Directed aspects of condensed type theory

Reid Barton
Carnegie Mellon University

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Outline

This talk is about joint work in progress with Johan Commelin.

1. Condensed sets (and condensed groupoids)
2. Etale and proper maps
3. Type-theoretic axioms for condensed sets and etale and proper maps
4. Directed univalence for the universe ODisc representing etale maps:

$$\prod (A\,B : \mathsf{ODisc}).\, \overrightarrow{\mathsf{Path}}_{\mathsf{ODisc}}(A, B) \simeq (A \to B)$$

Two main analogies:

- A condensed set is like a topological space.
- A condensed set is like a simplicial set, insofar as a topological space is like a preorder (via its specialization order).

Condensed sets

Condensed sets: summary

What:

- The category Cond of condensed sets is an "approximation" of the category Top of topological spaces by a topos.
- Introduced by Clausen–Scholze, Barwick–Haine, implicitly by Lurie.

Why:

- As a topos, Cond can serve as the target of an interpretation of ordinary dependent type theory.
- It also has natural higher versions (condensed groupoids, ...).
- Original motivation: better foundations for the ℓ-adic cohomology $H^*_{\mathrm{et}}(X, \mathbb{Q}_p)$ of a scheme X (Bhatt–Scholze).

Condensed sets: formal definition

We will actually define the category Cond^κ of κ-condensed sets, for a fixed uncountable regular cardinal κ (such as $\kappa = \omega_1$).

Theorem (Lurie)

If $\mathcal{C}$ is a regular (resp. extensive) category, then so is $\mathrm{Pro}^\kappa(\mathcal{C})$.

$$\begin{aligned} \mathrm{Pro}^\kappa(\mathrm{Fin}) &= \text{free completion of } \mathrm{Fin} \text{ under } \kappa\text{-small cofiltered limits} \\ \mathrm{Cond}^\kappa &= \text{sheaves on } \mathrm{Pro}^\kappa(\mathrm{Fin}) \text{ for the coherent topology} \\ &\quad\ \text{(generated by finitely jointly effective epimorphic families)} \end{aligned}$$

Since $\mathrm{Pro}^\kappa(\mathrm{Fin})$ is a small category, Cond^κ is a Grothendieck topos.

Relationship to topological spaces

Stone duality yields equivalences

$$\begin{aligned}\mathrm{Pro}^\kappa(\mathrm{Fin}) &\simeq \{\, \kappa\text{-small Boolean algebras} \,\} \\ &\simeq \{\, \text{Stone spaces with} < \kappa \text{ clopen subsets} \,\} \subseteq \mathrm{Top}.\end{aligned}$$

There are full embeddings, and a "realization—nerve adjunction"

$$\begin{tikzcd} & \mathrm{Pro}^\kappa(\mathrm{Fin}) \arrow[dl, "\mathrm{y}"'] \arrow[dr, "\lim"] & \\ \mathrm{Cond}^\kappa \arrow[rr, shift left] & & \mathrm{Top} \arrow[ll, shift left] \end{tikzcd}$$

so Cond^κ has the "correct" Hom-sets between κ-small limits of finite sets.

The "nerve" of a topological space X sends a profinite set S to

$$\mathrm{Hom}_{\mathrm{Top}}(S, X) = \text{the set of continuous functions from } S \text{ to } X.$$

Condensed groupoids, relationship to topoi

$$
\begin{aligned}
\mathrm{Cond}^{\kappa}_{\leq 0} &= \mathrm{Cond}^{\kappa} = \text{sheaves of sets on } \mathrm{Pro}^{\kappa}(\mathrm{Fin}) \\
\mathrm{Cond}^{\kappa}_{\leq 1} &= \text{stacks of groupoids on } \mathrm{Pro}^{\kappa}(\mathrm{Fin}) \\
\mathrm{Topoi}_{(2,1)} &= \text{the } (2,1)\text{-category of topoi and geometric morphisms}
\end{aligned}
$$

There is an analogous diagram:

$$
\begin{tikzcd}
 & \mathrm{Pro}^{\kappa}(\mathrm{Fin}) \arrow[dl, "\mathrm{y}"'] \arrow[dr, "\lim"] & \\
\mathrm{Cond}^{\kappa}_{\leq 1} \arrow[rr, shift left] & & \mathrm{Topoi}_{(2,1)} \arrow[ll, shift left]
\end{tikzcd}
$$

