HoTTEST seminar

On the fibration of algebras

Ahman, Castelnovo, Coraglia, Loregian, Martins-Ferreira, Reimaa

π day 2024

Fosco,
spring of 2022

A simple observation about simple Types:

let $\mathcal{C}$ be a cartesian category, we can build [J98, 1.3] the category $s(\mathcal{C})$ having

- objects (I, X) with $I, X : ob\,\mathcal{C}$
- homs $(J,Y) \to (I,X)$ are pairs $(u : J \to I,\ f : J \times Y \to X)$

which models simple types in context.

<u>RMK 1</u>

$$\begin{array}{c} s(\mathcal{C}) \\ \downarrow \\ \mathcal{C} \end{array} \qquad \begin{array}{c} (J,Y) \xrightarrow{(u,f)} (I,X) \\ \updownarrow \\ J \xrightarrow{u} I \end{array}$$

is a Grothendieck fibration with fiber $s(\mathcal{C})_I =: \mathcal{C}/\!/I$ the "simple slice"

$$\begin{array}{cc} (c, a : Fc) & \int F \\ \updownarrow & \downarrow \\ c & \mathcal{C} \end{array} \qquad \mathcal{C}^{op} \xrightarrow{F} \underline{Cat}$$

<u>Recall</u> $\mathrm{Fib}(\mathcal{C}) \simeq \mathrm{PFun}[\mathcal{C}^{op}, \underline{Cat}]$

$$\begin{array}{c} s(\mathcal{C}) \\ \downarrow \\ \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^{op} & \to & \underline{Cat} \\ I & & \mathcal{C}/\!/I \end{array}$$

[J98] B. Jacobs, "Categorical logic and type theory", 1998

RMK 1
$$\begin{tikzcd} s(\mathcal{C}) \arrow[d] \\ \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} (J,Y) \arrow[r, "{(u,f)}"] & (I,X) \arrow[d, mapsto] \\ J \arrow[r, "u"] & I \end{tikzcd}$$

is a Grothendieck fibration with fiber $s(\mathcal{C})_I =: \mathcal{C}/\!/I$

the "simple slice"

RMK 2 $s(\mathcal{C})_I = \mathcal{C}/\!/I = \mathrm{coKl}(\underbrace{I\times -}_{\text{the "coreader comonad"}})$

$$I\times - : \mathcal{C} \to \mathcal{C}$$
$$X \mapsto I\times X$$

hence the simple fibration is the result of the pasting of all (co)algebras for a parametric (co)monad

F: « IS THIS A THING? »

hence the simple fibration is the result of the pasting of all (co)algebras for a parametric (co)monad

TODAY'S PLAN

1. look for it elsewhere (spoiler: it appears in many different places!)
2. try to give a general theory of this phenomenon ← semidirect product?!
3. benefits of a general theory
4. applications to polynomials, automata, and more

1 look for it elsewhere

IS THIS A THING? YES!

denote $\mathcal{A}$ the category of parameters, $F : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ parametric endofunctor
(or co/monad)

$\mathcal{A} = \mathcal{X}$ with enough structure

- $F_A = A \times -$ COREADER COMONAD
  - $\xrightarrow{\text{coKl}}$ simple slice over A
  - $\xrightarrow{\text{coEM}}$ slice over A (streams)
- $F_A = A \to -$ READER MONAD
- $F_A = A \to (A \times -)$ STATE MONAD
- monoidal versions of the above
  e.g. $F_A = A \otimes -$ WRITER MONAD
- $F_A = A + -$ EXCEPTION MONAD

let's focus on the (co)algebras for a second denote $\mathcal{A}$ the category of parameters, $F : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ parametric endofunctor (or co/monad)

and consider $\mathcal{A} = \mathcal{X} = \underline{Set}$

endofunctors	coalgebras
$F_A = A \times -$	stream systems
$F_A = 2 \times -^A$	deterministic automata
$F_A = 2 \times (1 + -)^A$	partial automata
$F_A = 2 \times \mathcal{P}(-)^A$	non-deterministic automata
$F_{A,B} = (B \times -)^A$	Mealy automata
$F_{A,B} = B \times -^A$	Moore automata

