Locally Presentable Quasicategories

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1 Introduction

In these notes we will give an overview of the concept of local presentability as it arises in the theory of categories, model categories and quasicategories. Locally presentable categories are well-known and extremely useful. In the homotopy theoretic context we will consider, the analogues of locally presentable categories are combinatorial model categories and locally presentable quasicategories. In this introduction, we will refer to all three simply as locally presentable categories.

The essential idea of local presentability is the same in all three settings: locally presentable categories are generated under certain colimits by a small subcategory. This means that many properties of locally presentable categories are formally determined by this small subcategory.

Since objects in locally presentable categories all can be written as colimits of certain objects, we will investigate the process of formally adjoining colimits to a category. Theorems 3, 20 and 38 show that, in all three settings, given a category $\mathcal{A}$ we can construct its free cocompletion as a category of (simplicial) presheaves.

Intuitively, since locally presentable categories $\mathcal{C}$ are generated under colimits by a small subcategory $\mathcal{A}$, we might hope to recapture $\mathcal{C}$ by formally adjoining colimits to $\mathcal{A}$, and then quotienting by some set of relations to recover $\mathcal{C}$. We will see that versions of this plan can be put into practice to obtain representation theorems characterising locally presentable categories: these are Theorems 16, 25 and 48.

Both locally presentable quasicategories and combinatorial model categories are attempts to extend the idea of local presentability from categories to $(\infty, 1)$-categories. Using the representation theorems, we will see in Theorem 49 that these two approaches agree: a quasicategory is locally presentable if and only if it can be obtained from a combinatorial model category. This improves on the results we have already seen comparing models of homotopy theories: not only do quasicategories capture models such as relative categories - meaning that they capture the most general (and least usable) notions of a homotopy theory - but, for important model categories we are interested in studying, we get an explicit, usable correspondence. The notes will give a taste of the fact that, although they agree in the sense of Theorem 49, working with model categories and their underlying quasicategory can require relatively dissimilar technical methods.

The overall structure of these notes follows [Gro10, Section 3]. The material in Section 2 can be found in [AR94], Section 3 follows [Dug01b, Dug01a], and Section 4 follows [Lur09].

1.1 Notation and Terminology

We will denote the category of simplicial sets by **sSet** and the category of simplicially enriched categories by **SCat**. For a category $\mathcal{A}$, the category of **presheaves** will be denoted $\mathrm{Pre}(\mathcal{A}) = \mathbf{Set}^{\mathcal{A}^{op}}$, and **simplicial presheaves** will be denoted $\mathrm{sPre}(\mathcal{A}) = \mathrm{s}\mathbf{Set}^{\mathcal{A}^{op}}$.

Given objects $a, b \in \mathcal{A}$ in a simplicial category, we will denote their mapping space by $\underline{\mathcal{A}}(a, b)$, and the underlying set of 0-simplices by $\mathcal{A}(a, b)$. Simplicial sets with the Quillen model structure will be denoted $\mathbf{sSet_Q}$ and the weak equivalences will be called **Kan equivalences**; simplicial sets with the Joyal model structure will be denoted $\mathbf{sSet_J}$ and the weak equivalences called **Joyal equivalences**. The internal hom in **sSet** will be denoted $\mathrm{Fun}(X, Y)$. We will denote the **rigidification functor** by $\mathfrak{C} : \mathbf{sSet} \to \mathbf{SCat}$, and its right adjoint the **coherent nerve** by $\mathbf{N} : \mathbf{SCat} \to \mathbf{sSet}$.

For any model category $\mathcal{M}$, its category of bifibrant objects will be denoted $\mathcal{M}^{\circ}$.

2 Locally Presentable Categories

Definition 1. Let $\mathcal{C}$ be a category and $\mathcal{A} \subseteq \mathcal{C}$ a full subcategory. For any object $x \in \mathcal{C}$ the **canonical diagram of** x **with respect to** $\mathcal{A}$ is the projection functor:

$$\begin{array}{ccc} \mathcal{A} \downarrow x & \longrightarrow & \mathcal{C} \\ (a \to x) & \mapsto & a \end{array}$$

We call the colimit of this diagram the **canonical colimit of** x **with respect to** $\mathcal{A}$. We will write $\mathrm{Colim}(\mathcal{A} \downarrow x)$ for this colimit.

Example 2. We may consider any category $\mathcal{A}$ as a full subcategory of $\mathrm{Pre}(\mathcal{A})$ via the Yoneda embedding. Any functor $X \in \mathrm{Pre}(\mathcal{A})$ is isomorphic to its canonical colimit:

$$\mathrm{Colim}(\mathcal{A} \downarrow X) = \underset{\mathcal{A}(-,a) \to X}{\mathrm{Colim}} \mathcal{A}(-, a) \cong X$$

A full subcategory $\mathcal{A} \subseteq \mathcal{C}$ for which every $x \in \mathcal{C}$ is isomorphic to its canonical colimit relative to $\mathcal{A}$ is called **dense** in $\mathcal{C}$.

