INTRODUCTION TO TEST CATEGORIES

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These notes were taken and LaTeX'd by Chris Kapulkin, from Marek Zawadowski's lecture that was an introduction to the theory of test categories.

In modern homotopy theory it is common to work with the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ of simplicial sets instead of the category category **Top** of topological spaces. These categories are Quillen equivalent, however the former enjoys many good properties that the latter lacks and which make it a good framework to address many homotopy-theoretic questions. It is natural to ask: what is so special about the category Δ? In these notes we will try to characterize categories that can equally well as Δ serve as an environment for homotopy theory. The examples of such categories include, among others, the cube category and Joyal's Θ.

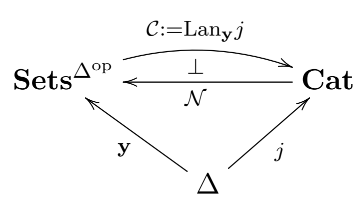
These notes are organized as follows. In sections 1 and 2 we review some standard results about the nerve functor(s) and the homotopy category, respectively. In section 3 we introduce test categories, provide their characterization, and give some examples. In section 4 we define test functors and as in the previous section: provide their characterization and some examples.

1. Background on Nerve Functors

The *nerve functor* $\mathcal{N} \colon \mathbf{Cat} \longrightarrow \mathbf{Sets}^{\Delta^{\mathrm{op}}}$ is given by:

$$\mathcal{N}(C)_n = \mathbf{Cat}(j[n], C),$$

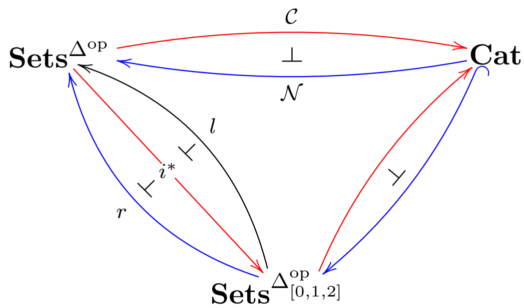
where $j \colon \Delta \hookrightarrow \mathbf{Cat}$ is the obvious inclusion. It has a left adjoint given by the left Kan extension:



Note that this basic setup depends only on **Cat** being cocomplete and j being an arbitrary covariant functor.

Let now $i \colon \Delta_{[0,1,2]} \hookrightarrow \Delta$ be the full subcategory with objects $[0]$, $[1]$, and $[2]$. The presheaf category $\mathbf{Sets}^{\Delta^{\mathrm{op}}_{[0,1,2]}}$ is also known as the category of *precategories*. The inclusion i determines a functor i^* and its two adjoints l and r between the corresponding presheaf categories. The adjunction $\mathcal{C} \vdash \mathcal{N}$

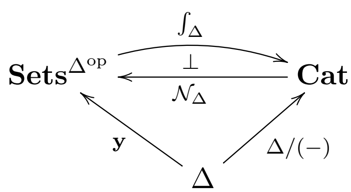
factors through $\mathbf{Sets}^{\Delta^{\mathrm{op}}_{[0,1,2]}}$ as follows:



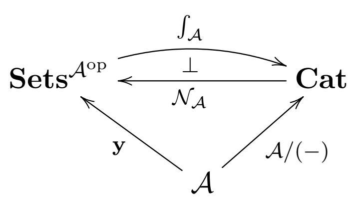
Instead of $j\colon \Delta \hookrightarrow \mathbf{Cat}$ we may consider the embedding given by: $[n] \mapsto \Delta/[n]$. This embedding determines another nerve functor $\mathcal{N}_\Delta\colon \mathbf{Cat} \longrightarrow \mathbf{Sets}^{\Delta^{\mathrm{op}}}$ defined via:

$$\mathcal{N}_\Delta(C)_n = \mathbf{Cat}(\Delta/[n], C).$$

This functor has a left adjoint given again by the left Kan extension; in this case this is the familiar functor $\int_\Delta\colon \mathbf{Sets}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Cat}$ taking a presheaf to its category of elements:



We shall note that this situation does not depend on any specific properties of the category Δ and in fact it might be replaced by an arbitrary small category $\mathcal{A}$:



where again $\mathcal{N}_\mathcal{A}(C)(a) = \mathbf{Cat}(\mathcal{A}/a, C)$ and $\int_\mathcal{A}$ takes a presheaf to its category of elements.

2. The homotopy category

From now on we will write $\widehat{\mathcal{A}}$ for the presheaf category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Sets}]$.

Let $\mathscr{H}$ denote the homotopy category of spaces i.e. the category $\mathbf{Top}[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the usual class of weak equivalences. An outstanding goal of algebraic topology is to describe the category $\mathscr{H}$ in some workable way.

