Descent & Univalence

Mathieu Anel

Carnegie Mellon University

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Abstract

The purpose of the talk is to explain the connection between

the notion of descent, characteristic of ∞-topoi,

and the notion of univalence, characteristic of HoTT.

Both of them being properties of the universe of a category with finite limits.

PLAN

I. Univalence
II. Descent
III. Logos theory
IV. Problems with ω and elementary higher topoi

Sizes

ω = countable cardinal
I need three inaccessible cardinals $\gamma > \beta > \alpha \geq \omega$

Cardinal	ω	α	β	γ
Size	finite	small	normal	large
Cat. of sets	*set*	*Set*	*SET*	–
Cat. of groupoids	*gpd*	*Gpd*	*GPD*	–
Cat. of categories	*cat*	*Cat*	*CAT*	–

By default a category is assumed to be of normal size, i.e. β-small.

– I –

Univalence

The universe of a lex category

I am going to start by some considerations of 1-category theory.

The universe of a lex category

Let C be a (normal) 1-category.

If C is lex (=has finite limits), there exists a (pseudo-)functor

$$\begin{array}{rcl} \mathbb{U} : C^{op} & \longrightarrow & CAT \\ X & \longmapsto & C_{/X} = \{Y \xrightarrow{r} X\} \\ f : X \to Y & \longmapsto & f^* : C_{/Y} \to C_{/X} \end{array}$$

A map $Y \to X$ is thought as a family of objects of C parametrized by X.

Intuitively, $\mathbb{U}$ classifies families of objects in C and all morphisms between them.

The universe of a lex category

The inclusion of the category of groupoids in the category of categories has two adjoints:

the internal and external groupoids of a category.

$$\begin{tikzcd} GPD \arrow[r, hook, shift left=0] & CAT \arrow[l, shift right=4ex, "ext\ (loc.)"'] \arrow[l, shift left=4ex, "int\ (core)"] \end{tikzcd}$$

(Remark: the internal groupoid is not natural with respect to non-invertible 2-arrows of CAT.)

The universe of a lex category

A variation of $\mathbb{U}$ is the internal groupoid U of $\mathbb{U}$ (its core)

$$\begin{array}{rcl} U : C^{op} & \longrightarrow & CAT \xrightarrow{int} GPD \\ X & \longmapsto & C^{(\text{core})}_{/X} = \{ Y \xrightarrow{r} X \text{ with only iso. as morphisms} \} \end{array}$$

Intuitively, U classifies families of objects in C and *isomorphism* between them.

I will call U the universe of C.

The universe of a lex category

As fibered categories over C,

$\mathbb{U}$ correspond to the codomain fibration and

U to the subfibration of spanned by cartesian maps only

$$\begin{tikzcd}
C^{\to}_{\text{cart}} \arrow[rr] \arrow[dr, "U", "cod." '] & & C^{\to} \arrow[dl, "\mathbb{U}" ', "cod."] \\
& C &
\end{tikzcd}$$

The universe of a lex category

The universe has a derivative

$$\begin{array}{rcl} \mathbb{U}' : C^{op} & \longrightarrow & CAT \\ X & \longmapsto & C_X = \{ Y \overset{s}{\underset{r}{\leftrightarrows}} X \} \end{array}$$

Intuitively, $\mathbb{U}'$ classifies families of pointed objects in C and all morphisms between them.

The second derivative $\mathbb{U}''$ would classify families of bi-pointed objects in C, etc.

The universe of a lex category

We can also consider the core of $\mathbb{U}'$

$$\begin{array}{rcccl} U' : C^{op} & \longrightarrow & CAT & \xrightarrow{int} & GPD \\ X & \longmapsto & C_X^{(\text{core})} = \{ Y \underset{r}{\overset{s}{\leftrightarrows}} X \text{ with only iso. as morphisms} \} & & \end{array}$$

Intuitively, U' classifies families of pointed objects in C and isomorphism between them.

I will call U' the derived universe of C.

The universe of a lex category

Forgetting the section induces a natural transformation

$$\upsilon : U' \quad \longrightarrow \quad U$$

called the universal family.

The universe of a lex category

An Z object of C defines a functor

$$\begin{aligned} \hat{B} : C^{op} &\longrightarrow SET \\ X &\longmapsto [X, B]\,. \end{aligned}$$

C is not assumed locally small, so the values are normal sets.

Using the embedding $SET \hookrightarrow GPD$ one can view faithfully $\hat{Z}$ as a functor

$$\begin{aligned} \hat{B} : C^{op} &\longrightarrow GPD \\ X &\longmapsto [X, B]\,. \end{aligned}$$

The universe of a lex category

A map

$$\chi_f : \hat{B} \quad \longrightarrow \quad U$$

in $[C^{op}, GPD]$, is the same thing as a map/family

$$f : E \quad \longrightarrow \quad B$$

in C.

