Cubical models of $(\infty, 1)$-categories

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Conclusions

Theorem

The category cSet *of cubical sets with connections carries a model structure that presents the homotopy theory of* $(\infty, 1)$*-categories, which is equivalent to the Joyal model structure via triangulation.*

References

- Kapulkin, Lindsey, Wong, *A co-reflection of cubical sets into simplicial sets with applications to model structures*, New York Journal of Mathematics 25 (2019), 627–641.
- D., Kapulkin, Lindsey, Sattler, *Cubical models of* $(\infty, 1)$*-categories*, 2020. arXiv:2005.04853

Model categories

A **model structure** on a bicomplete category consists of:

- $\xrightarrow{\sim}$ **weak equivalences**;
- $\rightarrowtail$ **cofibrations**;
- $\twoheadrightarrow$ **fibrations**

such that:

- All classes closed under retracts;
- Weak equivalences satisfy 2-out-of-3;
- Every map admits factorizations :

$$\overset{\sim}{\rightarrowtail} \ \twoheadrightarrow \qquad , \qquad \rightarrowtail \ \overset{\sim}{\twoheadrightarrow}$$

- A lift exists in any diagram

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow\rlap{\scriptstyle\text{cof}} & & \downarrow\rlap{\scriptstyle\text{fib}} \\ \cdot & \longrightarrow & \cdot \end{array}$$

where either vertical map is a weak equivalence.

Model categories

Given a model category $\mathcal{M}$, we can define:

- **homotopy category** $\operatorname{Ho}\mathcal{M}$ (obtained by inverting $\xrightarrow{\sim}$);
- **cofibrant** objects (those with $\varnothing \rightarrowtail X$);
- **fibrant** objects (those with $X \twoheadrightarrow *$);
- **cofibrant** and **fibrant replacement** ($X^{\mathrm{Cof}} \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} Y^{\mathrm{Fib}}$);
- **homotopies** between morphisms ($f \sim g$).

This allows us to characterize the homotopy category of $\mathcal{M}$ as:

$$\operatorname{Ho}\mathcal{M} \simeq \mathcal{M}_{\text{Cof-Fib}} / \sim$$

Quillen functors

A **Quillen adjunction** between model categories $\mathcal{M}$ and $\mathcal{N}$ is an adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{N}$$

such that:

- L preserves $\rightarrowtail$ and $\overset{\sim}{\rightarrowtail}$; equivalently,
- R preserves $\twoheadrightarrow$ and $\overset{\sim}{\twoheadrightarrow}$.

This induces $\mathrm{Ho}\mathcal{M} \rightleftarrows \mathrm{Ho}\mathcal{N}$ (the **derived adjunction**).

$L \dashv R$ is a **Quillen equivalence** if the derived adjunction is an equivalence.

Simplicial sets

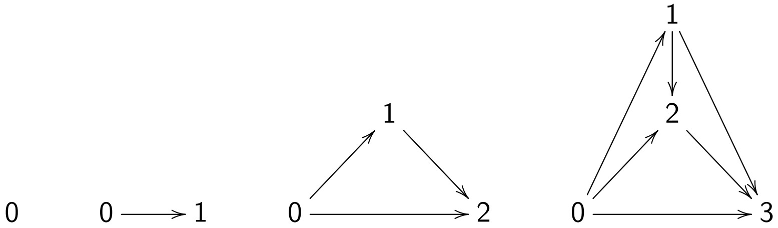
The **simplex category** Δ:

- objects are $[n] = \{0 \leq 1 \leq ... \leq n\}$;
- morphisms are order-preserving maps.

Simplicial sets are presheaves on Δ

$$\mathrm{sSet} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}),$$

and are pieced together from **standard simplices**:



Quillen model structure

The category sSet := $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ carries a model structure:

- cofibrations = monomorphisms;
- fibrant objects = Kan complexes (a.k.a. ∞-groupoids);
- weak equivalences = weak homotopy equivalences.

$$X \text{ Kan complex} \quad \Leftrightarrow \quad \begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array} \quad \text{for } 0 \leq k \leq n$$

Voevodsky's simplicial model of HoTT represents types as Kan complexes.

Quillen model structure

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- weak equivalences = weak homotopy equivalences.

A **homotopy** $H\colon f \sim g$ between $f, g\colon X \to Y$ is $H\colon \Delta^1 \times X \to Y$ restricting to $[f, g]$ at the endpoints.

A map $f\colon X \to Y$ of Kan complexes is a **homotopy equivalence** if there is $g\colon Y \to X$ with homotopies $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$.

A map $K \to L$ is a **weak homotopy equivalence** if $X^L \to X^K$ is a homotopy equivalence for each X Kan.

