CaTT: A type theory for weak ω-categories

Thibaut Benjamin

CEA LIST, work conducted at Ecole Polytechnique

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Introduction

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- I have conducted my PhD thesis about this theory.
- I am currently a postdoc at CEA LIST, where I work on runtime verification of C programs (in frama-c).

HoTT, Groupoids, and Brunerie's Type Theory

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- But we can abstract out this structure and define it as a minimal type theory (Brunerie)

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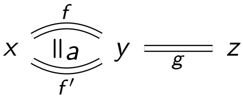
- These contexts describe *arbitrary equality situations* (a.k.a computads for weak ω-groupoids)



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  `  ⊢ a(f,g,h) : c(f,c(g,h))=c(c(f,g),h)`

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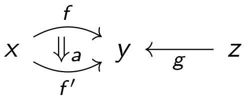

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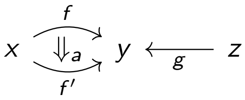

- We will see more general contexts later

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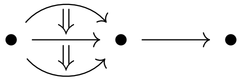

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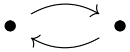

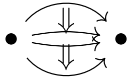

- We can recognize them algorithmically

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- Let's see this live !