The correspondence between the two is given by the (pseudo-)pullback square in $[C^{op}, GPD]$

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & U' \arrow[d, "\upsilon"] \\ \hat{B} \arrow[r, "\chi_f"'] & U \end{tikzcd}$$

The universe of a lex category

Proposition (Universal property of the universe)

For any map $f : E \to B$, there exists a unique *(pseudo-)cartesian square in $[C^{op}, GPD]$*

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & U' \arrow[d, "\upsilon"] \\ \hat{B} \arrow[r, "\chi_f"'] & U \end{tikzcd}$$

(in particular, υ is a representable natural transformation).

The universe of a lex category

$$\begin{tikzcd}
\{X \xrightarrow{s'} E\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & \{Y \overset{s}{\underset{r}{\leftrightarrows}} X\} \arrow[d, "\text{forget section}"] \\
\{X \xrightarrow{u} B\} \arrow[r, "\chi_f"'] & \{Y \xrightarrow{r} X\}
\end{tikzcd}$$

$$\chi_f(X \xrightarrow{u} B) \quad = \quad E \times_B X \to X$$

$$\begin{tikzcd}
E \times_B X \arrow[r] \arrow[d, "\chi_f(u)"] & E \arrow[d, "f"] \\
X \arrow[r, "u"] \arrow[ur, dashed, "s'"] \arrow[u, dashed, bend left, "s"] & B
\end{tikzcd}$$

The universe of a lex category

Given two objects X and Y in C, there is a presheaf of isomorphisms

$$
\begin{array}{rcl}
\underline{Iso}(X,Y) : C^{op} & \longrightarrow & SET \\
Z & \longmapsto & Iso_Z(Z \times X, Z \times Y) \\
 & & = \{\text{iso. } Z \times X \simeq Z \times Y \text{ in } C_{/Z}\}
\end{array}
$$

This presheaf is representable if C is LCC (locally cartesian closed).

$$
\begin{tikzcd}
\underline{Iso}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \llbracket Y,X \rrbracket \times \llbracket X,Y \rrbracket \times \llbracket Y,X \rrbracket \arrow[d, "(h,f,g) \mapsto (hf,fg)"] \\
1 \arrow[r, "(id_Y, id_X)"'] & \llbracket Y,Y \rrbracket \times \llbracket X,X \rrbracket
\end{tikzcd}
$$

The universe of a lex category

What is the fiber of the diagonal of U ?

$$\begin{tikzcd}
\Omega_{X,Y} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d, "\Delta"] \\
1 \arrow[r, "{(X,Y)}"'] & U \times U
\end{tikzcd}$$

$$\Omega_{X,Y} U = \ ?$$

The universe of a lex category

Computation shows that we have

$$\begin{tikzcd} \underline{Iso}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d, "\Delta"] \\ 1 \arrow[r, "{(X,Y)}"'] & U \times U. \end{tikzcd}$$

$$\Omega_{X,Y}U \quad = \quad \underline{Iso}(X,Y)$$

This says that, U is univalent. We shall come back to this.

The universe of a lex category

Recall that $\underline{Iso}(X, Y)$ is not assumed to be representable, but that it is if C is LCC.

In this case, we have a map

$$\begin{aligned} Iso : U \times U &\longrightarrow U \\ (X, Y) &\longmapsto Iso(X, Y) \end{aligned}$$

The universe of a lex category

The formula $\Omega_{X,Y} U = \underline{Iso}(X, Y)$ says that there exists a cartesian square

$$\begin{tikzcd}
U \arrow[r, "X \mapsto id_X \in Iso(X{,}X)"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\ulcorner", very near start] & U' \arrow[d, "v"] \\
U \times U \arrow[r, "(X{,}Y) \mapsto Iso(X{,}Y)"'] & U
\end{tikzcd}$$

This is another formulation of the univalence of U when C is LCC.

The universe of a lex category

In category theory, the univalence of U is not a condition, it is an obvious property.

Before Voevodsky, this property was never even given a name.

This is why it is so difficult for algebraic topologists/geometers to understand the univalent axiom (it is too obvious to them, they use it implicitely all the time, they would find absurd a setting where it would not hold).

