences and pro-fibrations, satisfies the axioms for a left proper closed simplicial model category. This model structure is cofibrantly generated.*

Proof. The factorization axiom **CM5** follows from Corollary 3 and Lemma 4. The lifting axiom **CM4** follows from Corollary 3. The other closed model axioms are automatically true. The function complex $\mathbf{hom}(X, Y)$ is the standard one, and one shows that, given a cofibration $A \to B$ of I-diagrams and a cofibration $K \to L$ of simplicial sets, then the map

$$(A \times L) \cup (B \times K) \to B \times L$$

is a cofibration which is a pro-equivalence if $A \to B$ is a pro-equivalence or $K \to L$ is a weak equivalence.

Left properness is trivial to verify, and cofibrant generation is a consequence of Lemma 1. $\square$

2 Subdivisions and simplex categories

Suppose that X is a simplicial set. The poset NX has objects given by all non-degenerate simplices $\sigma \in X$, and we say that $\sigma \leq \tau$ if σ is a member of the subcomplex of X which is generated by τ.

Suppose that K is a finite simplicial complex, in the sense that K is a subcomplex of some simplex Δ^N. Then $\sigma \leq \tau$ in NK if σ is a face of τ; furthermore, σ is a face of τ in a unique way. The nerve BNK is often said to be the *order complex* of the simplicial complex K [10]. The non-degenerate simplices σ of K are those for which the canonical map $\sigma : \Delta^n \to K$ is a monomorphism.

The *simplex category* $\mathbf{\Delta}/X$ for a simplicial set X has objects consisting of all simplicial set maps $\sigma : \Delta^n \to X$. The morphisms $\theta : \tau \to \sigma$ are commutative diagrams of simplicial set maps

$$\begin{array}{ccc} \Delta^m & & \\ {\scriptstyle\theta}\downarrow & \searrow^{\tau} & \\ & & X \\ & \nearrow_{\sigma} & \\ \Delta^n & & \end{array}$$

Suppose that X is a simplicial set and that L is a finite simplicial complex. A map $f : L \to X$ determines a functor $f : \mathbf{\Delta}/L \to \mathbf{\Delta}/X$ of simplex categories in the obvious way.

Since L is a simplicial complex, there is an inclusion functor $NL \to \mathbf{\Delta}/L$, and we have the following:

Lemma 6. *Suppose that L is a finite simplicial complex. Then the composite map*

$$\varinjlim_{\Delta^n \subset L} \Delta^n \to \varinjlim_{\Delta^n \to L} \Delta^n \xrightarrow{\cong} L$$

is an isomorphism. A simplicial set map $f : L \to X$ is defined by the composite functor

$$\tilde{f} : NL \to \mathbf{\Delta}/L \xrightarrow{f_*} \mathbf{\Delta}/X.$$

Lemma 6 says that a finite simplicial complex is a colimit of its non-degenerate simplices.

Proof. The r-simplices $x \in \Delta^n \xrightarrow{\sigma} L$ and $y \in \Delta^m \xrightarrow{\tau} L$ have the same image in K if and only if there is a diagram

$$\begin{array}{ccc} \Delta^r & \xrightarrow{x} & \Delta^n \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \Delta^m & \xrightarrow[\tau]{} & L \end{array}$$

Since L is a finite simplicial complex, the pullback $\Delta^m \times_L \Delta^n$ is a non-degenerate simplex of L if σ and τ are non-degenerate. It follows that the composite map

$$\varinjlim_{\Delta^n \subset L} \Delta^n \to \varinjlim_{\Delta^m \to L} \Delta^m \xrightarrow{\cong} L$$

is an isomorphism, and so there is a commutative diagram

$$\begin{array}{ccc}
\varinjlim_{\Delta^n \subset L} \Delta^n & \xrightarrow{\tilde{f}_*} & \varinjlim_{\Delta^n \to X} \Delta^n \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
L & \xrightarrow[f]{} & X
\end{array}$$

□

Corollary 7. *Suppose that $g : K \to L$ is a morphism of simplicial complexes. Then the map g is induced by a functor $g_* : NK \to NL$ which takes a non-degenerate simplex σ to the non-degenerate simplex which generates the subcomplex $\langle g(\sigma) \rangle$ of L.*

Proof. Consider the picture

$$\begin{array}{ccc}
\varinjlim_{\Delta^n \subset K} \Delta^n & \dashrightarrow & \varinjlim_{\Delta^m \subset L} \Delta^m \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
\varinjlim_{\Delta^n \to K} \Delta^n & \xrightarrow{g_*} & \varinjlim_{\Delta^m \to L} \Delta^m \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
K & \xrightarrow[g]{} & L
\end{array}$$

in which the vertical maps are isomorphisms by Lemma 6.