Example 2 allows us to show that the Yoneda embedding $y : \mathcal{A} \to \mathrm{Pre}(\mathcal{A})$ is the **free cocompletion** of any category $\mathcal{A}$:

Theorem 3. *Let $\mathcal{A}$ be a category. Given any cocomplete category $\mathcal{E}$ and any functor $F : \mathcal{A} \to \mathcal{E}$, there is a cocontinuous functor $\tilde{F} : \mathrm{Pre}(\mathcal{A}) \to \mathcal{E}$ and a natural isomorphism $\tilde{F} \circ y \cong F$. Moreover, this extension is unique up to isomorphism.*

Proof. Given any functor $F : \mathcal{A} \to \mathcal{E}$, we take $\tilde{F}$ to be its left Kan extension along the Yoneda embedding:

$$\tilde{F} = y_! F : \mathrm{Pre}(\mathcal{A}) \to \mathcal{E}$$

Because y is fully faithful, there is a natural isomorphism $y_! F \circ y \cong F$, so $y_! F$ is an honest extension of F to $\mathrm{Pre}(\mathcal{A})$. Since $\mathcal{E}$ is cocomplete we have a pointwise formula for $y_! F$:

$$y_! F(X) = \mathrm{Colim}\Big((\mathcal{A} \downarrow X) \longrightarrow \mathcal{A} \xrightarrow{F} \mathcal{E}\Big)$$

Using this formula, and the description in Example 2 of any presheaf as a colimit of representables, we can show that the following functor is a right adjoint for $y_! F$:

$$\begin{array}{rcl} R : \mathcal{E} & \longrightarrow & \mathrm{Pre}(\mathcal{A}) \\ e & \mapsto & \mathcal{E}(F(-), e) \end{array}$$

This shows that $y_! F$ is cocontinuous.

Finally, by Example 2, $\mathcal{A}$ is dense in $\mathrm{Pre}(\mathcal{A})$. Thus, cocontinuous functors from $\mathrm{Pre}(\mathcal{A})$ which agree on representables must agree on the whole of $\mathrm{Pre}(\mathcal{A})$. It follows that $y_! F$ is the essentially unique cocontinuous extension of F to $\mathrm{Pre}(\mathcal{A})$. □

Remark 4. The proof of Theorem 3 shows that precomposition with the Yoneda embedding induces an equivalence of categories:

$$y^* : \mathrm{Cocts}(\mathrm{Pre}(\mathcal{A}), \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{E})$$

where $\mathrm{Cocts}(\mathrm{Pre}(\mathcal{A}), \mathcal{E})$ denotes the category of cocontinuous functors from $\mathrm{Pre}(\mathcal{A})$ to $\mathcal{E}$.

Given an infinite cardinal λ, a diagram $J \to \mathcal{C}$ is called **λ-small** if $|\mathrm{Mor}(J)| < \lambda$. Recall that an infinite cardinal λ is **regular** if the category $\mathbf{Set}_{<\lambda}$ of sets with cardinality less than λ is closed under λ-small colimits. Equivalently, a regular cardinal λ cannot be expressed as a sum $\lambda = \sum_{i<\alpha} \gamma_i$ where $\gamma_i < \lambda$ and $\alpha < \lambda$.

From now on, λ will always denote a regular cardinal.

Definition 5. A category J is **λ-filtered** if any λ-small diagram in J has a cocone. That is, any λ-small diagram $D \to J$ extends to a diagram $D^{\triangleright} \to J$, where $D^{\triangleright}$ is obtained by freely adjoining a terminal object to D.

Definition 6. An object $c \in \mathcal{C}$ is called **λ-presentable** if the corepresented functor

$$\mathcal{C}(c, -) : \mathcal{C} \to \mathbf{Set}$$

preserves λ-filtered colimits. An object is called **finitely presentable** if it is $\aleph_0$-presentable.

Example 7. A set is λ-presentable in **Set** if and only if it has cardinality less than λ.

Example 8. A group is finitely presentable in **Grp** in the sense of Definition 6 if and only if it has a finite presentation in the group theoretic sense.

Definition 9. A category $\mathcal{C}$ is **locally λ-presentable** if it is cocomplete and there is a set $\mathcal{A} \subseteq \mathrm{Ob}(\mathcal{C})$ consisting of λ-presentable objects, such that every object in $\mathcal{C}$ can be expressed as a λ-filtered colimit of objects from $\mathcal{A}$. We call a category **locally presentable** if it is locally λ-presentable for some λ.

Example 10 ([AR94, Example 1.12])**.** For any small category $\mathcal{A}$, we can use the Yoneda lemma to show that all representable functors are finitely presentable in $\mathrm{Pre}(\mathcal{A})$. This can be used to show that $\mathrm{Pre}(\mathcal{A})$ is locally finitely presentable.

A locally λ-presentable category $\mathcal{C}$ has only a set of isomorphism classes of λ-presentable objects. Picking representatives for each isomorphism class we obtain a set $\mathcal{C}_\lambda \subseteq \mathrm{Ob}(\mathcal{C})$. We will also write $\mathcal{C}_\lambda$ for the full subcategory on these objects.

Remark 11 ([AR94, Prop 1.22]). If $\mathcal{C}$ is locally λ-presentable, then $\mathcal{C}_\lambda \subseteq \mathcal{C}$ is dense. Moreover, for any $c \in \mathcal{C}$, the slice category $(\mathcal{C}_\lambda \downarrow c)$ is λ-filtered. Thus, any object in a locally λ-presentable category can be written in a canonical way as a λ-filtered colimit of λ-presentable objects.

Definition 12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories admitting λ-filtered colimits. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **λ-accessible** if it preserves all λ-filtered colimits.

2.1 Reflective Localisation of Categories

Definition 13. Let $\mathcal{C}$ be a category and let S be a class of maps in $\mathcal{C}$.