The universe of a lex category

The universe U and its derivative U'

$$\begin{array}{rcl} U : C^{op} & \longrightarrow & GPD \\ X & \longmapsto & C_{/X}^{(\mathrm{core})} \end{array} \qquad \begin{array}{rcl} U' : C^{op} & \longrightarrow & GPD \\ X & \longmapsto & C_{X}^{(\mathrm{core})} \end{array}$$

cannot be representable (in general) for two reasons:

1. their values are groupoids and not sets,
2. their values are too big (Russel paradox).

The universe of a lex category

It is possible to solve issue 1 by considering ∞-categories.

All we used so far was that C was a lex 1-category.

But the same construction works if C is a lex ∞-category.

$$\begin{array}{rcl} U : C^{op} & \longrightarrow & \infty\text{-}GPD \\ X & \longmapsto & C_{/X}^{(\text{core})} \end{array} \qquad \begin{array}{rcl} U' : C^{op} & \longrightarrow & \infty\text{-}GPD \\ X & \longmapsto & C_{X}^{(\text{core})} \end{array}$$

From now on, C is going to be an $(\infty,1)$-category (in particular, it can still be a 1-category).

The universe of a lex ∞-category

In an ∞-category, the functor of points of an object Z take values in ∞-groupoids

$$\begin{array}{rcl} \hat{Z} : C^{op} & \longrightarrow & \infty\text{-}GPD \\ X & \longmapsto & [X, Z] \end{array}$$

This is now homogenous with the universe (and its derivative)

$$\begin{array}{rcccl} U : C^{op} & \longrightarrow & \infty\text{-}CAT & \xrightarrow{int} & \infty\text{-}GPD \\ X & \longmapsto & C_{/X}^{(\text{core})} = \{Y \xrightarrow{r} X \text{ and iso.}\} & & \end{array}$$

The universe of a lex ∞-category

The embedding $1\text{-}CAT \subset \infty\text{-}CAT$ preserves lex categories.

There is no canonical way to transform a lex 1-category C into a non-trivial lex ∞-category D where the universe of C could be representable.

We have to work by hand to find such a D.

Example: $C = Set$, $D = \infty\text{-}GPD$

The universe of a lex ∞-category

But there is still the size issue 2.

This was handled by Voevodsky by introducing univalent maps.

The universe of a lex ∞-category

A map $E \to B$ in C is univalent if its characteristic map is a monomorphism in the arrow category of $[C^{op}, \infty\text{-}GPD]$

$$\begin{tikzcd} \hat{E} \arrow[r, hook] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner", very near start] & U' \arrow[d, "\upsilon"] \\ \hat{B} \arrow[r, hook, "\chi_f"'] & U \end{tikzcd}$$

I will call a representable subobject of U such a monomorphism $\hat{B} \hookrightarrow U$.

The universe of a lex ∞-category

Given a univalent map $E \to B$, we have a cartesian square

$$\begin{tikzcd} \hat{B} \arrow[r, hook] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d, "\Delta"] \\ \hat{B} \times \hat{B} \arrow[r, hook] & U \times U \end{tikzcd}$$

(This is true for any mono.)

The universe of a lex ∞-category

In particular, given to points of B

$$\begin{tikzcd}
& \hat{B} \arrow[r, hook] \arrow[d, "\Delta"'] \arrow[dr, phantom, very near start, "\ulcorner"] & U \arrow[d, "\Delta"] \\
& \hat{B} \times \hat{B} \arrow[r, hook] & U \times U \\
1 \arrow[ur, "{(x,y)}"] \arrow[urr, "{(X,Y)}"'] & &
\end{tikzcd}$$

we have

$$\Omega_{x,y} B \quad = \quad \Omega_{X,Y} U.$$

The universe of a lex ∞-category

From previous computations, we deduce

$$\Omega_{x,y} B \quad = \quad \underline{Iso}(X, Y).$$

This is another way to state the condition defining univalent maps $E \to B$ (X and Y are the fibers at x and y).

In particular, $\underline{Iso}(X, Y)$ is representable.

The universe of a lex ∞-category

For any two points x and y in B, if $x = y$, then the corresponding fibers X and Y are isomorphic:

$$\Omega_{x,y}B \longrightarrow \underline{Iso}(X, Y).$$

The univalence condition says that the reciprocal is true: if two fibers X and Y are isomorphic, then they have to be the same fiber at a same point $x = y$.

The universe of a lex ∞-category

Intuitively, a map $E \to B$ is univalent iff it contains each of its fiber only once.

This is trickier than it sounds:

a map of sets with a single fiber $E \to \{\star\}$ is not univalent!

