From Higher Groups to Homotopy Surfaces

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Outline

Definitions

Recall the basic setup:

- *pointed connected* types B may be viewed as presenting higher groups, with *carrier* $\Omega B := (\mathrm{pt} =_B \mathrm{pt})$, and group structure induced from the identity types.

$$
\begin{aligned}
\infty\text{-Group} &:= (G : \mathrm{Type}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq \Omega BG) \\
&\simeq (G : \mathrm{Type}_{\mathrm{pt}}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq_{\mathrm{pt}} \Omega BG) \\
&\simeq \mathrm{Type}_{\mathrm{pt}}^{>0} \\
n\text{-Group} &:= (G : \mathrm{Type}_{\mathrm{pt}}^{<n}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G \simeq_{\mathrm{pt}} \Omega BG) \\
&\simeq \mathrm{Type}_{\mathrm{pt}}^{>0,\leq n}
\end{aligned}
$$

Stability

The more deloopings the merrier! (Recall that Eckmann-Hilton implies that double loop spaces are homotopy commutative.)

$$\begin{aligned}
(n,k)\mathrm{GType} &:= (G : \mathrm{Type}_{\mathrm{pt}}^{\leq n}) \times (B^k G : \mathrm{Type}_{\mathrm{pt}}^{\geq k}) \times (G \simeq_{\mathrm{pt}} \Omega^k B^k G) \\
&\simeq \mathrm{Type}_{\mathrm{pt}}^{\geq k, \leq n+k} \\
(n,\omega)\mathrm{GType} &:= \lim_k \, (n,k)\mathrm{GType} \\
&\simeq \big(B^{-}G : (k : \mathbb{N}) \to \mathrm{Type}_{\mathrm{pt}}^{\geq k, \leq n+k}\big) \\
&\qquad \times \big((k : \mathbb{N}) \to B^k G \simeq_{\mathrm{pt}} \Omega B^{k+1} G\big).
\end{aligned}$$

Infinite loop types in this way are precisely connective spectra ($n = \infty$).

Periodic table

Periodic table of k-tuply groupal n-groupoids.

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	sylleptic 3-group	$\cdots$	sylleptic ∞-group
4	— " —	— " —	symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

Lean formalization: Truncatedness

Many aspects have now been formalized in Lean (jww van Doorn, Rijke):

Theorem

Let $X : \mathrm{Type}_{\mathrm{pt}}^{\geq k}$ *be a* $(k-1)$*-connected, pointed type for some* $k \geq 0$*, and let* $Y : X \to \mathrm{Type}_{\mathrm{pt}}^{\leq n+k}$ *be a fibration of* $(n+k)$*-truncated, pointed types for some* $n \geq -1$*. Then the type of pointed sections,* $(x : X) \to_{\mathrm{pt}} Y\,x$*, is* n*-truncated.*

Corollary

Let $k \geq 0$ *and* $n \geq -1$*. If* X *is* $(k-1)$*-connected, and* Y *is* $(n+k)$*-truncated, then the type of pointed maps* $X \to_{\mathrm{pt}} Y$ *is* n*-truncated. In particular,* $\mathrm{hom}_{(n,k)}(G, H)$ *is an* n*-type for* $G, H : (n,k)\mathrm{GType}$*.*

Corollary

The type $(n,k)\mathrm{GType}$ *is* $(n+1)$*-truncated.*

Lean Formalization: Categorical equivalences

Theorem

We have the following equivalences of 1-categories (*for $k \geq 2$*):

$$(0,0)\mathrm{GType} \simeq \mathrm{Set}_{\mathrm{pt}};$$
$$(0,1)\mathrm{GType} \simeq \mathrm{Group};$$
$$(0,k)\mathrm{GType} \simeq \mathrm{AbGroup}.$$