[R19]

← more coming from the rich theory of (co)algebras

[R19] J. Rutten, "The method of coalgebra", 2019 not only the theory of automatas!

denote $\mathcal{A}$ the category of parameters, $F : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ parametric endofunctor (or co/monad)

- for $\mathcal{E}$ lccc

$$P_{-} : \mathrm{Poly}_I \times \mathcal{E}/I \longrightarrow \mathcal{E}/I$$

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I \quad \text{induces} \quad \mathcal{E}/I \xrightarrow{\Delta_s} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_t} \mathcal{E}/I$$

$$\Sigma_{-} \dashv \Delta_{-} \dashv \Pi_{-}$$

- (actually a special case of the reader)

$$\mathrm{Id} : [\mathcal{X}, \mathcal{X}] \times \mathcal{X} \to \mathcal{X}$$

$$\mathrm{Id}_F(-) := F(-)$$

but is this not just the theory of graded [FKM16] or "parametrised" [A09] monads?

YES AND NO

[FKM16] Fuji, Katsumata, Melliès, "Toward a formal theory of graded monads", 2016
[A09] Atkey, "Parametrised notions of computation", 2009

② try to give a general theory of this phenomenon

$$\begin{tikzcd} s(\mathcal{C}) \arrow[d] & (J,Y) \arrow[r, "(u,f)"] \arrow[d, mapsto] & (I,X) \\ \mathcal{C} & J \arrow[r, "u"] & I \end{tikzcd}$$

$$\boxed{s(\mathcal{C})_I = \mathcal{C}/\!/I = \mathrm{coKl}(I\times -)}$$

$$\boxed{\begin{array}{c} I\times - : \mathcal{C}\to\mathcal{C} \\ \\ \dfrac{\mathcal{C}\times\mathcal{C}\to\mathcal{C}}{\mathcal{C}\to[\mathcal{C},\mathcal{C}]} \end{array}}$$

denote $\mathcal{A}$ the category of parameters,

$F : \mathcal{A}\times\mathcal{X}\to\mathcal{X}$ parametric endofunctor (or co/monad)

two steps

$\boxed{1}$ out of a category of parameters, compute endofunctors

$$F : \mathcal{A}\to[\mathcal{X},\mathcal{X}]$$

$\boxed{2}$ out of an endofunctor, compute its algebras *

$$\mathrm{Alg}_{\mathcal{X}} : [\mathcal{X},\mathcal{X}]^{op}\longrightarrow \mathrm{Cat}$$

$$\begin{tikzcd} F & \mathrm{Alg}_{\mathcal{X}}(F) \arrow[d, "\alpha^*"] & FX \arrow[r, "x"] & X \\ G \arrow[u, Rightarrow, "\alpha"] & \mathrm{Alg}_{\mathcal{X}}(G) & GX \arrow[r, dashed, "\alpha^* x"] \arrow[dr, "\alpha_X"'] & X \\ & & & FX \arrow[r, "x"] & X \end{tikzcd}$$

* I know it's a different kind of algebra...

two steps

$\boxed{1}$ out of a category of parameters, compute endofunctors

$$F : \mathcal{A} \to [\mathcal{X},\mathcal{X}]$$

$\boxed{2}$ out of an endofunctor, compute its algebras

$$\mathrm{Alg}_{\mathcal{X}} : [\mathcal{X},\mathcal{X}]^{op} \longrightarrow \underline{\mathrm{Cat}}$$

$$\begin{array}{ccc} F & & \mathrm{Alg}_{\mathcal{X}}(F) \\ {\scriptstyle\alpha}\Uparrow & & \downarrow{\scriptstyle\alpha^*} \\ G & & \mathrm{Alg}_{\mathcal{X}}(G) \end{array}$$

$$\begin{tikzcd} \mathcal{A} \times_F \mathcal{X} \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Alg}_{\mathcal{X}} \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]} \end{tikzcd}$$