There are more isomorphisms $E \simeq E$ than paths in $B = \{\star\}$.

$$\Omega B \longrightarrow \underline{Iso}(E, E) \qquad = \qquad 1 \xrightarrow{\not\simeq} Aut(E)$$

The universe of a lex ∞-category

The only univalent maps in *Set* are

$$\varnothing \longrightarrow \{0\}$$

$$\{1\} \longrightarrow \{1\}$$

and

$$\{1\} \longrightarrow \{0, 1\}.$$

The only univalent maps in a 1-topos are the submaps of

$$1 \longrightarrow \Omega$$

(where Ω is the subobject classifier).

The universe of a lex ∞-category

More examples of univalent maps can be found in ∞-*Gpd*

Given a small ∞-groupoid E (for example a set), the group of symmetries $Aut(E)$ acts on E. I denote by $E /\!\!/ Aut(E)$ be the (homotopy) quotient.

$Aut(E)$ also acts on the point 1. The quotient $1 /\!\!/ Aut(E)$ is the classifying groupoid $BAut(E)$ of $Aut(E)$.

The map $E \to 1$ is equivariant for the action of $Aut(E)$ and induces a quotient map

$$E/Aut(E) \quad \longrightarrow \quad BAut(E).$$

The universe of a lex ∞-category

The map

$$E/Aut(E) \longrightarrow BAut(E).$$

is an example of a univalent maps (because $\Omega(BAut(E)) = Aut(E)$).

Example: If $E = \{1, \ldots, n\}$, we get

$$\{1, \ldots, n\}/\Sigma_n \longrightarrow B(\Sigma_n) = B(\Sigma_{n-1}) \longrightarrow B(\Sigma_n)$$

The universe of a lex ∞-category

The previous size issue 2 can be handled by asking that the universe U, even though it cannot be representable, can be approximated by representable objects.

Definition (EUM)

A lex category C is said to have enough univalent maps (EUM) if U is the union of all its representable subobjects.

Counter-Examples: posets, Set, 1-topoi, truncated ∞-categories

Examples: 1, ∞-Gpd, $[C, \infty\text{-}Gpd]$ (C locally small)

The universe of a lex ∞-category

Condition EUM says that for any map $X \to Y$, there exists a univalent map $E \to B$ and a cartesian square in C

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "\text{univ.}"] \\ Y \arrow[r] & B. \end{tikzcd}$$

Condition EUM implies also that *Iso*(X, X') is representable for any two objects of $C_{/Y}$. (This is close but weaker than being LCC.)

The universe of a lex ∞-category

All this depends only on a lex category C.

I have not used the existence of colimits so far.

The interaction of the universe with colimits is the matter of descent.

We will see that the condition EUM needs to be improved when colimits are involved.

– II –

Descent

Descent

Let C be a lex category with α-small colimits (cc lex category).

Recall that I have assumed $\alpha \geq \omega$, so "small" can mean "finite".

I will work directly in the setting of ∞-categories (and therefore drop the ∞ prefix).

Descent

Definition

C is said to have descent if the (categorical) universe

$$\begin{aligned} \mathbb{U} : C^{op} &\longrightarrow CAT \\ X &\longmapsto C_{/X} \end{aligned}$$

send colimits to limits: for any diagram $X : I \to C$

$$C_{/\operatorname{colim}_i X_i} \quad = \quad \lim_i \; C_{/X_i}$$

(where the limit in the right hand side is a pseudo-limit in CAT).

Descent

The category $\lim_i C_{/X_i}$ can be described as the category of cartesian diagrams over $X : I \to C$:

$$\lim C_{/X_i} = (C^I_{cart})_{/X}$$

where C^I_{cart} is the category of I-diagrams and cartesian morphisms.

$$\begin{tikzcd} Y \arrow[d, "cart"] & = & Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & Y_j \arrow[d] \\ X & & X_i \arrow[r] & X_j \end{tikzcd}$$

Descent

For any diagram $X : I \to C$, we get an adjunction

$$C_{/\operatorname{colim} X_i} \underset{cst_I}{\overset{\operatorname{colim}_I}{\leftrightarrows}} (C^I_{cart})_{/X_\bullet} = \lim C_{/X_i}.$$

where

$$\operatorname{cst}_I(Y \to \operatorname{colim} X)_i = Y_i$$

is defined by

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[d] \\ X_i \arrow[r] & \operatorname{colim} X_i. \end{tikzcd}$$

Descent

The descent condition $C_{/\operatorname{colim} X_i} = \lim_i C_{/X_i}$ is the statement that the adjunction

$$C_{/\operatorname{colim} X_i} \underset{cst_I}{\overset{\operatorname{colim}_I}{\leftrightarrows}} (C^I_{cart})_{/X_\bullet} = \lim C_{/X_i}.$$

is an equivalence.