Lean Formalization: Operations

decategorification $\mathrm{Decat} : (n,k)\mathrm{GType} \to (n-1,k)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle \|G\|_{n-1}, \|B^k G\|_{n+k-1}\rangle$

discrete categorification $\mathrm{Disc} : (n,k)\mathrm{GType} \to (n+1,k)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle G, B^k G\rangle$

looping $\Omega : (n,k)\mathrm{GType} \to (n-1,k+1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle \Omega G, B^k G\langle k\rangle\rangle$

delooping $\mathrm{B} : (n,k)\mathrm{GType} \to (n+1,k-1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G\rangle$

forgetting $F : (n,k)\mathrm{GType} \to (n,k-1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle G, \Omega B^k G\rangle$

stabilization $S : (n,k)\mathrm{GType} \to (n,k+1)\mathrm{GType}$
$\langle G, B^k G\rangle \mapsto \langle SG, \|\Sigma B^k G\|_{n+k+1}\rangle$,
where $SG = \|\Omega^{k+1}\Sigma B^k G\|_n$

Lean Formalization: (de)categorification

$\mathrm{Decat} \dashv \mathrm{Disc}$ with $\mathrm{Decat} \circ \mathrm{Disc} = \mathrm{id}$:

$k \setminus n$	0		1		2	$\cdots$	∞
0	pointed set	$\rightleftarrows$	pointed groupoid	$\rightleftarrows$	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	$\rightleftarrows$	2-group	$\rightleftarrows$	3-group	$\cdots$	∞-group
2	abelian group		braided 2-group		braided 3-group	$\cdots$	braided ∞-group
3	— ” —		symmetric 2-group		sylleptic 3-group	$\cdots$	sylleptic ∞-group
4	— ” —		— ” —		symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$
ω	— ” —		— ” —		— ” —	$\cdots$	connective spectrum

Lean Formalization: (de)looping

$\mathrm{B} \dashv \Omega$ with $\Omega \circ \mathrm{B} = \mathrm{id}$:

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

Lean Formalization: Stabilization

We also have:

- $S \dashv F$

Lemma (Wedge connectivity)

If $A : \mathrm{Type}_{\mathrm{pt}}$ is n-connected and $B : \mathrm{Type}_{\mathrm{pt}}$ is m-connected, then the map $A \vee B \to A \times B$ is $(n+m)$-connected.

Theorem (Freudenthal)

If $A : \mathrm{Type}_{\mathrm{pt}}^{>n}$ with $n \geq 0$, then the map $A \to \Omega\Sigma A$ is $2n$-connected.

Theorem (Stabilization)

If $k \geq n+2$, then $S : (n,k)\mathrm{GType} \to (n,k+1)\mathrm{GType}$ is an equivalence, and any $G : (n,k)\mathrm{GType}$ is an infinite loop space.

Examples

- $B\mathbb{Z} = \mathbb{S}^1$, other free groups on pointed sets, free abelian groups.
- Automorphism groups $\mathrm{Aut}\, a := (a = a)$ for $a : A$ with $\mathrm{BAut}\, a := \mathrm{im}(a : 1 \to A) = (x : A) \times \|a = x\|_{-1}$.
- Fundamental n-group of (A, a), $\Pi_n(A, a)$, with corresponding delooping $\mathrm{B}\Pi_n(A, a) = \|\mathrm{BAut}\, a\|_n$.
- Symmetric groups $S_n := \mathrm{Aut}([n])$, where $BS_n = \mathrm{BAut}([n])$ is the type of all (small) n-element sets. Colimit S_∞.
- Generally, if $G : \mathrm{Group}$, we can take BG to be the type of G-torsors.
- $\mathbb{S}^1 = \mathrm{B}\mathbb{Z}$ has delooping $\mathrm{B}^2\, \mathbb{Z}$, which we can take to be the type of oriented circles.
- $\mathcal{G}_n := \mathrm{Aut}(\mathbb{S}^{n-1})$ and $\mathcal{F}_n := \mathrm{Aut}(\mathbb{S}^n_{\mathrm{pt}})$. Colimits $\mathcal{G} \simeq \mathcal{F}$. Orientation preserving versions too.
- $\vdots$
- With cohesion, we should get $BO(n)$, $BU(n)$, etc.

