Multiverse models

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Motivation

Universe model: Fixed lcc category ("universe") $\mathcal{C}$ as model.
Multiverse model: *Category of* lcc (infinity) categories as model.

Proposition

Let

- $\mathcal{T}$ *be a dependent type theory*
- $\cdot \vdash \sigma$ *be a type in the empty context*
- $\mathcal{T}' \supseteq \mathcal{T}$ *be extension with constant* $\cdot \vdash c : \sigma$

Then:

$$\{\tau \mid (v : \sigma) \vdash \tau \textit{ in } \mathcal{T}\} \cong \{\tau' \mid \cdot \vdash \tau' \textit{ in } \mathcal{T}'\}$$

(Similarly for terms.) □

$\implies$ Contexts parametrize extensions of base type theory $\mathcal{T}$.

Seely: Dependent type theories $\sim$ lcc categories.
So: Contexts $\sim$ lcc categories?

From universe to multiverse models

	Universe model $\mathcal{C}$	... equivalently	Multiverse model
Ctx Γ	obj $\Gamma \in \mathrm{Ob}\,\mathcal{C}$	slice cat $\mathcal{C}_{/\Gamma}$	lcc cat Γ
Mor $\Delta \to \Gamma$	$f : \Delta \to \Gamma$	$f^* : \mathcal{C}_{/\Gamma} \to \mathcal{C}_{/\Delta}$ lcc over $\mathcal{C}$	$f : \Gamma \to \Delta$ lcc
Ty $\Gamma \vdash \sigma$	$\sigma : \mathrm{dom}\,\sigma \to \Gamma$	$\sigma \in \mathrm{Ob}\,\mathcal{C}_{/\Gamma}$	$\sigma \in \mathrm{Ob}\,\Gamma$
Tm $\Gamma \vdash s : \sigma$	$\sigma \circ s = \mathrm{id}$	$s : \mathrm{id}_\Gamma \to \sigma$ in $\mathcal{C}_{/\Gamma}$	$s : 1 \to \sigma$
Subst	$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \big\downarrow{\scriptstyle \sigma[f]} \lrcorner & & \big\downarrow{\scriptstyle \sigma} \\ \Delta & \longrightarrow & \Gamma \end{array}$	$\mathcal{C}_{/\Delta} \vdash f^*(s) : f^*(\sigma)$	$\Delta \vdash f(s) : f(\sigma)$
Ext	$\Gamma.\sigma = \mathrm{dom}\,\sigma$	$(\mathcal{C}_{/\Gamma})_{/\sigma} \cong \mathcal{C}_{/\,\mathrm{dom}\,\sigma}$	$\Gamma.\sigma = \Gamma_{/\sigma}$
$\Gamma \vdash \Sigma_\sigma\,\tau$	$\begin{array}{ccc} \Gamma.\sigma.\tau & & \\ \big\downarrow{\scriptstyle \tau} & \searrow^{\Sigma_\sigma \tau = \sigma \circ \tau} & \\ \Gamma.\sigma & \xrightarrow{\sigma} & \Gamma \end{array}$	$\Sigma_\sigma \dashv \sigma^* \rightsquigarrow \Sigma_\sigma\,\tau = \Sigma_\sigma(\tau)$	same
$\Gamma \vdash \Pi_\sigma\,\tau$	$\begin{array}{ccc} \Gamma.\sigma.\tau & & \Gamma.\Pi_\sigma\,\tau \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \Pi_\sigma\,\tau} \\ \Gamma.\sigma & \xrightarrow{\sigma} & \Gamma \end{array}$	$\sigma^* \dashv \Pi_\sigma \rightsquigarrow \Pi_\sigma\,\tau = \Pi_\sigma(\tau)$	same

Coherence problems

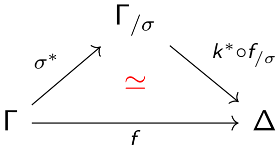
- Substitution does not commute *strictly* with type formers:

$$f(\mathrm{Eq}\, s_1\, s_2) \cong \mathrm{Eq}\, f(s_1)\, f(s_2)$$

- For ∞-categories also term substitution:

$$f(\mathrm{pair}\, s_1\, s_2) \simeq \mathrm{pair}\, f(s_1)\, f(s_2)$$

- Context extension has higher categorical universal property:



$$k : 1 \to f(\sigma) \text{ in } \Delta$$

Mismatch: 1-categorical syntax vs higher categorical semantics.

Model categories of lcc categories

Idea: Find higher equivalence $\mathcal{M} \simeq \mathrm{Lcc}$ such that $\mathcal{M}_0 \not\simeq \mathrm{Lcc}_0$ 1-categorically using model category theory.

1. Model category Lcc of *sketches for lcc (infinity) categories*.
2. Model category $\mathrm{sLcc} = \mathrm{Alg}(\mathrm{Lcc})$ of *algebraically fibrant lcc sketches/strict lcc categories*: Makes substitution and type/term formers commute.
3. Model category Coa(sLcc) of *algebraically cofibrant strict lcc categories*: Context extensions are equivalent to slice categories.

Lcc 1-categories: Solves coherence problem completely, model of extensional dependent type theory.
∞-categories: Step 3 breaks context extensions, only exist for *base* types and terms.