Recall that an adjunction is an equivalence if both functors are fully faithful.

Descent

The descent condition decomposes into two conditions

1. Colimits are universal if, for all $X : I \to C$, cst_I is fully faithful (= colim_I is a localization): for all $Y \to \mathrm{colim}\, X_i$

$$Y = \mathrm{colim}_i (Y \times_{\mathrm{colim}\, X_i} X_i)$$

(*decomposition then recomposition condition*)

2. Colimits are effective of, for all $X : I \to C$, colim_I is fully faithful: for all $E_i \to X_i$

$$E_i = (\mathrm{colim}_i\, E_i) \times_{\mathrm{colim}_i X_i} X_i .$$

(*composition then decomposition condition*)

Descent

The condition of universality of colimits is an easy one, it is satisfied as soon as C is LCC.

The condition of effectivity of colimits is more difficult.
We'll see below that the category *Set* does not have effective colimits.

The only 1-category satisfying it is $C = 1$.

Descent

In the case of a sum, the effectivity condition says that in the diagram

$$\begin{tikzcd} Y_1 \arrow[r, dashed] \arrow[d] & Y_1 \coprod Y_2 \arrow[d] & Y_2 \arrow[l, dashed] \arrow[d] \\ X_1 \arrow[r, dashed] & X_1 \coprod X_2 & X_2 \arrow[l, dashed] \end{tikzcd}$$

the two squares are cartesian.

Again, this means intuitively that pushing out does not touch the fibers.

Together with universality, this gives the extensivity of sums

$$C_{/X_1 \coprod X_2} = C_{/X_1} \times C_{/X_2}.$$

Descent

In the case of a pushout, the effectivity condition says that, in the cube

$$
\begin{tikzcd}
Y_0 \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\ulcorner", very near start] \arrow[drr, phantom, "\ulcorner", very near start] & & Y_2 \arrow[dr, dashed] \arrow[dd] \arrow[ddr, phantom, "\color{red}\ulcorner", very near start] & \\
& Y_1 \arrow[rr, dashed] \arrow[dd] \arrow[drr, phantom, "\color{red}\ulcorner", very near start] & & Y_\infty \arrow[dd] \\
X_0 \arrow[rr] \arrow[dr] & & X_2 \arrow[dr, dashed] & \\
& X_1 \arrow[rr, dashed] & & X_\infty
\end{tikzcd}
$$

if the back and left faces are cartesian, then so are the front and right faces.

Intuitively, this means that summing does not change the fibers.

Descent

Set does not satisfy descent: colimits are universal (*Set* is LCC) but not effective.

$$
\begin{tikzcd}
\{x, x'\} \arrow[d] & \{a, a'\} \coprod \{b, b'\} \arrow[l, "{(id,id)}"'] \arrow[r, "{(id,\sigma)}"] \arrow[d] \arrow[dl, phantom, very near start, "\lrcorner"] \arrow[dr, phantom, very near start, "\ulcorner"] & \{y, y'\} \arrow[d] & \operatorname{colim} Y_\bullet = 1 \arrow[d] \\
\{x\} & \{a, b\} \arrow[l] \arrow[r] & \{y\} & \operatorname{colim} X_\bullet = 1
\end{tikzcd}
$$

The fiber is everywhere two points.

But the fiber of the colimit map is a single point set.

Descent

∞-*Gpd* does satisfy descent.

Recall the recipe to compute the colimit of a diagram $X : I \to \infty\text{-}Gpd$:

1. compute the category of elements $\int_I X$
2. take its external groupoid $(\int_I X)^{ext}$

This is the colimit!

(In *Set*, the recipe would be to take connected components of $\int_I X$, which is also the π_0 of the colimit $(\int_I X)^{ext}$ in ∞-Gpd).

Descent

In the case of the pushout

$$\{x\} \longleftarrow \{a, b\} \longrightarrow \{y\}$$

The category of elements is

$$\begin{tikzcd}
& a \arrow[dl, bend right, "a=x"'] \arrow[dr, bend left, "a=y"] & \\
x & & y \\
& b \arrow[ul, bend left, "b=x"] \arrow[ur, bend right, "b=y"'] &
\end{tikzcd}$$

and the colimit is the circle S^1 in ∞-Gpd.