1. An object $z \in \mathcal{C}$ is called **S-local** if, for every map $s : a \to b$ in S, precomposition with s induces an isomorphism:

$$s^* : \mathcal{C}(b, z) \to \mathcal{C}(a, z)$$

2. A map $f : a \to b$ is called an **S-equivalence** if, for any S-local object z, precomposition with f induces an isomorphism:

$$f^* : \mathcal{C}(b, z) \to \mathcal{C}(a, z)$$

Exercise 14 ([Gro10, Prop 3.8])**.** Let $\mathcal{A} \subseteq \mathcal{C}$ be a reflective subcategory. Denote the inclusion by $R : \mathcal{A} \to \mathcal{C}$ and its left adjoint by $L : \mathcal{C} \to \mathcal{A}$. Let S be the set of all maps f in $\mathcal{C}$ for which $L(f)$ is an isomorphism. Then:

1. The essential image of $R : \mathcal{A} \to \mathcal{C}$ consists precisely of the S-local objects.
2. The S-equivalences are exactly the maps in S.

In the situation of Exercise 14, we claim that $\mathcal{A}$ is a localisation of $\mathcal{C}$ with respect to S. Since the right adjoint $R : \mathcal{A} \to \mathcal{C}$ is fully faithful, the counit of the adjunction $\varepsilon : L \circ R \Rightarrow \mathrm{id}_{\mathcal{A}}$ is an isomorphism. Thus, if we write η for the unit, the triangle identities for the adjunction imply that $L \circ \eta : L \Rightarrow L \circ R \circ L$ is also an isomorphism. That is, for any $c \in \mathcal{C}$, the map

$$L\eta_c : L(c) \to LRL(c)$$

is an isomorphism. That is, η_c is in S.

Let $F : \mathcal{C} \to \mathcal{E}$ be a functor taking every map in S to an isomorphism, and consider the functor:

$$\tilde{F} := F \circ R : \mathcal{A} \to \mathcal{E}$$

For any $c \in \mathcal{C}$, since η_c is in S, applying F gives an isomorphism:

$$F\eta_c : F(c) \to \tilde{F}L(c)$$

These give a natural isomorphism $F \cong \tilde{F} \circ L$.

Given any two functors $H, G : \mathcal{A} \to \mathcal{E}$ with $H \circ L \cong G \circ L$, using the isomorphism ε we obtain:

$$H \cong H \circ L \circ R \cong G \circ L \circ R \cong G$$

This shows that the functor $L : \mathcal{C} \to \mathcal{A}$ has the universal property of a localisation at S. With this in mind, we will refer to reflective subcategories as localisations:

Definition 15. Let $L : \mathcal{C} \rightleftarrows \mathcal{A} : R$ be an adjunction. We call L a **reflective localisation** if R is fully faithful, and **accessible** if in addition R is an accessible functor.

Theorem 16 ([AR94, Thm 1.46])**.** *A category $\mathcal{C}$ is locally λ-presentable if and only if it is equivalent to a reflective, λ-accessible localisation of* $\mathrm{Pre}(\mathcal{A})$ *for some small category $\mathcal{A}$.*

We will omit the proof of Theorem 16. Note, however, that if we start with a locally λ-presentable category $\mathcal{C}$, we may take $\mathcal{A} = \mathcal{C}_\lambda$.

3 Combinatorial Model Categories

Recall that a model category $\mathcal{M}$ is called **combinatorial** if it is both cofibrantly generated and locally presentable. Given any category $\mathcal{A}$, we may consider simplicial presheaves $\mathrm{sPre}(\mathcal{A})$ with the **projective model structure**, for which fibrations are given by pointwise Kan fibrations, and weak equivalences are pointwise Kan equivalences. This forms a combinatorial model category, with generating cofibrations given by maps of the form:

$$\mathcal{A}(-,a) \times \partial\Delta^n \to \mathcal{A}(-,a) \times \Delta^n$$

for all $a \in \mathcal{A}$ and $n \geq 0$. Here the simplicial sets $\partial\Delta^n$ and Δ^n induce constant simplicial presheaves, and the presheaf $\mathcal{A}(-,a)$ is thought of as a discrete simplicial presheaf. Note that, taking $n = 0$, we see that all representable functors are cofibrant.

Composing the Yoneda embedding with the discrete simplicial presheaf functor, we can see $\mathcal{A}$ as a full subcategory of $\mathrm{sPre}(\mathcal{A})$:

$$\mathcal{A} \xrightarrow{y} \mathrm{Pre}(\mathcal{A}) \longrightarrow \mathrm{sPre}(\mathcal{A})$$

We will denote this composite simply by $y : \mathcal{A} \to \mathrm{sPre}(\mathcal{A})$.

Given any $X \in \mathrm{sPre}(\mathcal{A})$, we can consider the canonical diagram of X with respect to two different full subcategories:

$$\begin{array}{ccc} \mathcal{A} \downarrow X & \longrightarrow & \mathrm{sPre}(\mathcal{A}) \\ (\mathcal{A}(-,a) \to X) & \mapsto & \mathcal{A}(-,a) \end{array} \qquad \begin{array}{ccc} \mathcal{A} \times \Delta \downarrow X & \longrightarrow & \mathrm{sPre}(\mathcal{A}) \\ (\mathcal{A}(-,a) \times \Delta^n \to X) & \mapsto & \mathcal{A}(-,a) \times \Delta^n \end{array}$$

We call the homotopy colimit of the second diagram the **canonical homotopy colimit of X with respect to $\mathcal{A}$**. We will denote it by $\mathrm{Hocolim}(\mathcal{A} \times \Delta \downarrow X)$.