Df// call U the "UNIVERSAL (SPLIT) FIBRATION OF ENDOFUNCTOR ALGEBRAS"

Df// fibrations p_F obtained this way we call "fibrations of endofunctor algebras"

$(A; X, x)$

A is an object in $\mathcal{A}$

$x : F_A X \to X$ is an algebra for F_A

and

$(A'; Y, y) \xrightarrow{(u,f)} (A; X, x)$

$u : A' \to A$, $f : Y \to X$ st

$$\begin{tikzcd}
F_{A'}Y \arrow[r, "F_{A'}f"] \arrow[d, "y"'] & F_{A'}X \arrow[r, "F_u X"] \arrow[dr] \arrow[d, phantom, "="] & F_A X \arrow[d] \\
Y \arrow[rr, "f"'] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
(A'; Y, y) \arrow[d, "(id, f)"', "vert"] \arrow[dr, bend left, "(u,f)"] & \\
(A'; X, x) \arrow[r, "cart", "(u, id)"'] & (A; X, x)
\end{tikzcd}$$

$$A' \xrightarrow[u]{} A$$

$$\begin{tikzcd}
\mathcal{A} \times_F \mathcal{X} \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Alg}_{\mathcal{X}} \arrow[d, "U"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X}, \mathcal{X}]}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \ltimes_F \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Alg}_{\mathcal{X}} \arrow[d, "\text{fib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \Theta_F \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{coAlg}_{\mathcal{X}} \arrow[d, "\text{opfib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \ltimes_F^{\mathrm{EM}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{EM}_{\mathcal{X}} \arrow[d, "\text{fib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]_{\eta,\mu}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \ltimes_F^{\mathrm{Kl}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Kl}_{\mathcal{X}} \arrow[d, "\text{opfib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]_{\eta,\mu}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \Theta_F^{\mathrm{EM}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{coEM}_{\mathcal{X}} \arrow[d, "\text{opfib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]_{\varepsilon,\delta}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \Theta_F^{\mathrm{coKl}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{coKl}_{\mathcal{X}} \arrow[d, "\text{fib}"] \\
\mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]_{\varepsilon,\delta}}
\end{tikzcd}$$

Fosco's question

fibration morphisms that we don't have...

$$
\begin{tikzcd}
\int \mathrm{Ke}_X \arrow[rr, "\times" description] \arrow[dr, "\text{opfib}"'] & & \int \mathrm{EM}_X \arrow[dl, "\text{fib}"] \\
& {[X,X]_{\eta,\mu}} &
\end{tikzcd}
$$

wrong variance!

... and some that we actually do have

$$
\begin{tikzcd}
\int \mathrm{EM}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Alg}^*_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathrm{Alg}_X \arrow[d] \\
{[X,X]_{\eta,\mu}} \arrow[r, hook] & {[X,X]_\eta} \arrow[r, hook] & {[X,X]}
\end{tikzcd}
$$

faithful but clearly not full (compatibility)

Why do we use the semidirect product notation?

Given groups $(H, \cdot_H, 1_H)$ and $(N, \cdot_N, 1_N)$
and $\varphi : H \to \mathrm{Aut}(N)$ group hom

$H \rtimes_\varphi N$ is the group $(H \times N, \bullet, (1_H, 1_N))$

$$(h,n) \bullet (h',n') := (hh', n\,\varphi(h)\, n')$$

Given categories $\mathcal{A}$ and $\mathcal{X}$
and $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$ functor

$\mathcal{A} \ltimes_F \mathcal{X}$ is the category with obj...

$$(u,f) \bullet (u',f') := (uu', \star)$$

$$(A''; Z, z) \xrightarrow{(u',f')} (A'; Y, y) \xrightarrow{(u,f)} (A; X, x)$$

$$(\star) \quad \begin{tikzcd} F_{A''}Z \arrow[r, "F_{u'}f'"] \arrow[d, "z"'] & F_{A'}Y \arrow[r, "F_u f"] \arrow[d, "y"'] & F_A X \arrow[d, "x"'] \\ Z \arrow[r, "f'"'] & Y \arrow[r, "f"'] & X \\ A'' \arrow[r, "u'"'] & A' \arrow[r, "u"'] & A \end{tikzcd}$$

Why do we use the semidirect product notation?