Joyal model structure

sSet carries another model structure:

- cofibrations = monomorphisms;
- fibrant objects = quasicategories (a.k.a. $(\infty, 1)$-categories);
- weak equivalences = weak categorical equivalences.

$$X \text{ quasicategory} \quad \Leftrightarrow \quad \begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array} \qquad \text{for } 0 < k < n$$

These are the **inner horns**, e.g.,

$$\begin{array}{ccc} & 1 & \\ \nearrow & & \searrow \\ 0 & & 2 \end{array}, \qquad \begin{array}{ccc} & 1 & \\ \nearrow & & \\ 0 & \longrightarrow & 2 \end{array}, \qquad \begin{array}{ccc} & 1 & \\ & & \searrow \\ 0 & \longrightarrow & 2 \end{array}$$

$$\Lambda^2_1 \qquad\qquad \Lambda^2_0 \qquad\qquad \Lambda^2_2$$

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Joyal model structure

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- cofibrations = monomorphisms;
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- weak equivalences = weak categorical equivalences.

The homotopies are given using

$$J = \begin{array}{ccc} \cdot & = & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & = & \cdot \end{array}$$

i.e. a homotopy of maps $X \to Y$ is $H \colon J \times X \to Y$.

A map $f \colon X \to Y$ of quasicategories is a **categorical equivalence** if there is $g \colon Y \to X$ with homotopies $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$.

A map $K \to L$ is a **weak categorical equivalence** if $X^L \to X^K$ is a categorical equivalence for each quasicategory X.

Cubical sets

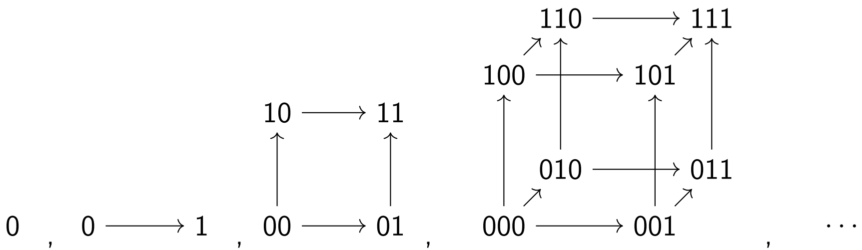
The **box category** $\square$:

- objects are $[1]^n = \{0 \leq 1\}^n$;
- morphisms are some subset of order-preserving maps.

Cubical sets are presheaves on $\square$

$$\mathsf{cSet} := \mathsf{Fun}(\square^{\mathrm{op}}, \mathsf{Set}),$$

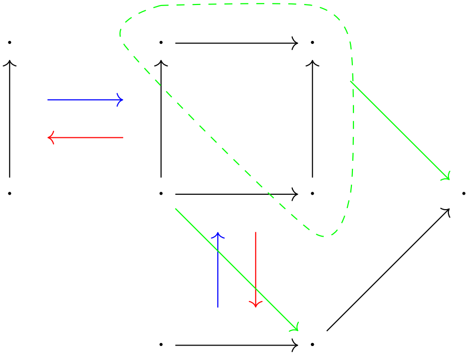
and are pieced together from **standard cubes**:



Cubical sets

In our work, maps in $\square$ are generated by:

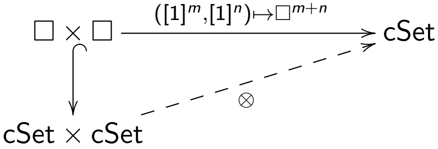
- **face** and **degeneracy** maps
- **connections** (max)



The geometric product

The Cartesian product of cubical sets is not well-behaved, e.g. $\square^1 \times \square^1 \neq \square^2$.

Instead we work with the **geometric product**.



Grothendieck model structure

cSet carries a model structure:

- cofibrations = monomorphisms;
- fibrant objects = cubical Kan complexes (a.k.a. ∞-groupoids);
- weak equivalences = weak homotopy equivalences.

$$X \text{ Kan complex} \quad \Leftrightarrow \quad \begin{array}{ccc} \sqcap^n_{k,\varepsilon} & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \square^n & & \end{array} \qquad \text{for } \varepsilon = 0, 1, \text{ and } 1 \leq k \leq n.$$

Grothendieck model structure

cSet carries a model structure:

- cofibrations = monomorphisms;
- fibrant objects = cubical Kan complexes (a.k.a. ∞-groupoids);
- weak equivalences = weak homotopy equivalences.

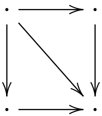
A **homotopy** of maps $X \to Y$ is $H\colon \square^1 \otimes X \to Y$.