The "nerve" of a topos $\mathcal{E}$ is the stack sending $S \in \mathrm{Pro}^{\kappa}(\mathrm{Fin})$ to

$$\mathrm{Hom}_{\mathrm{Topoi}}(\mathrm{Sh}(S), \mathcal{E})^{\simeq} = \text{the groupoid of } \mathcal{E}\text{-valued sheaves on } S.$$

Etale and proper maps of condensed sets

The Sierpinski space

Let's start with the situation for topological spaces.
(We could also tell the same story for topoi.)

Definition

The Sierpinski space is $\mathbb{S} = \{\bullet, \circ\}$, with $\varnothing$, $\mathbb{S}$, $\{\circ\}$ as its open sets.

We draw $\mathbb{S}$ like this:

$$\text{“closed point”} \quad \bullet \longrightarrow \circ \quad \text{“open point”}.$$

The map $\circ : 1 \to \mathbb{S}$ is the *classifier for open subspaces* (or open subsets):

$$\mathrm{Hom}(X, \mathbb{S}) \cong \{\, \text{open subsets of } X \,\}.$$

The specialization order of a topological space

Definition

Let X be a topological space and p, q points of X.
We say that q specializes to p and write $p \le q$ or $p \to q$ if:[1]

for every open $U \subseteq X$, if $p \in U$, then $q \in U$.

Clearly, $\le$ is a preorder on the points of X. Continuous maps preserve $\le$.
Open subspaces are upwards-closed under $\le$.
(In a Hausdorff space, the ordering $\le$ is trivial.)

Example

In $\mathbb{S} = \{\bullet \longrightarrow \circ\}$, $\bullet \le \circ$. This is the "universal" example: for any X,

$$\begin{aligned} \mathrm{Hom}(\mathbb{S}, X) \;&\cong\; \{\, (p,q) \in X \times X \mid p \le q \,\} \\ \gamma \;&\mapsto\; (\gamma(\bullet), \gamma(\circ)). \end{aligned}$$

[1] Notation in algebraic geometry is the opposite: $q \rightsquigarrow p$.

Local homeomorphisms

Definition

A map $f : Y \to X$ of topological spaces is a local homeomorphism if for every $y \in Y$, there exist open neighborhoods U of $f(y)$ and V of y such that f restricts to a homeomorphism $f : V \to U$.

Basic examples: open embeddings; $Y \to 1$ for Y discrete (i.e., a set).

For fixed X, there is an equivalence of categories

$$\mathrm{Sh}(X) \simeq \{\, \text{local homeomorphisms } f : Y \to X \,\}$$

given by the "etale space" construction.

Local homeomorphisms lift generalizations

Proposition

A local homeomorphism $f : Y \to X$ is right orthogonal to the inclusion $\{\bullet\} \subseteq \{\bullet \longrightarrow \circ\}$.

$$\begin{tikzcd}
\{\bullet \} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\{\bullet \longrightarrow \circ\} \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}$$

Proof.

Direct from the definitions, e.g. for existence, suppose $p \in Y$ and $f(p) \leq q'$ in X; then f restricts to a homeomorphism from some open nhd V of p to an open nhd U of $f(p)$, which must also contain q'; then V contains some q with $p \leq q$ and $f(q) = q'$. $\square$

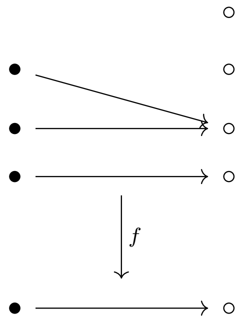
Local homeomorphisms over $\mathbb{S}$

$$\{\,\text{local homeomorphisms } f : Y \to \mathbb{S}\,\} \simeq \mathrm{Sh}(\mathbb{S}) \simeq \mathrm{Set}^{\bullet \to \circ}.$$

An object $g : A \to B$ of $\mathrm{Set}^{\bullet \to \circ}$ corresponds to a local homeomorphism built as a "mapping cylinder" using $\mathbb{S}$, like so:

$$A \xrightarrow{\quad g \quad} B$$

$$\begin{array}{c} A \times \mathbb{S} \amalg_{A \times \{\circ\}} B \times \{\circ\} \\ \Big\downarrow f \\ \mathbb{S} \end{array}$$



Proper maps

Definition

A map $f : Y \to X$ is proper (and separated)
if it has the "unique lifting property for ultrafilter convergence":

given an ultrafilter μ on Y and a point $x \in X$ to which $f(\mu)$ converges,
there is a unique lift y of x to which μ converges.