Descent

In the case of the pushout

$$\begin{tikzcd}
\begin{Bmatrix} a & b' \\ b & a' \end{Bmatrix} \arrow[r, "p_2", color=blue] \arrow[d, "p_1"', color=red] & \begin{Bmatrix} y \\ y' \end{Bmatrix} \\
\{x, x'\} &
\end{tikzcd}$$

The category of elements is

$$\begin{tikzcd}
 & & a' \arrow[dll, bend right, color=red] \arrow[drr, bend left, color=blue] & & \\
x' & & a \arrow[dll, bend right, color=red] \arrow[drr, bend left, color=blue] & & y' \\
x & b' \arrow[ul, bend left, color=red] \arrow[rrr, bend right, color=blue] & & & y \\
 & b \arrow[ul, bend left, color=red] \arrow[uurrr, bend right, color=blue] & & &
\end{tikzcd}$$

and the colimit is again a circle.

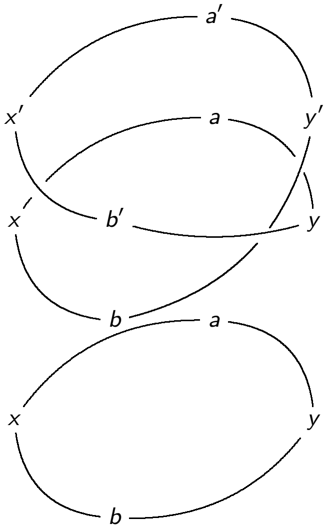
Descent

Let us come back to the previous example

$$\begin{tikzcd}
\{x, x'\} \arrow[d] & \{a, a'\} \coprod \{b, b'\} \arrow[l, "(id,id)"'] \arrow[r, "(id,\sigma)"] \arrow[d] \arrow[dl, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\llcorner", very near start] & \{y, y'\} \arrow[d] & Y_\bullet \arrow[d] \\
\{x\} & \{a, b\} \arrow[l] \arrow[r] & \{y\} & X_\bullet
\end{tikzcd}$$

Descent

The map between the colimits is the two-fold cover of the circle



$$\begin{tikzcd} \operatorname{colim} Y_\bullet \arrow[d] \\ \operatorname{colim} X_\bullet \end{tikzcd}$$

The fiber is now the same as in the diagram: two points.

Descent

The descent condition for $S = \infty\text{-}Gpd$ is

$$S_{/\operatorname{colim} X_i} = \lim\; S_{/X_i}$$

If K is a groupoid and $X : K \to C$ is the constant diagram 1, the descent condition is the homotopy Galois theorem (Toën, Shulman)

$$S_{/K} = S^K$$

(where $S^K = [K, S]$).

If $K = BG$ is connected,

$$S_{/BG} = S^{BG}.$$

says that an action of G is the same thing as a space over BG.

Descent

In $S = \infty\text{-}Gpd$, descent is equivalent to the homotopy Galois theorem:

$$S_{/\operatorname{colim} X_i} = S^{\operatorname{colim} X_i} = \lim\ S^{X_i} = \lim\ S_{/X_i}.$$

In more general settings, descent is motivated by working equivariantly: if a group G acts on an object X, with quotient $X /\!\!/ G$

$$C_{/(X /\!\!/ G)} = \{\text{actions of the groupoid } G \times X \rightrightarrows X \text{ in } C\}.$$

Other examples: $[C, S]$ (C locally small), lex localization of such...

Descent

The universe and univalent maps were defined using only finite limits in C.

The descent is a condition involving colimits in C.

More precisely descent is a property of compatibility between colimits and finite limits, akin to distributivity (we will see).

Descent

Definition

C is said to have core descent if the (core) universe

$$\begin{array}{rcl} U : C^{op} & \longrightarrow & GPD \\ X & \longmapsto & C^{(core)}_{/X} \end{array}$$

send colimits to limits: for any diagram $X : I \to C$

$$C^{(core)}_{/\operatorname{colim}_i X_i} = \lim_i C^{(core)}_{/X_i}$$

Descent

Proposition

If C has universal colimits (e.g. C is LCC) and core descent, then it has descent.

Proof.

If

$$\lim_i C_{/X_i} \xrightarrow{\text{colim}} C_{/\operatorname{colim}_i X_i}$$

is a localization, it is an equivalence iff it is conservative, but this is condition

$$C^{(\text{core})}_{/\operatorname{colim}_i X_i} = \lim_i C^{(\text{core})}_{/X_i}.$$

□

Descent

To connect univalence with descent, we need the following definition.

Recall that we fixed $\omega \leq \alpha < \beta$ and that a normal category is β-small.

Definition (EUM revisited)

A lex category C has α-enough univalent maps (α-EUM) if

1. *U is the union of representable sub-universes, and*
2. *this union is α-filtered.*

I don't know how to deduce this from condition EUM. (I think this is a mistake in the n-lab page on "elementary ∞-topoi", and in N. Rasekh paper.)