For example, the inclusion $\mathbf{CW} \hookrightarrow \mathbf{Top}$ of CW-complexes into the category of topological spaces induces an equivalence on the level of homotopy categories. There is also a Quillen equivalence between the categories $\widehat{\Delta}$ of simplicial sets and **Top**:

$$\widehat{\Delta} \underset{S}{\overset{|-|}{\rightleftarrows}} \mathbf{Top} \quad (\perp)$$

given by the geometric realization and the singular complex functors, which gives a third description of the homotopy category $\mathscr{H}$.

One can therefore ask whether the category **Cat** can be equipped with a class of weak equivalences $\mathcal{W}$ in such a way that $\mathbf{Cat}[\mathcal{W}^{-1}] \simeq \mathscr{H}$. This can be done using the nerve functor defined earlier. Which one? Well, it doesn't matter! We have $\mathcal{N}^{-1}(\mathcal{W}) = \mathcal{N}_{\Delta}^{-1}(\mathcal{W})$ and we can call this class of weak equivalences $\mathcal{W}_\infty$.

Let us try to understand the class $\mathcal{W}_\infty$ a bit better. It is not hard to show that it contains all equivalences of categories. A less obvious fact is that if a functor $F\colon C \longrightarrow D$ has an adjoint (either left or right), it belongs to $\mathcal{W}_\infty$. To see why this is the case let us note that there is a correspondence:

$$\left\{ \begin{array}{c} \text{natural transformations} \\ \varphi\colon F \Longrightarrow G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{functors } H\colon C \times [1] \longrightarrow D \text{ s.th.} \\ H(-,0) = F \text{ and } H(-,1) = G \end{array} \right\}$$

The nerve functor $\mathcal{N}$ takes $[1]$ to an object isomorphic to the representable $\Delta[1]$. It follows that a functor $F\colon C \longrightarrow D$ for which there exist a functor $G\colon D \longrightarrow C$ and two natural transformations: $\eta\colon 1_C \longrightarrow GF$ and $\varepsilon\colon FG \longrightarrow 1_D$ (not even satisfying the triangle equalities!) belongs to $\mathcal{W}_\infty$.

What this immediately tells us is that categories containing a terminal (or initial) object should be thought of as *contractible* or *aspherical* i.e. weakly equivalent to the terminal category (having one object and no non-identity arrows). This is the case since both the terminal and the initial object in a category C can be described as adjoints to the unique functor $C \longrightarrow \mathbf{1}$.

Even though the two nerve functors described in the previous section induce the same class of weak equivalences on **Cat** there is an important difference between them. While the adjunction $\int_\Delta \vdash \mathcal{N}_\Delta$ is a Quillen adjunction, the adjunction $\mathcal{C} \vdash \mathcal{N}$ is not. Of course, the former is more desirable than the latter.

So our goal for the remainder of this note is, roughly speaking, to find all categories $\mathcal{A}$ with an adjunction:

$$\widehat{\mathcal{A}} \underset{\mathcal{N}_\mathcal{A}}{\overset{\int_\mathcal{A}}{\rightleftarrows}} \mathbf{Cat} \quad (\perp)$$

such that:

- the adjunction $\int_\mathcal{A} \dashv \mathcal{N}_\mathcal{A}$ is Quillen.
- the notion is local (i.e. if $\mathcal{A}$ satisfies these conditions, then so does $\mathcal{A}/a$ for any $a \in \mathcal{A}$).

- the functor $\int_{\mathcal{A}}$ preserves at least finite products and possibly more structure.

3. Modelizers and test categories

We start by developing some formalism that will let us address the problem presented in the previous section.

Definition 1. A *modelizer* is a pair $(\mathbf{M}, \mathcal{W})$ where $\mathbf{M}$ is a category and $\mathcal{W} \subseteq \mathrm{Mor}(\mathbf{M})$ such that:

$$\mathbf{M}[\mathcal{W}^{-1}] \simeq \mathscr{H}.$$

Of course, finding 'all' modelizers is impossible, however if we restrict our attention only to presheaf categories, the question can be made canonical:

Problem 2. Describe all small categories $\mathcal{A}$ for which the adjunction $\int_{\mathcal{A}} \dashv \mathcal{N}_{\mathcal{A}}$ induces an equivalence of homotopy categories, where the weak equivalences in $\widehat{\mathcal{A}}$ are defined via $\mathcal{W}_{\widehat{\mathcal{A}}} := \int_{\mathcal{A}}^{-1}(\mathcal{W}_\infty)$.