Remark 17. When we write $\mathrm{Hocolim}(\mathcal{A} \times \Delta \downarrow X)$, we will always mean the Bousfield-Kan presentation of the homotopy colimit, which is available because $\mathrm{sPre}(\mathcal{A})$ is a simplicial model category. For a description of the Bousfield-Kan formula see [Dug01b].

Theorem 18 ([Dug01b, Prop 2.9])**.** *For any $X \in \mathrm{sPre}(\mathcal{A})$ the canonical homotopy colimit $\mathrm{Hocolim}(\mathcal{A} \times \Delta \downarrow X)$ is cofibrant, and the natural map:*

$$\mathrm{Hocolim}(\mathcal{A} \times \Delta \downarrow X) \longrightarrow \mathrm{Colim}(\mathcal{A} \times \Delta \downarrow X) \xrightarrow{\cong} X$$

is a weak equivalence. Thus, $\mathrm{Hocolim}(\mathcal{A} \times \Delta \downarrow X)$ is a cofibrant replacement for X in the projective model structure on $\mathrm{sPre}(\mathcal{A})$.

Note that the second map in Theorem 18 is an isomorphism by Example 2. In fact, Theorem 18 can be seen as an analogue of Example 2 for simplicial presheaves. We will now work towards the analogue of Theorem 3 in this setting, which will be Theorem 20:

Definition 19. Let $\mathcal{A}$ be a category and let $\mathcal{M}$ and $\mathcal{N}$ be model categories. Given functors $F : \mathcal{A} \to \mathcal{M}$ and $G : \mathcal{A} \to \mathcal{N}$, a **factorisation of F through G** consists of a left Quillen functor $L : \mathcal{N} \to \mathcal{M}$, together with a natural weak equivalence $\theta : L \circ G \overset{\sim}{\Rightarrow} F$.

Given two factorisations (L, θ) and (L', θ'), a **map of factorisations** consists of a natural transformation $\alpha : L \Rightarrow L'$ making the diagram below commute:

$$\begin{tikzcd} L \circ G \arrow[rr, Rightarrow, "\alpha \circ G"] \arrow[dr, Rightarrow, "\theta"'] & & L' \circ G \arrow[dl, Rightarrow, "\theta'"] \\ & F & \end{tikzcd}$$

Factorisations and the maps betweeen them form a category $\mathrm{Fact}_G (F)$.

Theorem 20 ([Dug01b, Prop 1.1])**.** *Let $\mathcal{A}$ be a category and let $\mathcal{M}$ be a model category. Given any functor $F : \mathcal{A} \to \mathcal{M}$, there is a factorisation of F through $y : \mathcal{A} \to \mathrm{sPre}(\mathcal{A})$. Moreover, the nerve of the category $\mathrm{Fact}_y (F)$ is contractible.*

Although we won't construct it here, if $\mathcal{M}$ is a simplicial model category there is a canonical choice for such a factorisation: we can find a left Quillen functor $L : \mathrm{sPre}(\mathcal{A}) \to \mathcal{M}$, whose right adjoint is given by:

$$\begin{aligned} R : \mathcal{M} &\longrightarrow \mathrm{sPre}(\mathcal{A}) \\ x &\mapsto \underline{\mathcal{M}}\left((F(-))^{cof}, x \right) \end{aligned}$$

Here $(-)^{cof}$ denotes cofibrant replacement in $\mathcal{M}$.

3.1 Bousfield Localisation of Combinatorial Model Categories

Definition 21. Let $\mathcal{M}$ be a simplicial model category, and let S be a set of cofibrations between cofibrant objects in $\mathcal{M}$.

1. An object $z \in \mathcal{M}$ is called **S-local** if it is fibrant and, for every map $s : a \to b$ in S, precomposition with s induces a Kan equivalence:
$$s^* : \underline{\mathcal{M}}(b, z) \to \underline{\mathcal{M}}(a, z)$$
2. A map $f : a \to b$ is called an **S-equivalence** if, for any S-local object z, precomposition with a cofibrant replacement of f induces a Kan equivalence:
$$f^* : \underline{\mathcal{M}}(b, z) \to \underline{\mathcal{M}}(a, z)$$

The conditions on the set S are only there for technical reasons: if we start with a general set of maps, we can take cofibrant replacements to obtain a set which meets these requirements. Thus, we can make sense of localisation at any set of maps S. Note that, since each S-local object is fibrant, any weak equivalence in $\mathcal{M}$ is an S-equivalence.

Definition 22. Let $\mathcal{M}$ be a simplicial model category and let S be a class of maps in $\mathcal{M}$. A **left Bousfield localisation** of $\mathcal{M}$ at S is a model structure $\mathcal{M}/S$ on the same underlying category such that:

1. The cofibrations of $\mathcal{M}/S$ are the same as those of $\mathcal{M}$.
2. The weak equivalences are the S-equivalences.

Since all weak equivalences in $\mathcal{M}$ are S-equivalences, $\mathcal{M}/S$ has more weak equivalences than the original model structure; since it has the same cofibrations, it follows that $\mathcal{M}/S$ has fewer fibrations than $\mathcal{M}$. The S-local objects will become the fibrant objects in the new model structure $\mathcal{M}/S$, but it is often difficult to give a complete characterisation of which fibrations in $\mathcal{M}$ remain fibrations in $\mathcal{M}/S$.