If $\sigma : \Delta^n \to K$ is a non-degenerate simplex, then the composite

$$\Delta^n \xrightarrow{\sigma} K \xrightarrow{g} L$$

has a unique factorization

$$\Delta^n \xrightarrow{s_\sigma} \Delta^r \xrightarrow{d_\sigma} L,$$

where s_σ is a codegeneracy and d_σ is a non-degenerate simplex of L. Write $g_*(\sigma) = d_\sigma$, and observe that the assignment $\sigma \mapsto g_*(\sigma)$ defines a functor $g_* : NK \to NL$. The codegeneracies s_σ and the functor g_* define the dotted arrow map in the diagram, in the sense that there are commutative diagrams

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{s_\sigma} & \Delta^r \\
{\scriptstyle in_\sigma}\downarrow & & \downarrow{\scriptstyle in_{g_*(\sigma)}} \\
\varinjlim_{\Delta^n \subset K} \Delta^n & \dashrightarrow & \varinjlim_{\Delta^m \subset L} \Delta^m
\end{array}$$

□

Remark 8. A similar argument shows that the inclusion $NL \subset \mathbf{\Delta}/L$ is a homotopy equivalence of categories for each finite simplicial complex L. Every simplex $\sigma : \Delta^n \to L$ has a canonical factorization $\sigma = d_\sigma s_\sigma$, and the assignment $\sigma \mapsto d_\sigma$ defines a functor $\mathbf{\Delta}/L \to NL$ which is inverse to the inclusion up to homotopy defined by the morphisms s_σ.

Corollary 9. *Suppose given maps*

$$K \xrightarrow{g} L \xrightarrow{f} X$$

where K and L are finite simplicial complexes. Then the composite $f \cdot g$ is induced by the composite functor

$$NK \xrightarrow{g_*} NL \xrightarrow{\tilde{f}} \mathbf{\Delta}/X.$$

The *subdivision* $\mathrm{sd}(\Delta^n)$ is defined to be the nerve of the corresponding order complex:

$$\mathrm{sd}(\Delta^n) = BN\Delta^n.$$

More generally, the *subdivision* $\mathrm{sd}(X)$ of a simplicial set X is defined by the assignment

$$\mathrm{sd}(X) = \varinjlim_{\Delta^m \to X} \mathrm{sd}(\Delta^m).$$

There is a map

$$\pi : \mathrm{sd}(X) \to BNX$$

that is natural in simplicial sets X, and is defined by the composites

$$\mathrm{sd}(\Delta^n) \xrightarrow{\cong} BN\Delta^n \xrightarrow{\sigma_*} BNX.$$

arising from the simplices $\Delta^n \to X$ of X.

Lemma 10. *The map $\pi : \mathrm{sd}(K) \to BNK$ is an isomorphism if K is a finite simplicial complex.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} \varinjlim_{\sigma \in NK} \mathrm{sd}\langle\sigma\rangle & \xrightarrow{\cong} & \mathrm{sd}(K) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ \varinjlim_{\sigma \in NK} BN\langle\sigma\rangle & \xrightarrow[\cong]{} & BNK \end{array}$$

where $\langle\sigma\rangle$ is the subcomplex of X that is generated by the simplex σ. The top horizontal map is an isomorphism, since K is a colimit of the subcomplexes $\langle\sigma\rangle$ which are generated by non-degenerate simplices σ by Lemma 6. The bottom horizontal map is an isomorphism, again since the intersection of two non-degenerate simplices of the simplicial complex K is a non-degenerate simplex.