(Equivalent to "universally closed + closed diagonal".)

Basic examples: closed embeddings; $Y \to 1$ for Y compact Hausdorff.

Proper maps lift specializations

Proposition

A proper map $f : Y \to X$ is right orthogonal to the inclusion $\{\circ\} \subseteq \{\bullet \longrightarrow \circ\}$.

$$\begin{tikzcd}
\{ \circ\} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\{\bullet \longrightarrow \circ\} \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}$$

Proof.

For p, q points of a topological space Z, the principal ultrafilter on q converges to p if and only if q specializes to p. So, the claim is an instance of the unique lifting property for ultrafilter convergence. $\square$

Proper maps over $\mathbb{S}$

We have an equivalence (?)

$$\{\,\text{proper maps } g : Y \to \mathbb{S}\,\} \simeq \mathsf{CHaus}^{\bullet\leftarrow\circ}.$$

Note that the arrow is backwards! We can see it must be so because the inclusion $\{\bullet\} \subseteq \{\bullet \longrightarrow \circ\}$ is closed, hence proper. Moreover, we saw that proper maps have unique liftings of specializations (not generalizations).

An object $g : A \longleftarrow B$ of $\mathsf{CHaus}^{\bullet\leftarrow\circ}$ corresponds to the proper map

$$\begin{tikzcd} A \times \{\bullet\} \amalg_{B\times\{\bullet\}} B \times \mathbb{S} \arrow[d] \\ \mathbb{S} \end{tikzcd}$$

The corresponding result for topoi is an instance of a recent theorem of Henry and Townsend.

Topological spaces versus posets

In summary: the functor

$$\mathrm{Pts} : \mathrm{Top} \to \mathrm{Preord}$$

sending a space to the specialization order on its points sends

$\mathbb{S} = \{\bullet \longrightarrow \circ\}$	to	$[1] = \{0 \longrightarrow 1\}$
an open subspace	to	an upwards-closed subset
a closed subspace	to	a downwards-closed subset
a local homeomorphism	to	a left fibration
a proper map	to	a right fibration

suggesting that local homeomorphisms and proper maps are, at least, reasonable analogues of left and right fibrations.

Left/right fibrations of posets are classified by the categories $\mathrm{Set}/\mathrm{Set}^{\mathrm{op}}$. Is there a (?something?) classifying local homeomorphisms/proper maps?

Etale and proper morphisms: formal definitions

Let's switch now to Cond ($= \mathrm{Cond}^\kappa$).

Definition

A morphism $f : Y \to X$ of Cond is etale if for any $S \in \mathrm{Pro}^\kappa(\mathrm{Fin})$ and morphism $h : S \to X$, the functor

$$\mathrm{Hom}_{\mathrm{Cond}_{/S}}(-, h^*Y) : (\mathrm{Pro}^\kappa(\mathrm{Fin})_{/S})^{\mathrm{op}} \to \mathrm{Set}$$

sends κ-small cofiltered limits to colimits.

Definition

A morphism $f : Y \to X$ of Cond is proper if it is a coherent morphism (in the sense of SGA).

Classifiers for etale and proper morphisms

Etale and proper morphisms are classified by *stacks* (of large groupoids) $\mathbb{E}\mathrm{t}$, $\mathbb{P}\mathrm{r}$ on Cond^κ:

$$\mathrm{Hom}_{\mathrm{Cond}^\kappa_{\leq 1}}(X, \mathbb{E}\mathrm{t}) = \{\, \text{etale morphisms } f : Y \to X \,\}$$
$$\mathrm{Hom}_{\mathrm{Cond}^\kappa_{\leq 1}}(X, \mathbb{P}\mathrm{r}) = \{\, \text{proper morphisms } f : Y \to X \,\}$$

This amounts to the fact that the property of being etale/proper is stable under base change and local on the base.