Lemma 23 shows that $\mathcal{M}/S$ has the universal property we expect from a localisation:

Lemma 23. *Let $\mathcal{M}$ be a simplicial model category and let S be a set of cofibrations between cofibrant objects. Let $L : \mathcal{M} \to \mathcal{N}$ be a left Quillen functor which takes maps in S to weak equivalences. Then L descends to a left Quillen functor $L : \mathcal{M}/S \to \mathcal{N}$.*

Definition 24. Let $\mathcal{M}$ be a model category. A **small presentation** for $\mathcal{M}$ consists of a small category $\mathcal{A}$, a left Quillen functor $L : \mathrm{sPre}(\mathcal{A}) \to \mathcal{M}$, and a set of cofibrations S between cofibrant objects in $\mathrm{sPre}(\mathcal{A})$, such that:

1. $L : \mathrm{sPre}(\mathcal{A}) \to \mathcal{M}$ takes maps in S to weak equivalences.
2. The induced left Quillen functor of Lemma 23, $L : \mathrm{sPre}(\mathcal{A})/S \to \mathcal{M}$, is a Quillen equivalence.

The following theorem can be seen as the analogue of Theorem 16 for combinatorial model categories:

Theorem 25 ([Dug01a, Thm 1.1])**.** *Every combinatorial model category $\mathcal{M}$ has a small presentation.*

For a general $\mathcal{M}$, the proof of Theorem 25 is relatively involved, but in the case where $\mathcal{M}$ is a simplicial model category we can give a description of the category $\mathcal{A}$ that is clearly analogous to Theorem 16. To do this, we need the following:

Lemma 26 ([Dug01a, Prop 4.7])**.** *Let $\mathcal{M}$ be a combinatorial model category, and let $\mathcal{M}_\lambda^{cof}$ be the full subcategory of $\mathcal{M}_\lambda$ on cofibrant objects. For sufficiently large regular cardinals λ the canonical map*

$$\mathrm{Hocolim}\left(\mathcal{M}_\lambda^{cof} \downarrow x\right) \to x$$

is a weak equivalence for any $x \in \mathcal{M}$.

Suppose $\mathcal{M}$ is a combinatorial simplicial model category, and choose a regular cardinal λ large enough that Lemma 26 holds. Then we can obtain a presentation for $\mathcal{M}$, taking $\mathcal{A} = \mathcal{M}_\lambda^{cof}$. As in Theorem 20, we obtain a left Quillen functor $L : \mathrm{sPre}\left(\mathcal{M}_\lambda^{cof}\right) \to \mathcal{M}$ whose right adjoint is given by:

$$\begin{array}{rcl} R : \mathcal{M} & \longrightarrow & \mathrm{sPre}\left(\mathcal{M}_\lambda^{cof}\right) \\ x & \mapsto & \underline{\mathcal{M}}(-, x) \end{array}$$

As in [Dug01a, Prop 3.2], we can find a set of maps S in $\mathrm{sPre}\left(\mathcal{M}_\lambda^{cof}\right)$ such that L descends to a Quillen equivalence:

$$L : \mathrm{sPre}\left(\mathcal{M}_\lambda^{cof}\right)/S \to \mathcal{M}$$

4 Locally Presentable Quasicategories

Definition 27. Let $\mathcal{M}$ be combinatorial simplicial model category. A **chunk** of $\mathcal{M}$ is a full subcategory $\mathcal{U} \subseteq \mathcal{M}$ satisfying:

1. For any $a \in \mathcal{U}$ and any finite set of maps $\phi_i : a \to b_i$ in $\mathcal{U}$, we have a factorisation:

$$\begin{tikzcd} & \bar{a} \arrow[dr, "q"] & \\ a \arrow[ur, "j"] \arrow[rr, "(\phi_i)"'] & & \prod_i b_i \end{tikzcd}$$

   with j a trivial cofibration, q a fibration, and $\overline{a} \in \mathcal{U}$. Moreover, this factorisation gives a simplicial functor in ϕ_i.

2. For any $a \in \mathcal{U}$ and any finite set of maps $\varphi_i : b_i \to a$ in $\mathcal{U}$, we have a factorisation:

$$\begin{tikzcd} & \bar{a} \arrow[dr, "q"] & \\ \coprod_i b_i \arrow[ur, "j"] \arrow[rr, "(\varphi_i)"'] & & a \end{tikzcd}$$

   with j a cofibration, q a trivial fibration, and $\overline{a} \in \mathcal{U}$. Moreover, this factorisation gives a simplicial functor in φ_i.

Given a chunk $\mathcal{U} \subseteq \mathcal{M}$, we will write $\mathcal{U}^\circ := \mathcal{M}^\circ \cap \mathcal{U}$.

For a simplicial category $\mathcal{A}$ and a combinatorial simplicial model category $\mathcal{M}$, we can form the simplicial category of simplicial functors $\mathcal{M}^{\mathcal{A}}$. Just as in the unenriched setting, we have the projective model structure on $\mathcal{M}^{\mathcal{A}}$: the weak equivalences are the pointwise Kan equivalences, and the fibrations the pointwise Kan fibrations. See [Lur09, Section A.3.3] for a discussion of model structures on enriched functor categories.