Definition 3. A category $\mathcal{A}$ as described in Problem 2 above will be called a *weak test category*.

There is a characterization of weak test categories due to Alexander Grothendieck. Before stating it we need an intermediate notion.

Definition 4. Let C be a small category. We say that C is *aspherical*, if the unique map $C \longrightarrow \mathbf{1}$ is in $\mathcal{W}_\infty$.

A presheaf $X \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$ is said to be *aspherical*, if the unique morphism $\int_{\mathcal{A}} X \longrightarrow \mathbf{1}$ is in $\mathcal{W}_\infty$.

Theorem 5 (Grothendieck)**.** *Let $\mathcal{A}$ be a small category. Then $\mathcal{A}$ is a weak test category if and only if for every small category C with a terminal object, the presheaf $\mathcal{N}_{\mathcal{A}}(C)$ is aspherical.*

However, being a weak test category is **not** a *local* notion i.e. it is not stable under slicing. For example, the category Δ_+ (of finite non-empty ordinals and strictly increasing functions) is a weak test category, but the slice $\Delta_+/[0]$ is not.

Definition 6. Let $\mathcal{A}$ be a small category. Then:

- $\mathcal{A}$ is a *local test category*, if for all $a \in \mathcal{A}$ the slice category $\mathcal{A}/a$ is a weak test category.
- $\mathcal{A}$ is a *test category*, if it is a weak test category and a local test category.
- $\widehat{\mathcal{A}}$ is an *elementary modelizer*, if $\mathcal{A}$ is a test category.

The local test categories may be characterized as follows.

Theorem 7. *Let $\mathcal{A}$ be a small category.*

(1) Then $\mathcal{A}$ is a local test category if and only if the presheaf $\mathcal{N}_{\mathcal{A}}([1])$ is locally asperical i.e. it is aspherical restricted to presheaves $\mathcal{A}/a$ for all $a \in \mathcal{A}$.

(2) *If $\mathcal{A}$ is a local test category and it is aspherical, then $\mathcal{A}$ is a test category.*

The reader familiar with topos theory probably recognizes that the presheaf $\mathcal{N}_{\mathcal{A}}([1])$ is in fact the *subobject classifier* in the presheaf topos $\widehat{\mathcal{A}}$. Indeed, we have a correspondence:

$$\left\{ \begin{array}{c} \text{Functors} \\ s\colon \mathcal{A}/a \longrightarrow \{0 \leq 1\} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Sieves} \\ \text{on } a \end{array} \right\}$$

that takes a functor $s\colon \mathcal{A}/a \longrightarrow \{0 \leq 1\}$ to $s^{-1}(0)$. It is easy to see that this set is closed under precomposition and so it is a sieve. We will therefore denote the presheaf $\mathcal{N}_{\mathcal{A}}([1])$ by $\Omega_{\widehat{\mathcal{A}}}$.

Examples 8. The examples of test categories include:

- the simplex category Δ.
- the cube category.
- Joyal's category Θ.
- the category $\mathbf{FinSets}_{\neq\emptyset}$ of non-empty finite sets.

In fact, any full small subcategory of **Cat** containing [0], [1], closed under finite products and not containing the empty category $\emptyset$ is a test category.

The main 'slogan' of test categories says:

Any test category is as good as Δ to do homotopy theory.

However, given a test category $\mathcal{A}$ we are not guaranteed that the functor $\int_{\mathcal{A}}$ preserves finite products at all (under any reasonable notion of preservation). We may therefore want to distinguish the test categories for which this is the case. In such test categories the product of presheaves will represent the product of the corresponding homotopy types.

Definition 9. Let $\mathcal{A}$ be a test category. We say that:

- $\mathcal{A}$ is a *strict test category*, $\int_{\mathcal{A}}\colon \widehat{\mathcal{A}} \longrightarrow \mathbf{Cat}$ preserves finite products up to a weak equivalence.
- $\widehat{\mathcal{A}}$ is a *strict modelizer*, if $\mathcal{A}$ is a strict test category.

Examples 10. All specific examples mentioned in 8 except the cube category are strict test categories.

4. Test functors

The notion of a test category generalizes the category Δ. There is a corresponding notion of a test functor that generalizes in the same fashion the nerve functor.