Furthermore, when X is a compact Hausdorff space, an etale/proper morphism to X is the same as a local homeomorphism/proper morphism in the sense of topology (up to a condition involving the cardinal κ).

Informally, $\mathbb{E}\mathrm{t}$ and $\mathbb{P}\mathrm{r}$ are the "spaces" (= condensed groupoids) of all sets and all compact Hausdorff spaces respectively.

Type-theoretic axioms for condensed sets

Related work

We have been influenced by synthetic topology (Martín Escardó), as well as synthetic domain theory and Abstract Stone Duality (Paul Taylor). Compared to synthetic topology, the main difference in our setting is that we postulate a classifier for etale maps (not only open embeddings) and also add a classifier for proper maps.

There is also a recent project "Synthetic Stone Duality" (Cherubini, Coquand, Geerligs, Moeneclaey) which ends up being closely related.

Setup

We assume a univalent universe of 0-types $\mathcal{U}$, closed under the standard type formers (including quotients by equivalence relations). The type $\mathcal{U}$ itself is a 1-type.

We postulate two subuniverses ODisc and CHaus of $\mathcal{U}$.
(ODisc = "overt discrete", CHaus = "compact Hausdorff".)
Formally, these are predicates $\mathcal{U} \to \mathsf{Prop}$, but we write $A : \mathsf{ODisc}$ to mean A belongs to the subuniverse ODisc. The intended interpretations are:

- ODisc classifies κ-small etale maps of Cond^κ ($\sim$ "κ-small sets"). This is **not** to address a size issue, but rather because only κ-small products of finite sets in Cond^κ are well-behaved!
- CHaus classifies proper maps of Cond^κ.

Theorem (B.–Commelin)

Under these interpretations, the axioms on the following slides hold:

“Formation” axioms

- ODisc and CHaus are each closed under “positive” type formers: 0, 1, $+$, $\times$, Σ, identity types, quotients of equivalence relations.
- ODisc and CHaus are closed under Π types indexed by *each other*.
- $\mathbb{N}$: ODisc.

The first two items have a “directed” flavor.

Collection axioms

The following axiom appears in Joyal and Moerdijk's Algebraic Set Theory, where it is called "collection" in reference to the axiom of set theory of the same name.

- Suppose given $X : \mathsf{ODisc}$, $Y : \mathcal{U}$, and a surjection $p : Y \to X$. Then there exists $X' : \mathsf{ODisc}$, a surjection $q : X' \to X$ and a lift $s : X' \to Y$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ X' \arrow[ur, "s"] \arrow[r, "q"] & X \end{tikzcd}$$

- Same axiom for the subuniverse CHaus.

(Corresponds to the "Local choice" axiom of Synthetic Stone Duality.)

Continuity axioms

By one of the formation axioms, the "Hom-functor" $\mathrm{Hom} : \mathcal{U}^{\mathrm{op}} \times \mathcal{U} \to \mathcal{U}$ restricts to

$$\mathrm{Hom} : \mathsf{CHaus}^{\mathrm{op}} \times \mathsf{ODisc} \to \mathsf{ODisc}.$$

- This restricted Hom commutes with ODisc filtered colimits in each variable separately:

$$\mathrm{Hom}(X, \operatorname*{colim}_{i \in I} A_i) = \operatorname*{colim}_{i \in I} \mathrm{Hom}(X, A_i)$$
$$\mathrm{Hom}(\lim_{i \in I^{\mathrm{op}}} X_i, A) = \operatorname*{colim}_{i \in I} \mathrm{Hom}(X_i, A)$$

  for I a filtered category internal to the subuniverse ODisc, and $A, A_i : \mathsf{ODisc}$, $X, X_i : \mathsf{CHaus}$.

  (There is an equivalent "dependent" version involving Π-types.)

Directed univalence

Directed paths

We write OProp for the universe of open propositions. Semantically, OProp corresponds to the topological space $\mathbb{S}$ viewed as a condensed set. Taking OProp as a "directed interval", we use it to define directed paths.