Examples: ∞-Gpd, ∞-topoi, $[C, \infty\text{-}Gpd]$ (C locally small).

Descent

Proposition

A cc lex ∞-category has core descent iff it has α-EUM.

Proposition

A cc LCC ∞-category has descent iff it has α-EUM.

Descent

Summary

for a cc lex category

$$\begin{tikzcd}
\begin{array}{c}\text{lex category}\\ +\ \text{EUM}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{lex category}\\ +\ \alpha\text{-EUM}\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{LCC category}\\ +\ \alpha\text{-EUM}\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{Isomorphism objects} & \text{core descent} & \text{descent}
\end{tikzcd}$$

– III –

Logos theory

Logos theory

Definition

A cc lex ∞-category is called a (α-)logos if it has descent.

A morphism of logoi $f^* : \mathcal{E} \to \mathcal{F}$ is simply a cc lex functor.

Examples:

1. $S = \infty\text{-}Gpd$
2. $[C, \infty\text{-}Gpd]$ (C small but also locally small)
3. β-small colimits of $[C, \infty\text{-}Gpd]$
4. free cocompletion $P(C)$ of a lex ∞-category

Logos theory

Definition

A Grothendieck logos is a logos which is (α-)presentable.

Let

$$\mathit{PresLogos} \subset \mathit{Logos}$$

be the subcategory of presentable logoi.

The opposite category of *PresLogos* is the category of ∞-topoi in the sense of Lurie's book

$$\mathit{Topos} = \mathit{PresLogos}^{op}.$$

Every logos is a β-small α-filtered colimit of Grothendieck logoi.

Logos theory

The notion of logos was introduced (with the presentability assumption) in Topo-logie (2019, Anel-Joyal) as the algebraic notion dual to the geometric notion of topos.

The motivation to introduce a more general notion are:

1. examples: gros topoi (don't need to be truncated anymore), Schulze pro-etale site...
2. the category *Logos* is better behaved than *PresLogos*,
3. a good structural analogy with commutative rings (presentable logoi are like finitely presented rings),
4. the general context of logoi is useful to encompass both Grothendieck and elementary topoi,
5. logos are higher analogs of pre-topoi.

Logos theory

But this comes at a price:

1. morphisms of logoi need not have a right adjoint
2. logoi need not be locally cartesian closed
3. logoi need not be locally small categories

Logos theory

A logos is a category with finite limits and small colimits.

We can forget these structures:

$$\begin{tikzcd}
Logos \arrow[r] \arrow[d] \arrow[dr] & CAT_{cc} \arrow[d] \\
CAT^{lex} \arrow[r] & CAT
\end{tikzcd}$$

Logos theory

Here is a nice feature of the general notion of logos.

Theorem (A.)

For any $\alpha > \omega$, the previous functors have left adjoints

$$
\begin{tikzcd}
Logos \arrow[r, shift right] \arrow[d, shift right] \arrow[dr, shift right] & CAT_{cc} \arrow[l, shift right, "Sym"'] \arrow[d, shift right] \\
CAT^{lex} \arrow[u, shift right, "\mathcal{P}"'] \arrow[r, shift right] & CAT \arrow[l, shift right, "(-)^{lex}"'] \arrow[u, shift right, "\mathcal{P}"'] \arrow[ul, shift right, "S[-]"']
\end{tikzcd}
$$

Moreover Logos is monadic *over CAT.*

Remark: The category of ω-logoi is not monadic over ∞-CAT (there is no free ω-logoi).

Logos theory

Analogy with commutative rings

$$
\begin{tikzcd}
\text{Commutative rings} \arrow[r, shift left] \arrow[d, shift left] & \text{Abelian groups} \arrow[l, shift left, "\mathrm{Sym}"'] \arrow[d, shift left] \\
\text{Commutative monoids} \arrow[u, shift left, "\mathbb{Z}."] \arrow[r, shift left] & \text{Sets} \arrow[l, shift left, "M(-)"'] \arrow[u, shift left, "\mathbb{Z}."] \arrow[ul, shift left, "\mathbb{Z}[-]"'] \arrow[from=ul, shift left]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Logoi} \arrow[r, shift left] \arrow[d, shift left] & \text{cocomplete categories} \arrow[l, shift left, "\mathrm{Sym}"'] \arrow[d, shift left] \\
\text{Lex categories} \arrow[u, shift left, "\mathcal{P}"] \arrow[r, shift left] & \text{Categories} \arrow[l, shift left, "(-)^{lex}"'] \arrow[u, shift left, "\mathcal{P}"] \arrow[ul, shift left, "S[-]"'] \arrow[from=ul, shift left]
\end{tikzcd}
$$