All maps $\pi : \mathrm{sd}\langle\sigma\rangle \to BN\langle\sigma\rangle$ are isomorphisms, again since K is a simplicial complex: if σ is a non-degenerate n-simplex of K, then the canonical map $\Delta^n \to \langle\sigma\rangle$ that is defined by σ is an isomorphism. □

Suppose again that $K \subset \Delta^N$ is a finite simplicial complex. The induced functor $NK \to N\Delta^N$ is a fully faithful imbedding, which induces a monomorphism $\mathrm{sd}(K) \to \mathrm{sd}(\Delta^N)$ of associated nerves. There is a total ordering on the non-degenerate simplices of Δ^N which extends the total ordering on its vertices, such that every k-simplex is less than every $(k+1)$-simplex for $0 \leq k \leq N-1$. In this way, there is a fully faithful imbedding $N\Delta^N \subset \mathbf{M}$ for some ordinal number M, and so there is a monomorphism $\mathrm{sd}(\Delta^N) \subset \Delta^M$. It follows from the resulting string of inclusions

$$\mathrm{sd}(K) \subset \mathrm{sd}(\Delta^N) \subset \Delta^M$$

that $\mathrm{sd}(K)$ is a finite simplicial complex.

The *last vertex* functor $N\Delta^n \to \mathbf{n}$ is defined by sending a non-degenerate simplex $\sigma : \mathbf{k} \to \mathbf{n}$ to $\sigma(k)$ — see [4], [6]. This functor induces a simplicial set map $\gamma : \mathrm{sd}\,\Delta^n \to \Delta^n$. The maps γ are natural in simplices Δ^n, and together induce a map

$$\gamma : \mathrm{sd}(X) \to X$$

which is natural in simplicial sets X. Composition of instances of this map defines the various natural maps

$$\mathrm{sd}^k(X) \xrightarrow{\gamma} \mathrm{sd}^{k-1}(X) \xrightarrow{\gamma} \dots \xrightarrow{\gamma} \mathrm{sd}(X) \xrightarrow{\gamma} X,$$

all of which will be denoted by γ, and called *subdivision maps*.

It is a consequence of the results of this section (specifically, Corollary 9) that if K is a finite simplicial complex and X is a simplicial set, then the string of simplicial set maps

$$\mathrm{sd}^k(K) \xrightarrow{\gamma} \mathrm{sd}^{k-1}(K) \xrightarrow{\gamma} \dots \xrightarrow{\gamma} \mathrm{sd}(K) \xrightarrow{\gamma} K \xrightarrow{f} X$$

is induced by the string of functors

$$N\,\mathrm{sd}^k(K) \xrightarrow{\gamma_*} N\,\mathrm{sd}^{k-1}(K) \xrightarrow{\gamma_*} \dots \xrightarrow{\gamma_*} N\,\mathrm{sd}(K) \xrightarrow{\gamma_*} NK \xrightarrow{\tilde{f}} \mathbf{\Delta}/X.$$

Suppose that $f : X \to Y$ is a map of Kan complexes, and replace f by a fibration in the usual way, by forming the pullback diagram

$$\begin{array}{ccccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I & \xrightarrow{d_1} & Y \\ \downarrow & & \downarrow{\scriptstyle d_0} & & \\ X & \xrightarrow[f]{} & Y & & \end{array}$$

Let π be the composite $d_1 \cdot f_*$. Then π is a fibration which is weakly equivalent to f.

Here, Y^I is the function complex $\mathbf{hom}(\Delta^1, Y)$, and the maps $d_0, d_1 : Y^I \to Y$ are defined by precomposition with the maps $d^0, d^1 : \Delta^0 \to \Delta^1$, respectively.

A solution of the lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{(\alpha,h_*)} & X \times_Y Y^I \\ \downarrow & \nearrow & \downarrow{\scriptstyle \pi} \\ \Delta^n & \xrightarrow[\beta]{} & Y \end{array}$$

is equivalent an extension of the adjoint diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & X \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle f} \\ (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & \xrightarrow[(h,\beta)]{} & Y \end{array} \tag{3}$$

to a diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta} & X \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^n \times \Delta^1 & \xrightarrow[H]{} & Y \end{array} \tag{4}$$

It follows that a simplicial set map $f : X \to Y$ between Kan complexes is a weak equivalence if and only if every diagram (3) extends to a diagram (4).