Actions

A G-action on $a : A$ is simply a homomorphism $G \to \mathrm{Aut}\, a$.
A G-type is a function $X : BG \to \mathrm{Type}$. Here we can form the

invariants $X^{hG} :\equiv (z : BG) \to X(z)$, also known as the *homotopy fixed points*, and the

coinvariants $X_{hG} :\equiv (z : BG) \times X(z)$, which is also known as *homotopy orbit space* or the *homotopy quotient* $X /\!\!/ G$.

Right and left adjoints to $A \mapsto A^{\mathrm{triv}}$ for $A : \mathrm{Type}$.

Proposition

Let $f : H \to G$ be a homomorphism of higher groups with delooping $Bf : BH \to_{\mathrm{pt}} BG$, and let $\alpha : \mathrm{hom}(X, Y)$ be a map of G-types. By composing with f we can also view X and Y as H-types, in which case we get a homotopy pullback square:

$$\begin{array}{ccc} X_{hH} & \longrightarrow & Y_{hH} \\ \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & Y_{hG}. \end{array}$$

Canonical actions

Every group G carries two canonical actions on itself:

the right action $G : BG \to \mathrm{Type}$, $G(x) = (\mathrm{pt} = x)$, and the

the adjoint action $G^{\mathrm{ad}} : BG \to \mathrm{Type}$, $G^{\mathrm{ad}}(x) = (x = x)$ (by conjugation).

We have $1 /\!/ G = BG$, $G /\!/ G = 1$ and $G^{\mathrm{ad}} /\!/ G = LBG := (\mathbb{S}^1 \to BG)$, the free loop space of BG.

Corollary

If $f : H \to G$ is a homomorphism of higher groups, then $G /\!/ H$ is equivalent to the homotopy fiber of the delooping $Bf : BH \to_{\mathrm{pt}} BG$, where H acts on G via the f-induced right action.

Projective spaces

Consider the sequence of actions $GM^n : BG \to \mathrm{Type}$ of G given by

$$GM^{-1}(x) := 0$$
$$GM^{n+1}(x) := (\mathrm{pt} = x) * GM^n(x) = G(x) * GM^n(x)$$

i.e., the iterated joins of the right action with itself (M is for Milnor). The types $GM^n(\mathrm{pt})$ are at least $(n+1)(k+2)-2$-connected if G is k-connected. Then the colimit $GM^\infty(\mathrm{pt}) = \varinjlim GM^n(\mathrm{pt})$ is contractible, so $GM^\infty /\!\!/ G = 1 /\!\!/ G = BG$.

We define the *projective spaces* for G to be $GP^n := GM^n /\!\!/ G$. Thus, $GP^{-1} = 0$, $GP^0 = 1$, $GP^1 = \Sigma G$, etc. (in general, GP^{n+1} is the mapping cone on the inclusion $GM^n(\mathrm{pt}) \to GP^n$).

The *real* and *complex projective spaces* are $\mathbb{R}\mathrm{P}^n := \mathcal{G}_1 P^n$ and $\mathbb{C}\mathrm{P}^n := \mathcal{S}\mathcal{G}_2 P^n$.

Orbit Stabilizer Theorem

Let X be a G-type.

Definition

Given $x : X$, the *stabilizer* of x is the group H_x with delooping $BH_x := \mathrm{im}(1 \to X \to X \mathbin{/\!/} G)$.
The *orbit* of x is the type $G \cdot x := (y : X) \times \|\langle \mathrm{pt}, x\rangle = \langle \mathrm{pt}, y\rangle\|_{-1}$.

Theorem (Orbit Stabilizer Theorem)

Given $x : X$, $G \mathbin{/\!/} H_x \simeq G \cdot x$.