Definition 28. Given a simplicial model category $\mathcal{A}$, a chunk $\mathcal{U} \subseteq \mathcal{M}$ is called an **$\mathcal{A}$-chunk** if the full subcategory $\mathcal{U}^{\mathcal{A}} \subseteq \mathcal{M}^{\mathcal{A}}$ is a chunk, where we consider $\mathcal{M}^{\mathcal{A}}$ with the projective model structure.

Example 29. For any simplicial category $\mathcal{A}$, any combinatorial simplicial model category $\mathcal{M}$ is an $\mathcal{A}$-chunk of itself. The factorisations we require can be obtained using the small object argument.

Theorem 30 ([Lur09, Cor A.3.4.14])**.** *Let $\mathcal{M}$ be a combinatorial simplicial model category and let $\mathcal{A}$ be a small simplicial category. Let $\mathcal{U} \subseteq \mathcal{M}$ be an $\mathcal{A}$-chunk. For any small simplicial category $\mathcal{D}$, the evaluation map*

$$ev : \mathcal{A} \times \left(\mathcal{U}^{\mathcal{A}}\right)^{\circ} \to \mathcal{U}^{\circ}$$

induces a bijection:

$$\mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathcal{D}, \left(\mathcal{U}^{\mathcal{A}}\right)^{\circ}\right) \xrightarrow{\cong} \mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathcal{A} \times \mathcal{D}, \mathcal{U}^{\circ}\right)$$

Theorem 31 ([Lur09, Prop 4.2.4.4])**.** *Let $\mathcal{M}$ be a combinatorial simplicial model category, and let X be a small simplicial set. Let $\phi : \mathfrak{C}X \to \mathcal{A}$ be a DK-equivalence with $\mathcal{A}$ a small simplicial category, and let $\mathcal{U} \subseteq \mathcal{M}$ be an $\mathcal{A}$-chunk.*

Then there is a Joyal equivalence:

$$\mathbf{N}\left(\left(\mathcal{U}^{\mathcal{A}}\right)^{\circ}\right) \to \mathrm{Fun}\left(X, \mathbf{N}\left(\mathcal{U}^{\circ}\right)\right)$$

Proof. Although we won't prove it, we can reduce to the case where $\mathcal{U}$ is small; this is possible using the proof of [Lur09, A.3.4.15]. For any simplicial set K we have the following chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Ho}\left(\mathbf{sSet}_J\right)\left(K, \mathbf{N}\left(\left(\mathcal{U}^{\mathcal{A}}\right)^{\circ}\right)\right) &\cong \mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathfrak{C}K, \left(\mathcal{U}^{\mathcal{A}}\right)^{\circ}\right) \\
&\cong \mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathcal{A} \times \mathfrak{C}K, \mathcal{U}^{\circ}\right) \\
&\cong \mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathfrak{C}X \times \mathfrak{C}K, \mathcal{U}^{\circ}\right) \\
&\cong \mathrm{Ho}\left(\mathbf{SCat}\right)\left(\mathfrak{C}\left(X \times K\right), \mathcal{U}^{\circ}\right) \\
&\cong \mathrm{Ho}\left(\mathbf{sSet}_J\right)\left(X \times K, \mathbf{N}\left(\mathcal{U}^{\circ}\right)\right) \\
&\cong \mathrm{Ho}\left(\mathbf{sSet}_J\right)\left(K, \mathrm{Fun}\left(X, \mathbf{N}\left(\mathcal{U}^{\circ}\right)\right)\right)
\end{aligned}$$

Most of these isomorphisms are self-explanatory. The second follows from Theorem 30, the fourth from the fact that $\mathfrak{C}$ preserves finite products up to weak equivalence, and the final isomorphism follows from the fact that the Joyal model structure on **sSet** is cartesian. The Yoneda lemma implies that we have an isomorphism:

$$\mathbf{N}\left(\left(\mathcal{U}^{\mathcal{A}}\right)^{\circ}\right) \xrightarrow{\cong} \mathrm{Fun}\left(X, \mathbf{N}\left(\mathcal{U}^{\circ}\right)\right)$$

is $\mathrm{Ho}\left(\mathbf{sSet}_J\right)$. Since both objects are bifibrant, this must be induced by a Joyal equivalence in **sSet**. ∎

Recall that the quasicategory of **spaces** is given by:

$$\mathcal{S} := \mathbf{N}(\mathcal{K}an) = \mathbf{N}\left(\mathbf{sSet}_Q^\circ\right)$$

Definition 32. Given any $X \in \mathbf{sSet}$, the quasicategory of **presheaves** on X is given by:

$$\mathcal{P}(X) := \mathrm{Fun}(X^{op}, \mathcal{S})$$

Corollary 33. *For any small simplicial set X we have a canonical Joyal equivalence:*

$$\mathbf{N}\left(\left(\mathbf{sSet}^{\mathfrak{C}X^{op}}\right)^\circ\right) \xrightarrow{\sim} \mathcal{P}(X)$$

Proof. This follows immediately from Theorem 31. □

Theorem 31 is an important step in proving the following result:

Theorem 34 ([Lur09, Cor 4.2.4.8])**.** *If $\mathcal{M}$ is a combinatorial simplicial model category then the associated quasicategory $\mathbf{N}(\mathcal{M}^\circ)$ admits all small limits and colimits.*

We will omit the proof of Theorem 34, since it would take us too far out of our way. However, Theorem 34 and Corollary 33 imply immediately that presheaf quasicategories have all small limits and colimits. Moreover, limits and colimits can be computed pointwise:

Theorem 35 ([Lur09, Cor 5.1.2.3])**.** *Let X be a simplicial set. A map*

$$p : K^{\triangleright} \to \mathcal{P}(X)$$

is a colimit diagram if and only if, for each vertex $x \in X$, the induced map

$$p_x : K^{\triangleright} \to \mathcal{S}$$

is a colimit diagram.

