Liquid Tensor Experiment
Joint work with

- Peter Scholze

The Lean community

- Adam Topaz
- Riccardo Brasca
- Patrick Massot
- Scott Morrison
- Kevin Buzzard
- Bhavik Mehta
- Filippo A.E. Nuccio
- Andrew Yang
- Damiano Testa
- Heather Macbeth
- Mario Carneiro
- many others
On 5 Dec 2020, Peter Scholze posted a challenge:
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Check the main theorem of liquid vector spaces
On 5 Dec 2020, Peter Scholze posted a challenge:

*Check the main theorem of liquid vector spaces*

*...on a computer*
Nine days later ...
1998 Liquid Tension Experiment
1998  Liquid Tension Experiment

1999  Liquid Tension Experiment 2
1998  Liquid Tension Experiment

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2020  Dec 05: “Liquid Tensor Experiment”,
      Peter Scholze

2021 Apr 16: release LTE 3

tracks include:

Solid Resolution Theory,
Beating the Odds,
Shades of Hope

2022 Jul 14: complete formal verification of
main theorem of liquid vector spaces
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2022  Jul 14: complete formal verification of main theorem of liquid vector spaces
Why liquid vector spaces?

1. Embeds real/complex manifolds into a unifying category of geometric objects: *analytic spaces*

2. Makes algebraic methods available in functional analysis
Topological algebraic problem

\[ \mathbb{R}^\delta \rightarrow \mathbb{R} \]
A condensed set is a functor

\[ \text{Profinite}^{\text{op}} \to \text{Set} \]

satisfying a certain sheaf condition.

“Yoneda” gives: \( \text{Top} \to \text{Cond}(\text{Set}) \)
Condensed sets (2)

CompHaus is equivalent to qcqs objects in Cond(Set).

Weakly Hausdorff compactly generated $X$ are roughly the same as quasi-separated condensed sets.
Condensed abelian groups

Cond(Ab) is very nice:

abelian category satisfying (AB3), (AB4), (AB5), (AB3*), and even (AB4*) and (AB6).
Analytic spaces

An analytic ring is a condensed ring together with a “completion functor” for condensed $R$-modules.
Analytic spaces

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Examples: discrete rings, $p$-adic rings
Analytic spaces

– An analytic ring has an analytic spectrum
Analytic spaces

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- Gluing these spectra gives analytic spaces
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– Examples: schemes, formal schemes, Berkovich spaces, adic spaces
Analytic spaces

- An analytic ring has an analytic spectrum
- Gluing these spectra gives *analytic spaces*
- Examples: schemes, formal schemes, Berkovich spaces, adic spaces
- What about real/complex manifolds?
Let $S$ be a profinite set

- $\mathbb{R}[S]$ is the space of linear combinations of Dirac measures
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- We want to complete it
Let $S$ be a profinite set

- $\mathbb{R}[S]$ is the space of linear combinations of Dirac measures
- We want to complete it
- First candidate: $\mathcal{M}(S)$, the space of signed Radon measures
Functional analysis

- What we want:

\[
\begin{align*}
S & \xrightarrow{f} V \\
\mathcal{M}(S) & \xrightarrow{} \int_{S} f(s) \, d\mu
\end{align*}
\]
Functional analysis

– What we want:

\[ \mathbb{R}[S] \xrightarrow{f} V \]

\[ \exists! \mu \mapsto \int_S f(s) \, d\mu \]

in other words:

\[ \text{Hom}(\mathbb{R}[S], V) = \text{Hom}(\mathcal{M}(S), V). \]
Functional analysis

– What we want:

\[ \exists! \mu \rightarrow \int_s f(s) \, d\mu \]

\[ \mathcal{M}(S) \]

in other words:

\[ \text{Hom}(\mathbb{R}[S], V) = \text{Hom}(\mathcal{M}(S), V) \].

– We also want \( \text{Ext}^i(\mathcal{M}(S), V) = 0 \) for \( i > 0 \)
Functional analysis

– What we want:

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– We also want \( \text{Ext}^i(\mathcal{M}(S), V) = 0 \) for \( i > 0 \)

– But wait! For which kind of spaces \( V \)?
Functional analysis

– The formalism of analytic rings forces non-locally-convex topological vector spaces into the picture
Functional analysis

- The formalism of analytic rings forces non-locally-convex topological vector spaces into the picture.