Given groups $(H, \cdot_H, 1_H)$ and $(N, \cdot_N, 1_N)$
and $\varphi : H \to \mathrm{Aut}(N)$ group hom

$H \ltimes_\varphi N$ is the group $(H \times N, \bullet, (1_H, 1_N))$

$$(h,n) \bullet (h', n') := (hh', n\,\varphi(h)\,n')$$

for G group, define the holomorph

$$\mathrm{Hol}(G) := \mathrm{Aut}(G) \ltimes G$$

when $\mathrm{Aut}(G) \xrightarrow{\mathrm{Id}} \mathrm{Aut}(G)$

Given categories $\mathcal{A}$ and $\mathcal{X}$
and $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$ functor

$\mathcal{A} \ltimes_F \mathcal{X}$ is the category with obj...

$$(u, f) \bullet (u', f') := (uu', *)$$

for $\mathcal{X}$ category, the "holomorph" is

$$[\mathcal{X}, \mathcal{X}] \ltimes \mathcal{X} \cong \int \mathrm{Alg}_{\mathcal{X}}$$

when $[\mathcal{X}, \mathcal{X}] \xrightarrow{\mathrm{Id}} [\mathcal{X}, \mathcal{X}]$

we can characterize fibrations of EM-algebras:

$\boxed{\text{Thm 0}}$ $\begin{array}{c}\mathcal{E}\\ {\scriptstyle p}\downarrow\\ \mathcal{A}\end{array}$ is a fib of EM-algebras $\iff$ there exists a fib. morphism

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "p"'] & & \mathcal{A} \times \mathcal{X} \arrow[dl, "\pi_A"] \\ & \mathcal{A} & \end{tikzcd}$$

which is monadic as a 1-cell in $\underline{\mathrm{Fib}}(\mathcal{A})$

$\Updownarrow$

H has a fibered left adjoint L and $EM(HL) \cong p$

<u>Rmk</u> in the case of $\mathcal{A} \times^{EM} \mathcal{X}$,

free algebra

$$\begin{tikzcd} \mathcal{A} \times^{EM} \mathcal{X} \arrow[r, shift right] & \mathcal{A} \times \mathcal{X} \arrow[l, bend right, shift right] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

⟨ param, carrier ⟩

③ benefits of a general theory

~~it's pretty~~

we can address different problems all at once reindexing

$$\mathrm{Alg}_X G \xleftarrow{\alpha^*} \mathrm{Alg}_X F$$

$$G \underset{\alpha}{\Rightarrow} F$$

(co) limits in the fibers $\rightarrow$ overall (co) completeness

adjoints to reindexing & (co) continuity the following <u>heavily</u> rely on $\mathcal{A} \ltimes \mathcal{X}$ being fibered over $\mathcal{A}$

$\boxed{\text{Thm 1}}$ let $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]_{M/M}$ be a parametric monad. ← straight forward from Thm 0

The forgetful $\mathcal{A} \ltimes_F \mathcal{X} \to \mathcal{A} \times \mathcal{X}$ is monadic.

<u>corollary</u> then it creates limits

$\boxed{\text{Thm 2}}$ let $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]_{M/M}$ be a parametric monad such that

F preserves filtered colimits separately in each parameter.

Then if $\mathcal{X}$ is cocomplete, so is $\mathcal{A} \ltimes \mathcal{X}$.

$\boxed{\text{Thm 3}}$ Let $\mathcal{X}$ be κ-accessible and F only restricted to κ-accessible functors $\mathcal{X} \to \mathcal{X}$.

Then each reindexing α^* has a left adjoint Σ_α.