If X and Y are not Kan complexes, then $f : X \to Y$ is a weak equivalence if and only if all diagrams (3) extend to diagrams (4) after subdivision. In other words, given a diagram (3), there is some $k \geq 0$ such that the composite diagram

$$\begin{array}{ccccc} \mathrm{sd}^k(\partial\Delta^n) & \xrightarrow{\gamma} & \partial\Delta^n & \xrightarrow{\alpha} & X \\ {\scriptstyle d_{0*}}\downarrow & & & & \downarrow{\scriptstyle f} \\ \mathrm{sd}^k((\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\})) & \xrightarrow[\gamma]{} & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & \xrightarrow[(h,\beta)]{} & Y \end{array}$$

extends to a diagram

$$\begin{array}{ccccc} \mathrm{sd}^k(\Delta^n) & \xrightarrow{\gamma} & \Delta^n & \xrightarrow{\theta} & X \\ {\scriptstyle d_{0*}}\downarrow & & & & \downarrow{\scriptstyle f} \\ \mathrm{sd}^k(\Delta^n \times \Delta^1) & \xrightarrow[\gamma]{} & \Delta^n \times \Delta^1 & \xrightarrow[H]{} & Y \end{array}$$

This follows from the fact that the map $f : X \to Y$ is a weak equivalence if and only if the induced map $\mathrm{Ex}^\infty X \to \mathrm{Ex}^\infty Y$ is a weak equivalence of Kan complexes — see section III.4 of [4].

3 General diagram categories

Suppose that I is a fixed (but arbitrary) small index category.

The definitions of Section 1 persist for the category $s\mathbf{Pre}^I$ of I-diagrams of simplicial presheaves with their natural transformations. A *cofibration* is a sectionwise monomorphism. A *pro-equivalence* $X \to Y$ is a natural transformation which induces a weak equivalence of simplicial sets

$$f^* : \underrightarrow{\mathrm{holim}}_{\,i\in I}\ \mathbf{hom}(Y_i, Z) \to \underrightarrow{\mathrm{holim}}_{\,i\in I}\ \mathbf{hom}(X_i, Z)$$

for all injective fibrant objects Z..Finally, a *pro-fibration* is a map which has the right lifting property with respect to maps which are cofibrations and pro-equivalences.

We prove, in this section, an analogue of Proposition 5, which asserts that these definitions give the category $s\mathbf{Pre}^I$ the structure of a left proper closed simplicial model category. The main result is Theorem 12 below.

Suppose that X is an I-diagram of simplicial presheaves and that the simplicial presheaf Z is injective fibrant. Suppose that K is a finite simplicial complex.

A simplicial set map $f : K \to \underrightarrow{\mathrm{holim}}_{\,I^{op}}\ \mathbf{hom}(X, Z)$ is induced by a functor

$$\tilde{f} : NK \to \mathbf{\Delta}/\ \underrightarrow{\mathrm{holim}}_{\,I^{op}}\ \mathbf{hom}(X, Z),$$

according to Lemma 6. The homotopy colimit can be identified with the diagonal of the nerve of a simplicial category $H_I(X, Z)$ with morphisms in simplicial degree n having the form

$$\begin{array}{ccc} X(i) \times \Delta^n & \xrightarrow{\alpha_* \times 1} & X(j) \times \Delta^n \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

where $\alpha : i \to j$ is a morphism of I. Observe that there is a forgetful functor

$$\pi : H_I(X, Z) \to I.$$

An n-simplex D of the homotopy colimit consists of a functor $\alpha : \mathbf{n} \to I$ and a diagram

$$\begin{array}{c} X(\alpha(0)) \times \Delta^n \to X(\alpha(1)) \times \Delta^n \to \dots \to X(\alpha(n-1)) \times \Delta^n \to X(\alpha(n)) \times \Delta^n \\ \searrow \quad \searrow \quad \swarrow \quad \swarrow_{\tau} \\ Z \end{array}$$

or alternatively an n-simplex in the nerve of the slice category $X \times \Delta^n/Z$. Note (this is standard) that the simplex is completely determined by the functor α and the map τ.