- $V$ needs to be a $p$-Banach space for $p < 1$: a complete TVS whose topology is induced by a $p$-norm:

\[ \|\lambda v\| = |\lambda|^p \|v\| \]
Functional analysis

– The entropy function

\[ \ell^1 \rightarrow \ell^2 \]

\[(x_n)_n \mapsto (x_n \log |x_n|)_n\]

can be used to show that \( \mathcal{M}(S) \) does not satisfy the required universal property
Functional analysis

- The entropy function

\[ \ell^1 \to \ell^2 \]

\[ (x_n)_n \mapsto (x_n \log |x_n|)_n \]

can be used to show that \( M(S) \) does not satisfy the required universal property.

- We need to use \( M_{p'}(S) \) instead:
  the subspace of signed Radon measures satisfying some \( \ell^{p'} \)-convergence condition.
Theorem (Clausen–Scholze)

Let $0 < p' < p < 1$ be real numbers,
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Let $0 < p' < p < 1$ be real numbers, let $S$ be a profinite set, and let $V$ be a $p$-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of $p'$-measures on $S$. 
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Let $0 < p' < p < 1$ be real numbers, let $S$ be a profinite set, and let $V$ be a $p$-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of $p'$-measures on $S$.

Then

$$\text{Ext}^i_{\text{Cond}(\text{Ab})}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$. 
Liquid vector spaces

Fix a real parameter $0 < p \leq 1$

Then $p$-liquid vector spaces form a full subcat $\text{Liq}_p \subset \text{Cond}(\mathbb{R})$ with good properties …
Liquid vector spaces (2)

- $\text{Liq}_p$ is an abelian category
Liquid vector spaces (2)

- $\text{Liq}_p$ is an abelian category

- Closed under limits, colimits, extensions

- If $p' \leq p$, then $\text{Liq}_p \subset \text{Liq}_{p'}$

Examples:
- Banach spaces, nuclear Fréchet spaces
- Liquid tensor product is compatible with topological tensor product of nuclear Fréchet
Liquid vector spaces (2)

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- Examples:
  
  Banach spaces, nuclear Fréchet spaces
- Liquid tensor product is compatible with
  
  topological tensor product of nuclear Fréchets
Summary:

Liquid analytic ring structure on $\mathbb{R}$
The Experiment
– “spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it”
Scholze (LTE blogpost, 2020)

- “spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it”

- “proof has some very unexpected features … very much of arithmetic nature”
– “spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it”

– “proof has some very unexpected features . . . very much of arithmetic nature”

– “I think this may be my most important theorem to date”
Scholze (LTE blogpost, 2020)

- “spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it”

- “proof has some very unexpected features … very much of arithmetic nature”

- “I think this may be my most important theorem to date”

- “I think nobody else has dared to look at the details, and so I still have some small lingering doubts”
Theorem (Clausen–Scholze)

Let $0 < p' < p < 1$ be real numbers, let $S$ be a profinite set, and let $V$ be a $p$-Banach space.

Let $\mathcal{M}_{p'}(S)$ be the space of $p'$-measures on $S$.

Then

$$\operatorname{Ext}^i_{\operatorname{Cond(\text{Ab})}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$. 
First target (Thm 9.4 of Analytic.pdf)

Fix $0 < r < r' < 1$.

For any $m$, there exists a $k$ and $c_0$ such that for all profinite sets $S$ and $r$-normed $\mathbb{Z}[T^\pm]$-modules $V$ the system of complexes

$$\hat{C}_c^\bullet: \hat{V}(\overline{M}_{r'}(S)_{\leq c})^{T-1} \to \hat{V}(\overline{M}_{r'}(S)^{2}_{\leq k_1 c})^{T-1} \to \ldots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$. 
Progress report

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– Some statements/proofs of lemmas and auxiliary definitions were changed
– Detailed blueprint
– Answer to Question 9.9 of Analytic.pdf
– Alternative to Breen–Deligne resolutions
– Effort to conceptualize parts of the proof
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Theorem (Breen–Deligne)

There exists a functorial resolution of an abelian group \( A \) of the form

\[
\cdots \rightarrow \bigoplus_{j=1}^{n_j} \mathbb{Z}[A^{r_{i,j}}] \cdots \rightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0
\]

where all \( n_j \) and \( r_{i,j} \) are natural numbers.
Theorem (Breen–Deligne)

There exists a functorial resolution of an abelian group $A$ of the form

$$
\cdots \to \bigoplus_{j=1}^{n_j} \mathbb{Z}[A^{r_{i,j}}] \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0
$$

where all $n_j$ and $r_{i,j}$ are natural numbers.