<u>corollary</u> $U : \int \mathrm{Alg}_{\mathcal{X}} \to [\mathcal{X}, \mathcal{X}]_{\kappa\mathrm{Acc}}$ is a bifibration

4 applications to polynomials, automata, and more

$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I$ induces $\mathcal{E}/I \xrightarrow{\Delta_s} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_t} \mathcal{E}/I$

$\Sigma_- \dashv \Delta_- \dashv \Pi_-$

Thm ([MP00]) if a lccc $\mathcal{E}$ has w-types, then so do all of its slices $\mathcal{E}/I$

$W(f) =$ initial algebra of $P_f : \mathcal{E} \xrightarrow{\Delta_!} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_!} \mathcal{E}$

In [GH09] a simpler proof is given: the adjoint pair $\mathcal{E} \rightleftarrows \mathcal{E}/I$ ($\perp$) lifts to the algebras, and left adjoints...

We want to see whether that is a "bifibration"-like property.

[MP00] Moerdijk, Palmgren, "Well founded trees in categories", 2000
[GH09] Gambino, Hyland, "Wellfounded trees and dependent polynomial functors", 2009

TO POLYNOMIALS /2

for simplicity, say $\mathcal{E} = \underline{Set}$

$$\begin{array}{c} \int \mathrm{Alg}_{\underline{Set}/I} \\ \Big\downarrow {\scriptstyle \text{bifib by Thm3}} \\ [\underline{Set}/I, \underline{Set}/I]_{\omega\text{-acc}} \end{array}$$

$$\begin{tikzcd} \mathrm{Poly}_I \arrow[r] \arrow[dr] & {[\underline{Set}/I, \underline{Set}/I]} \\ & {[\underline{Set}/I, \underline{Set}/I]_{\omega\text{-acc}}} \arrow[u, hook] \end{tikzcd}$$

$\Longrightarrow$

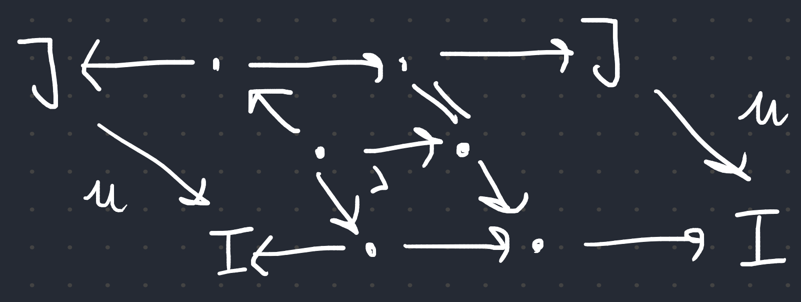
$$\begin{array}{c} \mathrm{Poly}_I \ltimes \underline{Set}/I \\ \Big\downarrow {\scriptstyle \text{bifib}} \\ \mathrm{Poly}_I \end{array}$$

$\overset{?}{\Longrightarrow}$

variance issues?

"too" 2-dimensional?

$$\begin{array}{c} \mathrm{Poly} \ltimes \underline{Set}/\_ \\ \Big\downarrow {\scriptstyle \text{bifib}} \\ \mathrm{Poly}^{\Delta}_{\square} \end{array}$$



$J \searrow I$

$$\begin{array}{c} \mathrm{Poly}^{\Delta}_{\square} \\ \Big\downarrow {\scriptstyle \text{bifib}} \\ \underline{Set} \end{array}$$

follows from [K16, 11.1.12]

and conclude the result at once

$$\begin{array}{c} \mathrm{Poly} \ltimes \underline{Set}/\_ \\ \Big\downarrow {\scriptstyle \text{bifib}} \\ \mathrm{Poly}^{\Delta}_{\square} \\ \Big\downarrow {\scriptstyle \text{bifib}} \\ \underline{Set} \end{array}$$