Given such an object, if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map then the simplex $\theta^* D$ is defined by the composite functor $\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\alpha} I$ and the composite diagram

$$\begin{array}{ccccccc}
X(\alpha(\theta(0))) \times \Delta^m & \to & X(\alpha(\theta(1))) \times \Delta^m & \to \dots \to & X(\alpha(\theta(m-1))) \times \Delta^m & \to & X(\alpha(\theta(m))) \times \Delta^m \\
\downarrow {\scriptstyle 1\times\theta} & & \downarrow {\scriptstyle 1\times\theta} & & \downarrow {\scriptstyle 1\times\theta} & & \downarrow {\scriptstyle 1\times\theta} \\
X(\alpha(\theta(0))) \times \Delta^n & \to & X(\alpha(\theta(1))) \times \Delta^n & \to \dots \to & X(\alpha(\theta(m-1))) \times \Delta^n & \to & X(\alpha(\theta(m))) \times \Delta^n \\
& \searrow & \searrow & & \swarrow & \swarrow & \\
& & & Z & & &
\end{array}$$

Alternatively, the n-simplex above is a functor $\alpha : \mathbf{n} \to I$ and a natural transformation $f : (X \cdot \alpha) \times \Delta^n \to Z$. Then $\theta^*(\alpha, f)$ is the pair consisting of the composite functor $\alpha \cdot \theta$ together with the composite natural transformation

$$(X \cdot \alpha \cdot \theta) \times \Delta^m \xrightarrow{1\times\theta} (X \cdot \alpha \cdot \theta) \times \Delta^n \xrightarrow{f\cdot\theta} Z.$$

Write

$$E_I(X, Z) = \mathbf{\Delta} / \underrightarrow{\operatorname{holim}}_{I^{op}} \mathbf{hom}(X, Z)$$

to make the notation easier to deal with.

Suppose that $i : X \to Y$ is a monomorphism of I-diagrams. Let $j : K \subset L$ be an inclusion of finite simplicial complexes, and consider a diagram

$$\begin{array}{ccc}
K & \longrightarrow & \underrightarrow{\operatorname{holim}}_{I^{op}} \mathbf{hom}(Y, Z) \\
\downarrow & & \downarrow \\
L & \longrightarrow & \underrightarrow{\operatorname{holim}}_{I^{op}} \mathbf{hom}(X, Z)
\end{array}$$

Converting to functors by using the methods of the last section (Corollary 9) gives the diagram of functors

$$\begin{array}{ccc}
NK & \xrightarrow{\omega} & E_I(Y, Z) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i^*} \\
NL & \xrightarrow[\beta]{} & E_I(X, Z)
\end{array}$$

The functor i^* is defined by restriction to X. The diagram consists of a functor ω whose restriction to X extends to a functor β that is defined on NL.

There is a functor

$$v_Y : E_I(Y, Z) \to s\mathbf{Pre},$$

which takes any n-simplex $f : Y \cdot \alpha \times \Delta^n \to Z$ to the simplicial set $Y(\alpha(n)) \times \Delta^n$ (which is the colimit of $Y \cdot \alpha \times \Delta^n$). This functor v_Y is independent of Z: if

$Z \to W$ is a simplicial set map, then the diagram

$$\begin{array}{ccc} E_I(Y,Z) & \longrightarrow & E_I(Y,W) \\ & {\scriptstyle v_Y}\searrow \quad \swarrow{\scriptstyle v_Y} & \\ & s\mathbf{Pre} & \end{array}$$

commutes.

The functor $\omega : NK \to E_I(Y,Z)$ can be identified with a natural transformation

$$v_Y \cdot \omega \to Z.$$

Taking the colimit $L_K(Y)$ of the composite functor

$$NK \xrightarrow{\omega} E_I(Y,Z) \xrightarrow{v_Y} s\mathbf{Pre}$$

therefore defines a functor $\omega_* : NK \to E_I(Y, L_K(Y))$ and a simplicial set map $f_\omega : L_K(Y) \to Z$ such that the diagram of functors

$$\begin{array}{ccc} NK & \xrightarrow{\omega_*} & E_I(Y, L_K(Y)) \\ & {\scriptstyle \omega}\searrow & \downarrow{\scriptstyle f_{\omega *}} \\ & & E_I(Y,Z) \end{array}$$

commutes.