Logos theory

And with frame theory

$$\begin{tikzcd}
\text{Frames} \arrow[r, shift right] \arrow[d, shift left] \arrow[dr, shift right] & \text{Sup-lattice} \arrow[l, shift right, "\mathsf{Sym}"'] \arrow[d, shift left] \\
\text{Meet-lattice} \arrow[u, shift left, "\bigvee"] \arrow[r, shift right] & \text{Posets} \arrow[l, shift right, "(-)^\wedge"'] \arrow[u, shift left, "\bigvee"] \arrow[ul, shift right, "\underline{2}[-]"']
\end{tikzcd}$$

Logos theory

The free logos $S[C]$ on a category C is constructed by

1. completing C for finite limits C^{lex}
2. completing C^{lex} for small limits $S[C] = P(C^{lex})$

If C is small, so is C^{lex} and

$$S[C] = P(C^{lex}) = [(C^{lex})^{op}, S].$$

Every logos is a left exact localization of a free logos.

Logos theory

Descent = Distributivity

<table>
<tr><th colspan="2">Descent</th><th>Commutative ring</th></tr>
<tr><td>Universality colimits</td><td>$Y = \operatorname{colim}_i \left(Y \times_{\operatorname{colim}_j X_j} X_i \right)$</td><td>distributivity relation
$y \sum_j x_j = \sum_i y x_i$</td></tr>
<tr><td>Effectivity colimits</td><td>given $\begin{array}{ccc} Y_i & \longrightarrow & Y_j \\ \downarrow & \ulcorner & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$
$Y_i = (\operatorname{colim}_j Y_j) \times_{\operatorname{colim}_j X_j} X_i$
(not a consequence of universality)</td><td>given elements x_i and y_i such that
$y_i x_j = x_i y_j$
$y_i \sum_j x_j = x_i \sum_j y_j$
(consequence of distributivity)</td></tr>
</table>

Logos theory

Logos theory	*Commutative algebra*
α	ω
general logos	arbitrary ring
Grothendieck logos	finitely presented ring
"bounded" $\mathcal{E}$-logos $\mathcal{E} \to \mathcal{F}$	finitely presented morphism $A \to B$
polynomial functor $P : U' \longrightarrow U$	exponential function $exp(x) = \sum \frac{x^n}{n!}$ (not a polynomial)
universe $U = P(1)$ (not representable)	Euler number $e = exp(1) = \sum \frac{1}{n!}$ (not algebraic)

– IV –

Problems with ω and elementary higher topoi

Problems with ω

One question about ∞-topoi is to find a generalization of elementary topoi.

The way I understand this problem is to find some kind of finite version of a higher topos.

I'm going to finish on a few thought about this, and share my pessimism about the problem.

Please prove me wrong!

Problems with ω

The way I understand the problem to define higher elementary topoi is to the fill the gap in the following analogy table

1-categorical setting	*∞-categorical setting*
Grothendieck topos	presentable logos
elementary topos	?
pre-topos	general logos

Problems with ω

I know two examples of elementary 1-logoi that are not Grothendieck topoi.

1. the category *FinSet* of finite sets
2. the effective topos (that I don't understand enough to say anything about it)

Problems with ω

The trouble with higher "elementary topoi" start with this remark:

the category *Fin* of finite homotopy types does not have fiber products, nor dependent products.

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & S^1 \end{tikzcd} \qquad \prod_{S^1} S^1 = S^1 \times \mathbb{Z}$$

(This is the obstruction for the monadicity of ω-logoi mentioned earlier.)

Problems with ω

An inaccessible cardinal α is a cardinal such that sets of size $< \alpha$ are stable by

- dependent sums (regular cardinal)
- and dependent products.

An ∞-inaccessible cardinal α is a cardinal such that ∞-groupoids of size $< \alpha$ are stable by

- dependent sums (∞-regularity)
- and dependent products.

ω is not ∞-inaccessible.

But it is still ∞-regular.

Problems with ω

The ordinal ω suffers other important drawbacks in the higher setting (some of them I mentioned already):

1. the notion of ω-logos is not monadic,
2. finite CW complexes do not have finite limits (= ∞-accessibility),
3. finite CW complexes have infinite homotopy invariants,
4. coherent homotopy types (type with finite homotopy) are infinite cell-complexes,
5. the computation of the image and Postnikov truncations of a morphism use countable colimits, and
6. the splitting of idempotents is also countable colimit.

Problems with ω

For all these reasons, I do not think a reasonable notion of logos could be found by asking only for finite colimits.

Some ∞-inaccessible cardinal has to be involved.

Thanks!