Proof uses homotopy theory.
The MacLane $Q'$-construction is a functorial complex of an abelian group $A$ of the form

$$Q'(A): \cdots \rightarrow \mathbb{Z}[A^{2^i}] \cdots \rightarrow \mathbb{Z}[A^4] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0$$

with the property that:
The MacLane $Q'$-construction is a functorial complex of an abelian group $A$ of the form

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with the property that:

if $\text{Ext}^i(Q'(A), B) = 0$ for all $i \geq 0$, 

Lemma

The MacLane $Q'$-construction is a functorial complex of an abelian group $A$ of the form

\[ Q'(A) : \quad \cdots \to \mathbb{Z}[A^{2^i}] \cdots \to \mathbb{Z}[A^4] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0 \]

with the property that:

if $\text{Ext}^i(Q'(A), B) = 0$ for all $i \geq 0$,

then $\text{Ext}^i(A, B) = 0$ for all $i \geq 0$. 
Human-friendly proof?

Can we find a proof that does not need computer verification?

I will present some thoughts

(j/w Reid Barton)
The technical key ingredient involved a chain complex of objects:

\[ \mathbb{R}_{\geq 0}^{\text{op}} \to \text{NormAbGrp} \]

The theorem claims that this complex is “exact” in some sense.
The presence of functors

\[ \mathbb{R}_{\geq 0}^{\text{op}} \to \text{NormAbGrp} \]

and the metric on objects in NormAbGrp suggests that we can try to mix sheaf semantics and continuous logic.
Suppose we have

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

with \( A, B, C : \mathbb{R}^{\geq 0}_{\text{op}} \rightarrow \text{NormAbGrp} \).
Sheaf semantics

Suppose we have

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

with \( A, B, C : \mathbb{R}^{\geq 0}_{\text{op}} \rightarrow \text{NormAbGrp}. \)

Then \( \forall b : B, g(b) = 0 \vdash \exists a : A, f(a) = b \)
interpretes to

\[ \exists k \geq 1, \forall s \gg 0, \ldots \]

\[ \forall b \in B_{ks}, g(b) = 0 \implies \exists a \in A_s, f(a) = b|_s \]
Continuous logic

Suppose we have

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

with \( A, B, C \in \text{NormAbGrp} \).
Continuous logic

Suppose we have

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

with \( A, B, C \in \text{NormAbGrp} \).

Then \( \forall b : B, g(b) = 0 \vdash \exists a : A, f(a) = b \)
interpretes to

\[ \exists K, \ldots \]

\[ \forall b \in B, \forall \varepsilon > 0, \exists a \in A, \| f(a) - b \| \leq K \| g(b) \| + \varepsilon \]
Normed exactness

Combine the sheaf semantics and continuous logic

Result:
“exactness” interpretes to “normed exactness” as needed for the chain complex in theorem 9.4.
We can “compile” the proofs of homological algebra results into the “normed-and-sheafy” setting.

Examples: snake lemma, long exact sequence, spectral sequence
A diamond “modality”?

At one point in the global proof, this strategy fails.

We need to make a certain norm estimate.

And it is not simply the interpretation of an internal reasoning step.

We think it is related to some modal operator. But there are some speedbumps. WIP!
The proof is certainly not “a formality”.

At crucial points one needs:

– Combinatorics (Gordan’s lemma)
– Results about cohomology of profinite sets
Lessons
State of the art maths can be formalized in a reasonable amount of time.
Lessons (2)

Proof assistant

Lean showed to be a powerful tool for managing complex proofs.
Lessons (2)

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Lean showed to be a powerful tool for managing complex proofs
Partial lessons (3)

What makes the proof tick?

Why does it pass through arithmetic?

In which logic should the proof work?