Write $j : L_K(Y) \to \mathcal{L}(L_K(Y))$ for the natural injective fibrant model of the simplicial presheaf $L_K(Y)$. The map $f_\gamma : L_K(Y) \to Z$ factors through a map $\mathcal{L}(L_K(Y)) \to Z$.

Suppose given a commutative diagram of inclusions

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \downarrow & & \downarrow \\ L & \longrightarrow & L' \end{array} \qquad (5)$$

of finite complexes, and suppose that $i : X \to Y$ is a cofibration of I-diagrams such that all diagrams

$$\begin{array}{ccc} NK & \xrightarrow{\omega} & E_I(Y,Z) \\ \downarrow & & \downarrow \\ NL & \xrightarrow[\beta]{} & E_I(X,Z) \end{array} \qquad (6)$$

extend to diagrams

$$\begin{array}{ccc} NK' & \longrightarrow & E_I(Y,Z) \\ \downarrow & & \downarrow \\ NL' & \longrightarrow & E_I(X,Z) \end{array} \qquad (7)$$

after subdivision, for all injective fibrant Z.

This means that, for each diagram (6) there is a $k \geq 0$ such that the diagram

$$\begin{array}{ccccc} N\,\mathrm{sd}^k(K) & \xrightarrow{\gamma_*} & NK & \xrightarrow{\omega} & E_I(Y,Z) \\ \downarrow & & & & \downarrow \\ N\,\mathrm{sd}^k(L) & \xrightarrow[\gamma_*]{} & NL & \xrightarrow[\beta]{} & E_I(X,Z) \end{array}$$

extends to a diagram

$$\begin{array}{ccc} N\,\mathrm{sd}^k(K') & \longrightarrow & E_I(Y,Z) \\ \downarrow & & \downarrow \\ N\,\mathrm{sd}^k(L') & \longrightarrow & E_I(X,Z) \end{array}$$

A commutative diagram (6) is, equivalently, a diagram of simplicial presheaf maps

$$\begin{array}{ccc} L_K X & \longrightarrow & L_L X \\ \downarrow & & \downarrow \\ L_K Y & \longrightarrow & Z \end{array} \tag{8}$$

and to say that diagram (6) extends to a diagram (7) amounts to the assertion that there is a diagram

$$\begin{array}{ccc} L_{K'} X & \longrightarrow & L_{L'} X \\ \downarrow & & \downarrow \\ L_{K'} Y & \longrightarrow & Z \end{array}$$

which restricts to the given diagram (8) along the maps

$$\begin{array}{ccccc} L_K Y & \longleftarrow & L_K X & \longrightarrow & L_L X \\ \downarrow & & \downarrow & & \downarrow \\ L_{K'} Y & \longleftarrow & L_{K'} X & \longrightarrow & L_{L'} X \end{array}$$

In other words, we require the existence of a lifting in the diagram

$$\begin{array}{ccc} L_K Y \cup_{L_K X} L_L X & \longrightarrow & Z \\ \downarrow & \nearrow & \\ L_{K'} Y \cup_{L_{K'} X} L_{L'} X & & \end{array} \tag{9}$$

for all injective fibrant Z.

The requirement that (6) extends to (7) up to subdivision amounts to the existence of a number $k \geq 0$ such that the lift exists in the diagram

$$\begin{array}{ccccc} L_{\mathrm{sd}^k(K)}Y \cup_{L_{\mathrm{sd}^k\,K}(X)} L_{\mathrm{sd}^k(L)}X & \longrightarrow & L_K Y \cup_{L_K X} L_L X & \longrightarrow & Z \\ \downarrow & & & \nearrow & \\ L_{\mathrm{sd}^k(K')}Y \cup_{L_{\mathrm{sd}^k(K')}X} L_{\mathrm{sd}^k(L')}X & & & & \end{array}$$

It is enough to solve the extension problem in the case where Z is an injective fibrant model of the pushout $L_K Y \cup_{L_K X} L_L X$, since all extension problems (9) are solved by the existence of extensions for this fibrant model.