Lcc sketches

1. Choose base model category $\mathcal{C}$.
   - 1-categories: Cat
   - ∞-categories: sSet with Joyal model structure or sSet^+
2. Define set of cofibrant *shape* objects for markings:

   Model category $\mathcal{C}^I$ of *marked objects* (Isaev):
   - $X \in \mathrm{Ob}\,\mathrm{C}$
   - for $K \in I$ some $k : K \to X$ are distinguished/*marked*
3. Localize $\mathcal{C}^I$ (left Bousfield)

$\implies$ fibrant objects reduced to lcc (infinity) categories

Algebraically fibrant objects

Definition

Let

- $\mathcal{M}$ model category
- J generating set of trivial cofibrations.

An *algebraically fibrant object* consists of

- $X \in \operatorname{Ob} \mathcal{M}$
- Operator ℓ_X:

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle j}\downarrow & \nearrow_{\ell_X(j,a)} & \\ B & & \end{array}$$

Morphisms are maps $X \to Y$ commuting with ℓ_X and ℓ_Y.

Theorem (Nikolaus, Bourke)

Let $\mathcal{M}$ be a combinatorial model category. Then $G : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ is monadic, the induced model structure on $\mathrm{Alg}(\mathcal{M})$ exists, and G is a Quillen equivalence.

Better: Use (e.g. simplicially) enriched lifting operator.

Strict lcc (infinity) categories

- Existence of universal objects (e.g. pullbacks) in fibrant objects of Lcc is encoded via right lifting property against trivial cofibrations
- Objects X of $\mathrm{sLcc} = \mathrm{Alg}(\mathrm{Lcc})$ are lcc categories equipped with canonical choice of universal objects
- Morphisms $X \to Y$ preserve canonical choice on the nose: strict lcc functors

Theorem (1-categorical)

sLcc *carries cwf structure supporting extensional equality types and finite product types.*

Theorem (∞-categorical)

sLcc *carries cwf structure supporting weak identity types and weak finite product types.*

"Weak" types: Computation rule holds up to term of identity type.

Context extension in sLcc

Context extension vs slice category:

$$\begin{array}{ccc} \langle x \rangle & \longrightarrow & \langle v : 1 \to x \rangle \\ {\scriptstyle x \mapsto \sigma}\downarrow & \ulcorner & \downarrow \\ \Gamma & \longrightarrow & \Gamma.\sigma \end{array}$$

in sLcc

$$\begin{array}{ccc} \{x\} & \longrightarrow & \{v : 1 \to x\} \\ {\scriptstyle x \mapsto \sigma}\downarrow & \mathrm{ho}\ulcorner & \downarrow \\ G(\Gamma) & \longrightarrow & G(\Gamma_{/\sigma}) \end{array}$$

in Lcc

- Γ cofibrant $\implies \Gamma.\sigma \simeq \Gamma_{/\sigma}$
- If so and $\Gamma.\sigma \vdash \tau$ then

  $$\tau \in \Gamma.\sigma \xrightarrow{\simeq} \Gamma_{/\sigma} \xrightarrow{\Pi_\sigma} \Gamma$$

  defines $\Gamma \vdash \Pi_\sigma \tau$
- Depends on choice of $\Gamma.\sigma \simeq \Gamma_{/\sigma}$: Not substitution stable
- Thus: Make cofibrancy part of structure

Algebraically cofibrant objects

Definition

An *algebraically cofibrant object* in a model category $\mathcal{M}$ is a coalgebra for a cofibrant replacement monad.

Intuition for $\mathcal{M} = \mathrm{sLcc}$: Every non-strict lcc functor $f : G(\Gamma) \to G(\Delta)$ has a *canonical* isomorphic/homotopic strictification $f^s : \Gamma \to \Delta$.

Theorem (Ching & Riehl)

Let $\mathcal{M}$ be a suitable model category. Then the induced model structure on $\mathrm{Coa}(\mathcal{M})$ exists, and $\mathrm{Coa}(\mathcal{M}) \to \mathcal{M}$ is a left Quillen equivalence.

Theorem (1-categorical)

$\mathrm{Coa}(\mathrm{sLcc})$ *carries cwf structure supporting extensional dependent type theory.*

Theorem (∞-categorical)

$\mathrm{Coa}(\mathrm{sLcc})$ *carries cwf structure supporting weak dependent products along base types.*

Problem: $\mathrm{Coa}(\mathrm{sLcc})$ not closed under context extensions in sLcc, only if type is detected by map in $\mathrm{Coa}(\mathrm{sLcc})$.

Outlook

Problem

Solve context extension issue for ∞-categories.

Problem

Show that multiverse models interpret *type classifiers*.

- Context extension by type variable.
- Not classified by type in base context, avoids Girard's paradox.
- Parametricity?
- 1-categorically: Exists in sLcc, not in $\mathrm{Coa}(\mathrm{sLcc})$.
- Without Π-types: τ-categories as in *Categories, Allegories*.

Problem

General coherence constructions using model category theory:

- $\mathrm{sSet} = \mathrm{Alg}_J(\mathrm{ssSet}^+)$ modulo coherence?
- Consider $\mathrm{sLex}' = \{\mathcal{C} \in \mathrm{sLex} \mid \mathcal{C}$ has strict pullbacks along *both* legs$\}$. Is $\mathrm{sLex}' \subseteq \mathrm{sLex}$ a right Quillen *equivalence*?

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