Given a simplicial category $\mathcal{C}$, we can form a fibrant replacement for $\mathcal{C}$ in **SCat** by applying Ex^∞ to each mapping space. Taking mapping spaces in this fibrant replacement induces a simplicial functor:

$$\begin{array}{ccc} \mathcal{C}^{op} \times \mathcal{C} & \longrightarrow & \mathcal{K}an \\ (a,b) & \mapsto & \mathrm{Ex}^\infty\left(\underline{\mathcal{C}}(a,b)\right) \end{array}$$

Given any $X \in \mathbf{sSet}$ we can consider this map for $\mathfrak{C}X$:

$$\mathfrak{C}(X^{op} \times X) \to \mathfrak{C}X^{op} \times \mathfrak{C}X \to \mathcal{K}an$$

Under the adjunction $\mathfrak{C} \dashv \mathbf{N}$ this corresponds to:

$$X^{op} \times X \to \mathcal{S}$$

This, in turn, corresponds under the internal hom adjunction to a map which we call the **Yoneda embedding**:

$$y : X \to \mathcal{P}(X)$$

Theorem 36 (Yoneda Lemma for Quasicategories)**.** *For any $X \in \mathbf{sSet}$, the Yoneda embedding $y : X \to \mathcal{P}(X)$ is fully faithful.*

Proof. Let $\mathcal{C}$ be a fibrant replacement for $\mathfrak{C}X^{op}$ in **SCat**. We can factor y as follows:

$$X \xrightarrow{j} \mathbf{N}\left(\left(\mathbf{sSet}^{\mathcal{C}}\right)^{\circ}\right) \xrightarrow{\sim} \mathrm{Fun}\left(X^{op}, \mathbf{N}\left(\mathcal{K}an\right)\right)$$

The second map is the Joyal equivalence of Theorem 31, so we need only show that j is fully faithful. However, j may be written as the composite:

$$X \to \mathbf{N}\left(\mathfrak{C}X\right) \to \mathbf{N}\left(\mathcal{C}^{op}\right) \to \mathbf{N}\left(\left(\mathbf{sSet}^{\mathcal{C}}\right)^{\circ}\right)$$

The map $X \to \mathbf{N}\left(\mathcal{C}^{op}\right)$ is the derived unit of the Quillen equivalence $\mathfrak{C} \dashv \mathbf{N}$, so it is a Joyal equivalence. The final map is obtained by applying the homotopy coherent nerve to the simplicially enriched Yoneda embedding:

$$y : \mathcal{C}^{op} \to \mathbf{sSet}^{\mathcal{C}}$$

Note that, since $\mathcal{C}$ is fibrant, the Yoneda embedding does indeed factor through the bifibrant objects of $\mathbf{sSet}^{\mathcal{C}}$. By the simplicially enriched Yoneda lemma, this map induces isomorphisms on mapping spaces. $\square$

We now give the analogue of Example 2 for quasicategories:

Lemma 37 ([Lur09, Lemma 5.1.5.3])**.** *Let X be a small simplicial set, and identify it with its image in $\mathcal{P}(X)$ under the Yoneda embedding. For any vertex $F \in \mathcal{P}(X)$ the map*

$$\left(X \downarrow F\right)^{\triangleright} \to \mathcal{P}(X)$$

which takes the cone point to F, is a colimit diagram.

This leads to the quasicategorical analogue of Theorem 3:

Theorem 38 ([Lur09, Thm 5.1.5.6])**.** *Let X be a small simplicial set and let $\mathcal{C}$ be a quasicategory admitting all small colimits. Composition with $y : X \to \mathcal{P}(X)$ induces a Joyal equivalence:*

$$y^* : \mathrm{Cocts}\left(\mathcal{P}(X), \mathcal{C}\right) \xrightarrow{\sim} \mathrm{Fun}\left(X, \mathcal{C}\right)$$

Here $\mathrm{Cocts}\left(\mathcal{P}(X), \mathcal{C}\right)$ denotes the full subcategory of $\mathrm{Fun}\left(\mathcal{P}(X), \mathcal{C}\right)$ on the functors that preserve small colimits.

Definition 39. Let X be a simplicial set and consider a vertex $x \in X$. We have the following composite:

$$X \xrightarrow{y \times x} \mathcal{P}(X) \times X^{op} \longrightarrow \mathcal{S}$$

Denote this map by $y_x : X \to \mathcal{S}$. We call this the functor **corepresented** by x.

Fix a regulat cardinal λ. We can now make quasicategorical versions of Definitions 5, 6 and 9:

Definition 40. A quasicategory $\mathcal{C}$ is called **λ-filtered** if, for every λ-small simplicial set K and every map $f : K \to \mathcal{C}$, there is an extension of f to a diagram:

$$\overline{f} : K^{\triangleright} \longrightarrow \mathcal{C}$$

We call an arbitrary simplicial set λ-filtered if it is Joyal equivalent to a λ-filtered quasicategory.