We return to the cases of (3) and (4), which concern the cases of (5) given by the countable list of diagrams

$$\begin{array}{ccc} K = \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ L = (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & \longrightarrow & \Delta^n \times \Delta^1 \end{array}$$

where $n \geq 0$. We also suppose that the regular cardinals $\lambda > \alpha$ are chosen as above.

Then for a fixed diagram (5), or fixed $n \geq 0$, the list of associated extension problems (9) is determined by the size of the set of functors $NL \to I$. The category I is λ-bounded, while there are countably many finite complexes $L = \Delta^n \times \Delta^1$ of interest, and so the entire list of relevant extension problems is λ-bounded.

Lemma 11. *Suppose that the regular cardinals $\lambda > \alpha$ are chosen as above. Suppose that the monomorphism $i : X \to Y$ is a pro-equivalence, and that A is a λ-bounded subobject of Y. Then there is a λ-bounded subobject $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a pro-equivalence.*

Proof. Consider the commutative diagram of inclusions of I-diagrams

$$\begin{array}{ccc} A \cap X & \longrightarrow & X \\ \downarrow & & \downarrow i \\ A & \longrightarrow & Y \end{array}$$

where A is a λ-bounded subobject of Y. Suppose that the lifting problem (9) can be solved up to subdivision for $i : X \to Y$, and that Z is the fibrant model $\mathcal{L}(L_K Y \cup_{L_K X} L_L X)$. Consider the diagram

$$\begin{array}{ccccc} L_K A \cup_{L_K(A\cap X)} L_L(A \cap X) & \longrightarrow & L_K Y \cup_{L_K X} L_L X & \longrightarrow & Z \\ \downarrow & & \downarrow & \nearrow_{\theta} & \\ L_{K'} A \cup_{L_{K'}(A\cap X)} L_{L'}(A \cap X) & \xrightarrow[j]{} & L_{K'} Y \cup_{L_{K'} X} L_{L'} X & & \end{array}$$

The image of the composite $\theta \cdot j$ is λ-bounded, and $Z = \mathcal{L}(L_K Y \cup_{L_K X} L_L X)$ is a filtered colimit of the subobjects $\mathcal{L}(L_K C \cup_{L_K(C\cap X)} L_L(C\cap X))$, as C varies through the λ-bounded subobjects of Y, so the image of $\theta \cdot j$ factors through $\mathcal{L}(L_K A_1 \cup_{L_K(A_1\cap X)} L_L(A_1 \cap X))$ for some λ-bounded $A_1 \subset Y$ with $A \subset A_1$. This can be done simultaneously for the full λ-bounded list of extension problems.

Repeat this construction λ times, and let $B = \varinjlim_{t<\lambda} A_t$. Then the map

$$\varinjlim_{t<\lambda} \mathcal{L}(L_K A_t \cup_{L_K(A_t\cap X)} L_L(A_t \cap X)) \to \mathcal{L}(L_K B \cup_{L_K(B\cap X)} L_L(B\cap X))$$

is an isomorphism, and there is a commutative diagram

$$\begin{array}{ccc} L_K B \cup_{L_K(B\cap X)} L_L(B\cap X) & \xrightarrow{j} & \mathcal{L}(L_K B \cup_{L_K(B\cap X)} L_L(B\cap X)) \\ \downarrow & \nearrow & \\ L_{K'} B \cup_{L_{K'}(B\cap X)} L_{L'}(B\cap X) & & \end{array}$$

where j is the fibrant model map. This holds for all relevant extension problems, so that the map $B \cap X \to B$ is a pro-equivalence. $\square$

Lemma 11 is the generalization of Lemma 1, to the case of arbitrary small index categories I. Proposition 5 follows from Lemma 1, via a sequence of formal steps given by Lemma 2, Corollary 3 and Lemma 4. These same results apply to the present case of arbitrary I-diagrams, starting from Lemma 11, giving the following result:

Theorem 12. *Suppose that I is a small category I, and that $\mathcal{C}$ is a Grothendieck site. The category $s\mathbf{Pre}^I$ of I-diagrams in simplicial presheaves on $\mathcal{C}$, together with the classes of cofibrations, pro-equivalences and pro-fibrations, satisfies the axioms for a left proper closed simplicial model category. This model structure is cofibrantly generated.*

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