[K16] J. Kock, "Notes on polynomial functors", 2012

To DIPARAMETRIC COMPUTATIONS/1

Ex let $L : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{Y}$, $R : \mathcal{A}^{op} \times \mathcal{Y} \longrightarrow \mathcal{X}$ st for each $A : \mathcal{A}$

$$\mathcal{Y}(L(A,X),Y) \cong \mathcal{X}(X, R(A,Y)) \text{ natural in } A, X, Y$$

A diparametric monad as in [A09] is a functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{T} [\mathcal{X},\mathcal{X}]$ s.t.
when $\mathcal{A}$ is regarded as a free $[\mathcal{X},\mathcal{X}]$-enriched category $\underline{\mathcal{A}}$
and T as a profunctor
$T : \underline{\mathcal{A}} \nrightarrow \underline{\mathcal{A}}$ is a monad in $\underline{\mathrm{Prof}}[\mathcal{X},\mathcal{X}]$

monad laws = extranatural transformations + axioms

The category of diparametric free algebras
is $\pi \mathrm{Ke}(T)$ with objects (A,X) and morphisms $(A,X) \to (A',X')$
are $X \to T(A,A',X')$ in $\mathcal{X}$

[A09] Atkey, "Parametrised notions of computation", 2009

To DIPARAMETRIC COMPUTATIONS/2

Ex let $L : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{Y}$, $R : \mathcal{A}^{op} \times \mathcal{Y} \longrightarrow \mathcal{X}$ st for each $A : \mathcal{A}$

$$\mathcal{Y}(L(A,X),Y) \cong \mathcal{X}(X, R(A,Y)) \text{ natural in } A, X, Y$$

$(\mathbb{L} \dashv \mathbb{R})_A := \{ L_A \dashv R_A \mid A : \mathcal{A} \}$

$T : \mathcal{A}^{op} \times \mathcal{A} \to [\mathcal{C}, \mathcal{C}]$

$(A', A) \longmapsto R_A L_{A'}$

$(A, A'; x)$ with $x : R_A L_{A'} X \to X$

$$\begin{tikzcd}
\mathrm{Alg}(\mathbb{L} \dashv \mathbb{R})_{\mathcal{A}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{C},\mathcal{C}] \times \mathcal{C}} \arrow[d] \\
\mathcal{A}^{op} \times \mathcal{A} \arrow[r, "T"'] & {[\mathcal{C},\mathcal{C}]}
\end{tikzcd}$$

Prop there is a comparison functor

$$K : \mathrm{Alg}(\mathbb{L} \dashv \mathbb{R})_{\mathcal{A}} \longrightarrow \Pi \mathcal{K}\ell(T)$$

OTHER

- to show (co)completeness of categories of automata

  [BLLL23] for Mealy and Moore a. ✓

- apply to (co)induction techniques such as in [HJ98]

- develop the (huge) amount of algebra this theory seems to suggest:

if $\mathcal{A}$ has $\varnothing$, $\mathcal{X}$ has 1 then F induces

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left=1.5, "\text{free alg. at initial}"] \arrow[r, phantom, "\perp"] & \mathcal{A}\times\mathcal{X} \arrow[l, shift left=1.5, "\text{forgetful}"] \arrow[r, shift left=1.5, "\text{param.}"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left=1.5, "\text{terminal in the fiber}"] \end{tikzcd}$$

a "$\varnothing$-sequence" (= composes to $\begin{tikzcd} {} \arrow[r, shift left=1.5, "\varnothing"] \arrow[r, phantom, "\perp"] & {} \arrow[l, shift left=1.5, "1"] \end{tikzcd}$)

"extensions"?

torsion theories?

[BLLL23] Boccali, Laretto, Loregian, Luneia, "completeness for categories of generalized automata", 2023

[HJ98] Hermida, Jacobs, "Structural induction and coinduction in a fibrational setting", 1998

Still in progress,
suggestions are welcome!

Still in progress,
suggestions are welcome!

lots of examples

lots of results

too many directions
(= no direction)

do you have any
problems we can
throw this tech.
at?

everything seems very
algebraic, do you feel
it would be reasonable
to try to formalize this in HoTT?
what obstacles can we expect?

Thank you for listening.