Definition

For X any type (not necessarily in $\mathcal{U}$) and $p, q : X$, we define

$$\overrightarrow{\mathsf{Path}}_X(p, q) := \sum (\gamma : \mathsf{OProp} \to X).\, \gamma(\bot) = p \times \gamma(\top) = q.$$

This corresponds to the original definition: a specialization relation $p \leq q$ in a topological space X corresponds to a continuous map $\gamma : \mathbb{S} \to X$ with $\gamma(\bullet) = p$ and $\gamma(\circ) = q$.

Non-composability of paths

Beware! For a general type X, there need not be a function

$$\mathsf{pathcomp} : \overrightarrow{\mathsf{Path}}_X(p,q) \times \overrightarrow{\mathsf{Path}}_X(q,r) \to \overrightarrow{\mathsf{Path}}_X(p,r).$$

In this respect, condensed type theory is similar to simplicial type theory (and unlike the original setting of topological spaces). This might be an unavoidable consequence of having all constructions of dependent type theory available on general types.

Directed univalence

Theorem (B.–Commelin)

$$\prod (A\, B : \mathsf{ODisc}).\, \overrightarrow{\mathsf{Path}}_{\mathsf{ODisc}}(A, B) \simeq (A \to B)$$

as an internal theorem, proved from the axioms listed earlier.

External plausibility argument:

$$\begin{aligned}
&\{\, \text{global sections of } \mathsf{OProp} \to \mathsf{ODisc} \,\} \\
&= \mathbb{E}\mathrm{t}(\mathbb{S}) \\
&= \{\, \text{local homeomorphisms } f : Y \to \mathbb{S} \,\} \\
&= \mathrm{Sh}(\mathbb{S}) \\
&= \{\, \text{morphisms of } \mathrm{Set} \,\} \\
&= \{\, \text{global sections of } \textstyle\sum (A\, B : \mathsf{ODisc}).\, A \to B \,\}.
\end{aligned}$$

About the proof

In “ordinary” univalence

$$\prod (A\, B : \mathcal{U}).\, (A = B) \simeq (A \simeq B),$$

we get the forward direction “for free”, from the induction principle for $=$ (or in cubical type theory, from the transport operations).

However, in our setting, there is no obvious function in either direction

$$\overrightarrow{\mathsf{Path}}_{\mathsf{ODisc}}(A, B) \to (A \to B)$$

or

$$(A \to B) \to \overrightarrow{\mathsf{Path}}_{\mathsf{ODisc}}(A, B).$$

I will briefly sketch the construction in the reverse direction.

The reverse construction

Given $g : A \to B$ with $A, B : \mathsf{ODisc}$, how to construct a directed path $\gamma : \mathsf{OProp} \to \mathsf{ODisc}$ with $\gamma(\bot) = A$, $\gamma(\top) = B$?

Externally (in empty context, say) g corresponds to an actual function $g : A \to B$ between sets; we saw earlier that the corresponding local homeomorphism over $\mathbb{S}$ was given by the "mapping cylinder" construction

$$f : A \times \mathbb{S} \amalg_{A \times \{\circ\}} B \times \{\circ\} \longrightarrow \mathbb{S}.$$

The corresponding construction in type theory turns out to be

$$\gamma : \mathsf{OProp} \to \mathsf{ODisc}, \quad \gamma(P) := \sum (b : B).\, P * \left(\sum (a : A).\, g(a) =_B b \right)$$

where $P * C$ denotes the join of the types P and C: the pushout of

$$P \longleftarrow P \times C \longrightarrow C.$$

An alternative construction

Lemma

For P : OProp and C : ODisc, the canonical map

$$P * C \to (\neg P \to C)$$

is an isomorphism.

This turns out to be an instance of the continuity lemma.

Corollary

$$\begin{aligned} &\sum (b : B).\, P * \Big(\sum (a : A).\, g(a) =_B b \Big) \\ \cong &\sum (b : B).\, \neg P \to \Big(\sum (a : A).\, g(a) =_B b \Big). \end{aligned}$$

Having both constructions is useful for proving directed univalence.

Conclusion

Directedness and directed univalence is not just about $(\infty, 1)$-categories.

Question

To what extent can these different settings (e.g., condensed sets, simplicial spaces) be fit into a common framework?

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To what extent can these different settings (e.g., condensed sets, simplicial spaces) be fit into a common framework?

Thank you for listening!