Proof.

The map $BH_x \to_{\mathrm{pt}} BG$ is the projection from $(z : BG) \times (y : X\ z) \times \|\langle \mathrm{pt}, x\rangle = \langle z, y\rangle\|_{-1}$ to BG, so the fiber over pt is $G \cdot x$. □

Central extensions

The cohomology of a higher group G is simply the cohomology of its delooping BG. Indeed, for any spectrum A, we define

$$H^k_{\mathrm{Grp}}(G, A) :\equiv \|BG \to_{\mathrm{pt}} B^k A\|_0.$$

Of course, to define the k'th cohomology group, we only need the k-fold delooping $B^k A$.
If $A : (\infty, 2)\mathrm{GType}$ is a braided ∞-group, then we have the second cohomology group $H^2_{\mathrm{Grp}}(G, A)$, and an element $c : BG \to_{\mathrm{pt}} B^2 A$ gives rise to a *central extension*

$$BA \to BH \to BG \xrightarrow{c} B^2 A,$$

Example

The central extension $1 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 1$ is classified by the map $BC_n \to B^2\mathbb{Z}$ that sends an n-element set with a cyclic ordering to the canonical oriented circle obtained by gluing.

Outline

Motivation

From “Open Problems” on the HoTT Wiki

‘Similarly to the torus, consider the projective plane, Klein bottle, … as discussed in the book (sec 6.6). Show that the Klein bottle is not orientable. (This requires defining “orientable”.)’

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From “Open Problems” on the HoTT Wiki

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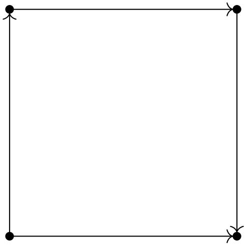
Attempted solution

We can define the tangent bundle $\tau : \mathrm{KB} \to B\mathcal{G}_2$. Looking at cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients, we see that τ doesn’t lift to $BS\mathcal{G}_2$.

This is not satisfactory!

Tangent bundle of the Klein bottle



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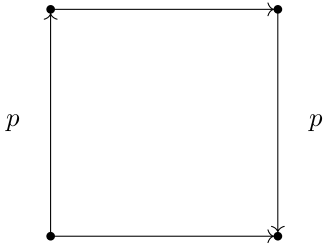

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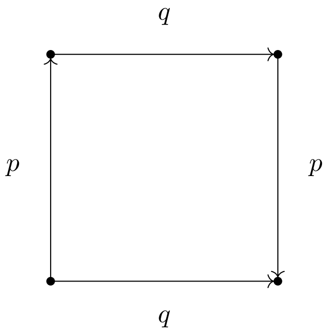

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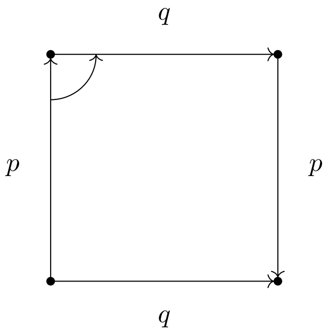

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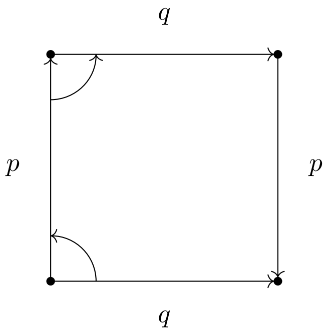

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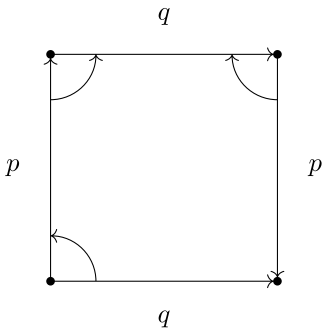