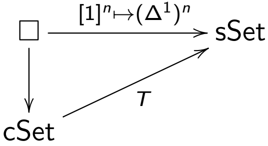
A map $f\colon X \to Y$ of cubical Kan complexes is a **homotopy equivalence** if there is $g\colon Y \to X$ with homotopies $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$.

A map $K \to L$ is a **weak homotopy equivalence** if $X^L \to X^K$ is a homotopy equivalence for each cubical Kan complex X.

Comparing cSet and sSet: Triangulation

We define $T\colon \mathrm{cSet} \to \mathrm{sSet}$ by Kan extension:





T has a right adjoint U given by $(UX)_n = \mathrm{sSet}((\Delta^1)^n, X)$.

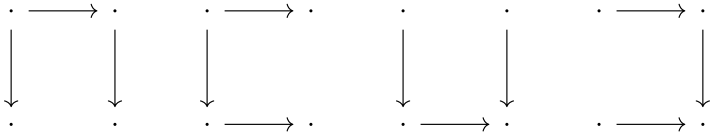
Theorem (Cisinski)

$T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on cSet *and the Quillen model structure on* sSet*.*

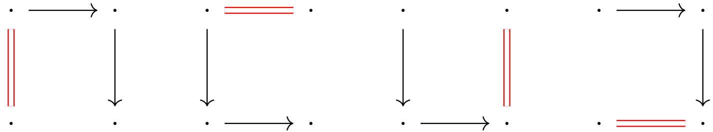
Inner open boxes

Goal: construct a cubical analogue of $\mathrm{sSet}_{\mathrm{Joyal}}$, Quillen-equivalent via triangulation.

What's an **inner** open box?



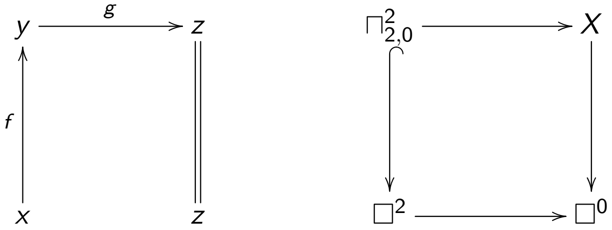
Solution: require critical edges to be degenerate!



Cubical quasicategories

A **cubical quasicategory** is $X \in \mathrm{cSet}$ having the RLP against inner open box fillings.

In particular, this lets us "compose" edges.



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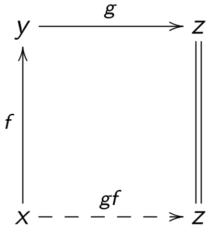
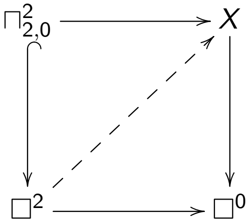




The cubical Joyal model structure

Theorem

cSet *carries a model structure:*

- *Cofibrations are monomorphisms;*
- *Fibrant objects are cubical quasicategories;*
- *Weak equivalences are weak categorical equivalences.*

The cubical Joyal model structure

Homotopies are given using

$$K \quad = \quad \begin{array}{ccccc} \cdot & \longrightarrow & \cdot & = & \cdot \\ \| & & \downarrow & & \| \\ \cdot & = & \cdot & \longrightarrow & \cdot \end{array}$$

i. e. a homotopy of maps $X \to Y$ is $H \colon K \otimes X \to Y$.

A map $f \colon X \to Y$ of cubical quasicategories is a **categorical equivalence** if there is $g \colon Y \to X$ with homotopies $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$.

A map $K \to L$ is a **weak categorical equivalence** if $X^L \to X^K$ is a categorical equivalence for each cubical quasicategory X.

Application: the fundamental theorem

Theorem (Fundamental Theorem of Category Theory)

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories $\Leftrightarrow$ it is full, faithful, and essentially surjective.

Theorem

A cubical map $f \colon X \to Y$ of cubical quasicategories is a categorical equivalence $\Leftrightarrow$

- *it induces a homotopy equivalence* $\mathrm{Map}(x, y) \to \mathrm{Map}(fx, fy)$*;*
- *it is essentially surjective on vertices.*

Here, the mapping space is defined by

$$\mathrm{Map}(x, y)_n = \left\{ \square^{n+1} \xrightarrow{\sigma} X \mid \sigma\partial_{n+1,0} = x \text{ and } \sigma\partial_{n+1,1} = y \right\}$$

This definition gives a more workable approach than $\mathrm{Hom}^{\mathrm{R}}$ and $\mathrm{Hom}^{\mathrm{L}}$ from "Higher Topos Theory".