We would like to call a functor $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ a weak test functor, if for the induced functor

$$\mathcal{N}_i\colon \mathbf{Cat} \longrightarrow \widehat{\mathcal{A}}$$

given by

$$C \longmapsto \mathbf{Cat}(i(-), C)$$

we have $\mathcal{N}_i(\mathcal{W}_\infty) \subseteq \mathcal{W}_{\widehat{\mathcal{A}}}$ and $\mathcal{N}_i$ induces an equivalence of categories

$$\mathbf{Cat}[\mathcal{W}_\infty^{-1}] \xrightarrow{\simeq} \widehat{\mathcal{A}}[\mathcal{W}_{\widehat{\mathcal{A}}}].$$

The examples we have in mind are of the form $\mathcal{A} \longrightarrow \mathbf{Cat}$ given by $a \mapsto \mathcal{A}/a$.

Sadly, there is no nice characterization of such functors. In search for such a characterization we start with the following definition.

Definition 11. Let $\mathcal{A}$ be a weak test category. A functor $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ is a *weak test functor*, if:

(1) $i(a)$ is aspherical for all $a \in \mathcal{A}$.
(2) $\mathcal{N}_i(\mathcal{W}_\infty) \subseteq \mathcal{W}_{\widehat{\mathcal{A}}}$.

Remark 12. If $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ is full and satisfies condition (2) of definition 11, then i is a weak test functor.

Remark 13. Let $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ be a weak test functor. Then the functor $\mathcal{N}_i$ induces an equivalence of homotopy categories:

$$\mathbf{Cat}[\mathcal{W}_\infty^{-1}] \longrightarrow \widehat{\mathcal{A}}[\mathcal{W}_{\widehat{\mathcal{A}}}^{-1}]$$

with the homotopy inverse given by $\mathrm{Lan}_{\mathbf{y}} i\colon \widehat{\mathcal{A}} \longrightarrow \mathbf{Cat}$.

For this notion of a weak test functor, there exists a nice characterization.

Theorem 14. *Let $\mathcal{A}$ be a weak test category and $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ a functor such that $i(a)$ is aspherical for every $a \in \mathcal{A}$. Then i is a weak test functor if and only if $\mathcal{N}_i(C)$ is aspherical for any small aspherical category C.*

Accordingly, there is a notion of local test functor and a characterization of them.

Definition 15. A functor $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ is a *local test functor*, if for any $a \in \mathcal{A}$ the composite:

$$\mathcal{A}/a \longrightarrow \mathcal{A} \longrightarrow \mathbf{Cat}$$

is a weak test functor.

Theorem 16. *Let $\mathcal{A}$ be a small category and $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ a functor such that $i(a)$ is aspherical for all $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $\mathcal{A}$ is a local test category and i is a local test functor.
(2) $\mathcal{N}_i(\mathcal{C})$ is locally aspherical for all aspherical categories $\mathcal{C}$.

If moreover $\mathcal{A}$ is known to have a terminal object, then these conditions are equivalent to:

(3) $\mathcal{N}_i([1]) = \Omega_{\widehat{\mathcal{A}}}$ is locally aspherical.

It is now natural to expect that the test functors will be defined as the ones that are both weak test functors and local test functors. But the following proposition shows that such a definition would contain some redundancy.

Proposition 17. *If $\mathcal{A}$ is a weak test category and $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ is a local test functor, then i is a weak test functor.*

Definition 18. A functor $i\colon \mathcal{A} \longrightarrow \mathbf{Cat}$ satisfying the assumptions of Proposition 17 will be called a *test functor*.

Examples 19. The following are examples of test functors:

- the paradigmatic example $\Delta \hookrightarrow \mathbf{Cat}$
- the inclusion $\mathcal{A} \hookrightarrow \mathbf{Cat}$ for any full subcategory $\mathcal{A} \subseteq \mathbf{Cat}$ containing the categories $\mathbf{1}$, $[1]$; closed under finite products and not containing the empty category $\emptyset$.
- $\mathcal{P}_{\neq\emptyset}\colon \mathbf{FinSets}_{\neq\emptyset} \longrightarrow \mathbf{Cat}$, where $\mathcal{P}_{\neq\emptyset}(X) = \mathcal{P}(X) \setminus \{\emptyset\}$.

The inclusion $\Delta_+ \hookrightarrow \mathbf{Cat}$ is only a weak test functor, **not** a test functor.

5. Further topics

By inspection of the proofs of the above theorems, one can see that the choice of $\mathcal{W}_\infty$ is only one among many possible. Thus instead of working with the fixed class of weak equivalences $\mathcal{W}_\infty$ in **Cat**, one can try to axiomatize the classes $\mathcal{W}$ for which the above theorems remain true.

Furthermore, if $\mathcal{A}$ is a test category, then the presheaf category $\widehat{\mathcal{A}}$ admits a canonical model structure in which the weak equivalences are maps $\mathcal{W}_{\widehat{\mathcal{A}}}$ as described above and cofibrations are precisely (pointwise) monomorphisms.