Definition 41. Let $\mathcal{C}$ be a quasicategory. An object $x \in \mathcal{C}$ is **λ-presentable** if the corepresented functor $y_x : \mathcal{C} \to \mathcal{S}$ preserves λ-filtered colimits.

In [Lur09], λ-presentable objects are called **λ-compact**.

Definition 42. A locally small quasicategory $\mathcal{C}$ is called **locally λ-presentable** if $\mathcal{C}$ admits all small colimits and contains an essentially small, full subcategory $\mathcal{A} \subseteq \mathcal{C}$, consisting of objects that are λ-presentable, such that $\mathcal{A}$ generates $\mathcal{C}$ under λ-filtered colimits.

In [Lur09], locally λ-presentable quasicategories are simply called **λ-presentable**.

Definition 43. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories admitting λ-filtered colimits. A functor $f : \mathcal{C} \to \mathcal{D}$ is called **λ-accessible** if it preserves all λ-filtered colimits.

4.1 Reflective Localisation of Quasicategories

We start with the quasicategorical analogue of Definitions 13 and 21:

Definition 44. Let $\mathcal{C}$ be a quasicategory, and let S be a collection of maps in $\mathcal{C}$.

1. An object $z \in \mathcal{C}$ is called **S-local** if, for every map $s : a \to b$ in S, precomposition with s induces a Kan equivalence:

$$s^* : \mathrm{Map}_{\mathcal{C}}(b, z) \to \mathrm{Map}_{\mathcal{C}}(a, z)$$

2. A map $f : a \to b$ is called an **S-equivalence** if, for any S-local object z, precomposition with f induces a Kan equivalence:

$$f^* : \mathrm{Map}_{\mathcal{C}}(b, z) \to \mathrm{Map}_{\mathcal{C}}(a, z)$$

Definition 45. Let $L : \mathcal{C} \rightleftarrows \mathcal{A} : R$ be an adjunction between quasicategories. We call L a **reflective localisation** if R is fully faithful, and **accessible** if in addition R is an accessible functor.

Exercise 46 ([Lur09, Prop 5.5.4.2])**.** Let $\mathcal{C}$ be a quasicategory, and let $L : \mathcal{C} \to \mathcal{A}$ be a reflective localisation. Denote the inclusion by $R : \mathcal{A} \to \mathcal{C}$. Let S be the set of all maps f in $\mathcal{C}$ for which $L(f)$ is an equivalence. Then:

1. The essential image of $R : \mathcal{A} \to \mathcal{C}$ consists precisely of the S-local objects.

2. Every S-equivalence in $\mathcal{C}$ belongs to S.

The proof of Exercise 46 is formally the same as Exercise 14.

Exercise 46 shows that any reflective localisation of a quasicategory can be understood as a localisation at a collection of maps S. If we assume $\mathcal{C}$ is a locally presentable quasicategory, then we can show that localising at any set of maps S gives a reflective, accessible localisation:

Theorem 47 ([Lur09, Prop 5.5.4.15])**.** *Let $\mathcal{C}$ be a locally presentable quasicategory and let S be a set of maps in $\mathcal{C}$. Let $\mathcal{A} \subseteq \mathcal{C}$ be the full subcategory on the S-local objects. Then the inclusion $R : \mathcal{A} \to \mathcal{C}$ is accessible, and it has a left adjoint $L : \mathcal{C} \to \mathcal{A}$. Moreover, a map f in $\mathcal{C}$ is an S-equivalence if and only if $L(f)$ is an equivalence.*

Theorem 47 gives us a great deal of control over reflective, accessible localisations of locally presentable quasicategories. We will now state the quasicategorical analogue of Theorems 16 and 25:

Theorem 48 ([Lur09, Thm 5.5.1.1])**.** *A quasicategory $\mathcal{C}$ is locally presentable if and only if it is a reflective, accessible localisation of $\mathcal{P}(\mathcal{A})$ for some small quasicategory $\mathcal{A}$.*

Finally, we can use Theorem 48 to sketch a proof of the result mentioned in the introduction:

Theorem 49 ([Lur09, Prop A.3.7.6])**.** *Let $\mathcal{C}$ be a quasicategory. Then $\mathcal{C}$ is locally presentable if and only if there is a combinatorial simplicial model category $\mathcal{M}$ and an equivalence:*

$$\mathcal{C} \simeq \mathbf{N}(\mathcal{M}^\circ)$$

Proof. By Theorem 48, $\mathcal{C}$ is locally presentable if and only if it is a reflective, accessible localisation of $\mathcal{P}(\mathcal{A})$ for some small quasicategory $\mathcal{A}$. By Theorem 47, every such localisation has the form $\mathcal{P}(\mathcal{A})/S$ for a set of maps S.

By Corollary 33, we have an equivalence:

$$\mathcal{P}(\mathcal{A}) \simeq \mathbf{N}\left(\left(\mathbf{sSet}^{\mathfrak{C}\mathcal{A}^{op}}\right)^\circ\right)$$

Moreover, reflective accessible localisations of $\mathcal{P}(\mathcal{A})$ correspond to left Bousfield localisations of $\mathbf{sSet}^{\mathfrak{C}\mathcal{A}^{op}}$. This gives a correspondence between locally presentable quasicategories and the combinatorial simplicial model categories that arise as a localisation of simplicial presheaves. But by Theorem 25, every combinatorial simplicial model category is equivalent to one of this form. □

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