Triangulation

Theorem
$T : \mathrm{cSet}_{\mathrm{Joyal}} \rightleftarrows \mathrm{sSet}_{\mathrm{Joyal}} : U$ *is a Quillen adjunction.*

It would be hard to show directly that $T \dashv U$ is a Quillen equivalence.

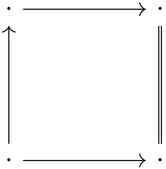
We show $Q : \mathrm{sSet} \rightleftarrows \mathrm{cSet} : \int$ is a Quillen equivalence, and that the derived functors of T and Q are inverses.

The functor $Q^\bullet\colon \Delta \to \mathrm{cSet}$

Define quotients of the standard cubes:

$Q^0 = \cdot \qquad\qquad Q^1 = \cdot \longrightarrow \cdot$

$Q^2 =$



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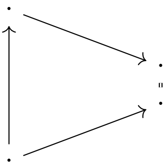
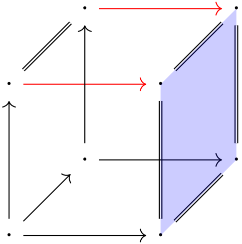


$Q^3 =$



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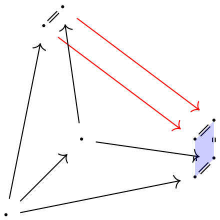
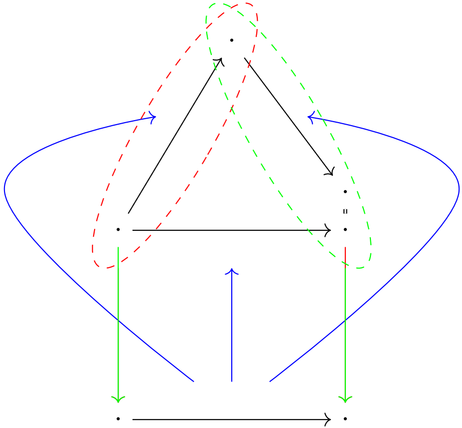


The functor $Q^\bullet \colon \Delta \to \mathrm{cSet}$

Crucially using connections, we have:

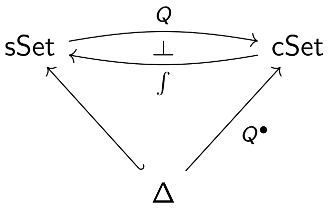


i.e., the Q^n's form a co-simplicial object!

The functor $Q^\bullet \colon \Delta \to \mathrm{cSet}$

$Q^\bullet \colon \Delta \to \mathrm{cSet}$ defines a functor sending $[n]$ to Q^n.

Using $Q^\bullet$, we obtain an adjunction:



The right adjoint $\int$ is given by $(\int X)_n = \mathrm{cSet}(Q^n, X)$.

The adjunction $Q \dashv \int$

Theorem (Kapulkin-Lindsey-Wong)

$Q \dashv \int$ *defines a co-reflective inclusion of* sSet *into* cSet.

$$\mathrm{sSet} \underset{\int}{\overset{Q}{\underset{\bot}{\rightleftarrows}}} \mathrm{cSet}$$

I.e.:

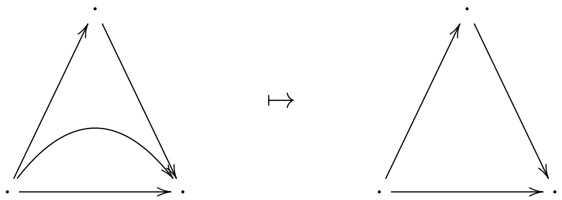
- The functor $Q\colon \mathrm{sSet} \to \mathrm{cSet}$ is fully faithful.
- For each $X \in \mathrm{sSet}$, the unit $\eta_X \colon X \to \int QX$ is an isomorphism.

For each $X \in \mathrm{cSet}$, the counit $Q\int X \to X$ is a monomorphism.
$Q\int X$ is the "maximal simplicial set contained in X".

$T \dashv U$ as a Quillen equivalence

We can show that $Q \dashv \int$ is a Quillen equivalence.

We have a natural weak categorical equivalence $TQ \implies \mathrm{id}$:



This gives a natural isomorphism of the derived functors, showing that $T \dashv U$ is a Quillen equivalence as well.

Other results

- Two more models of $(\infty, 1)$-categories: **marked cubical sets** and **structurally marked cubical sets**.
- Theory of **cones in cubical sets**.
- New proof that $T \dashv U$ is a Quillen equivalence between the Quillen and Grothendieck model structures.
- New proof of the fundamental theorem for quasicategories.