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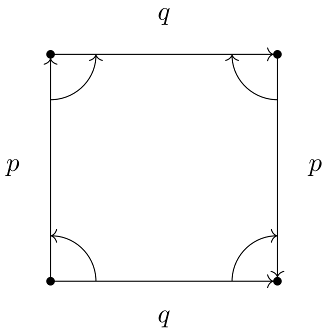
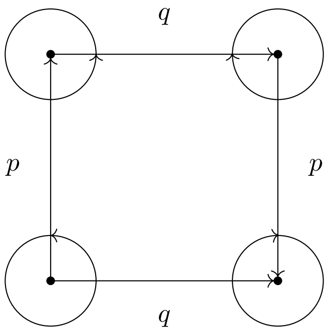


Tangent bundle of the Klein bottle



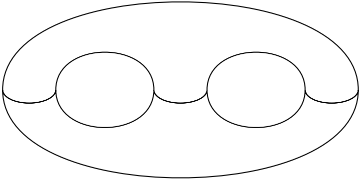
$$\tau : \mathrm{KB} \to B\mathcal{G}_2$$

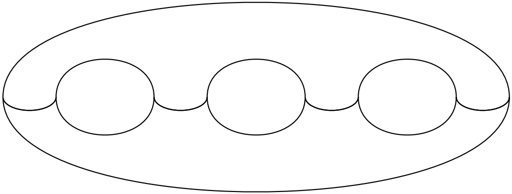
A real solution

Desiderata:

- Define the type of surfaces, Surf.
- Define the type of oriented surfaces, $\mathrm{OredSurf}$ with a forgetful map $\mathrm{OredSurf} \to \mathrm{Surf}$.
- Then the type of orientable surfaces, $\mathrm{ObleSurf}$, is the image.
- Prove that $\mathrm{OredSurf} \to \mathrm{ObleSurf}$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle.
- $\vdots$
- Prove the classification of surfaces theorem. (Constructively?).

Some surfaces





Poincaré Duality Surfaces

Definition

A n-*dimensional Poincaré duality type* is a type $X : \mathrm{FinType}$ (i.e., merely equivalent to a finite complex) together with an *orientation class* $w : X \to B\mathcal{G}_1$, and a class $[X] : H_n(X; \mathbb{Z}_w)$ called the *fundamental class* such that the cap product map

$$- \cap [X] : H^i(X; \mathbb{Z}\pi_1 X) \to H_{n-i}(X; \mathbb{Z}\pi_1 X_w)$$

is an isomorphism.

Theorem

For any X, the type of pairs $(w, [X])$ making X an n-dimensional Poincaré duality type is contractible.

Theorem (Eckmann-Müller-Linnell)

Every 2-dimensional Poincaré duality type in classical homotopy theory is equivalent to a closed surface.

Haves and needs

Luckily, we have most of the components already:

- We have the type $\mathrm{FinCell}$ of finite cell complexes with realization map $\mathrm{FinCell} \to \mathrm{Type}$ with image $\mathrm{FinType}$.
- We know that (so far: unparametrized) cohomology of a finite complex can be computed via cellular cohomology (jww Favonia).
- We have the cup product and thus also cap product maps for integral coefficients, and hence with a little more work for free abelian coefficients over a family of finite sets.

We still need:

- Extend the work on cellular cohomology to parametrized versions and to homology.
- Construct the usual surfaces. (Perhaps Hurewicz would be handy to construct the fundamental classes.)
- Prove the classification theorem. In particular, we need to go from a 2-dim. PD type X to its Spivak stable normal bundle, and prove Spivak's theorem on spherical fibrations and get a tangent bundle. Prove basic theorems establishing that orientations of the tangent bundle correspond to orientation of X.

Outlook

- Proposal for another big formalization project.
- Is it better to define surfaces using spectrum-level PD?
- Generalize to surfaces with boundary in order to study braid groups and mapping class groups, and ... homological stability?
- In smooth/analytic cohesion, relate to smooth/analytic surfaces.
- Brown's theorem on computability of homotopy groups of finite complexes?

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