Towards Higher Universal Algebra in Type Theory

HoTT Electronic Seminar Talks

Eric Finster

December 6, 2018

Voevodsky's Vision for Univalent Mathematics

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h-level 0 The Mathematics of Cantor

- Sets and structured sets

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Problem

How can we describe structures on homotopy types without recourse to a "strict" equality?

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  - (∞, n)-categories, ∞-double categories, ...
  - Even if these can be reduced to simplicial methods, will this be an efficient way to describe them?
  - Can we describe a natural class of higher structures *directly*?

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- Special cases of this definition are
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  2. Type-valued diagrams on $(\infty, 1)$-categories
  3. Corollary: simplicial types are definable in MLTT with coinduction.

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- The definition of algebra relies on a construction which is not yet completely formalized (though it is sketched ...)
- Hence the complete definition of simplicial type is not yet finished.
- The "on paper" definition of algebra, however, is completely transparent. I do not expect any difficulties in finishing it other than the fact that it is somewhat long.

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- For $i : I$, an element $f : \mathrm{Op}\, i$ represents an operation whose *output* sort is i.
- For $f : \mathrm{Op}\, i$ and $j : I$, an element $p : \mathrm{Param}\, f\, j$ represents an *input* parameter of sort j.

Representations of Operations

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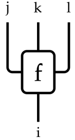

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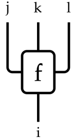

- However, we specifically allow for higher homotopy both in the operations and the parameters

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A polynomial P : Poly I generates an associated type of *trees*.

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Definition

The inductive family $\mathrm{Tr}\ P : I \to \mathit{Type}$ has constructors:

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\begin{aligned}
\mathrm{lf} &: (i : I) \to \mathrm{Tr}\ P\ i \\
\mathrm{nd} &: \{i : I\} \to (f : \mathrm{Op}\, P i) \\
&\to (\phi : (j : J)(p : \mathrm{Param}\, f\, j) \to \mathrm{Tr}\, P\, j) \\
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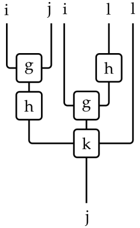

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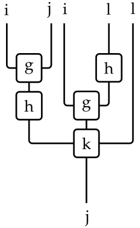
We can represent trees both *geometrically* and *algebraically*



$$k(h(g(i, j), g(i, h(l)), l) : j$$

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Let $P : \mathrm{Poly}\, I$ be a polynomial $w : \mathrm{Tr}\, P\, i$ a tree and $f : \mathrm{Op}\, P\, i$ an operation. A *frame* from w to f is a family of equivalences

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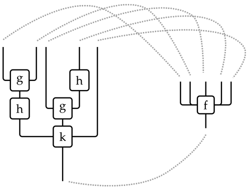

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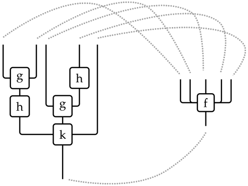

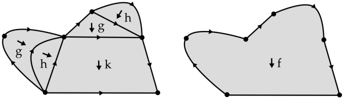

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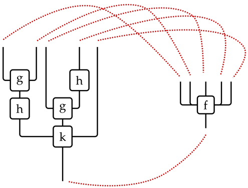

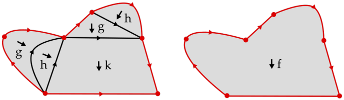

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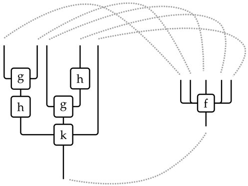

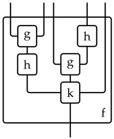

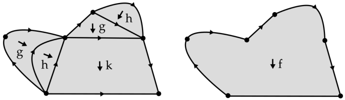

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Let P : Poly I be a polynomial w : Tr $P\ i$ a tree and f : Op $P\ i$ an operation. A *frame* from w to f is a family of equivalences

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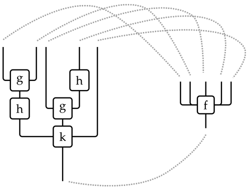

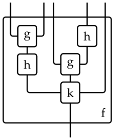

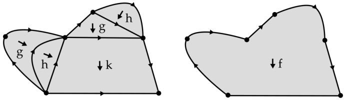

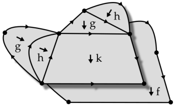

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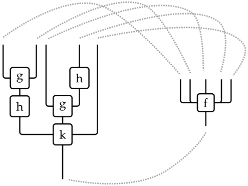

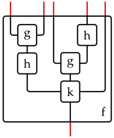

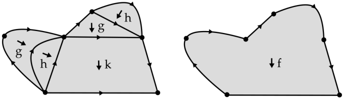

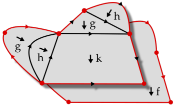

Polynomial Relations

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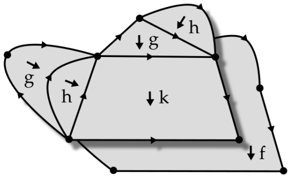

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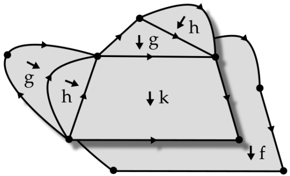

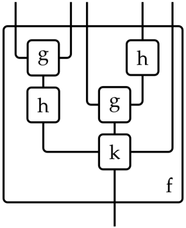

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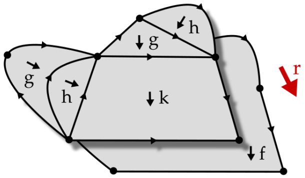

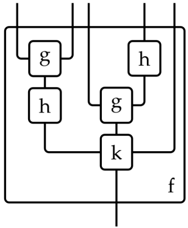

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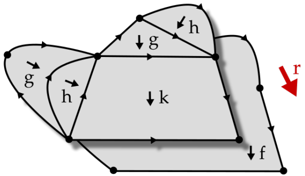

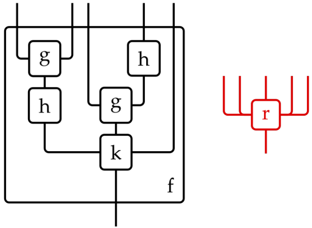

The Slice of a Polynomial by a Relation

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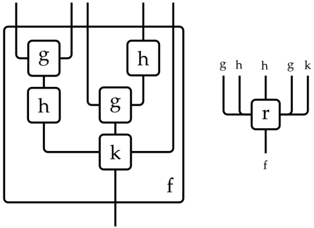
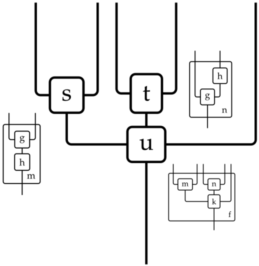


Trees in the Slice Polynomial



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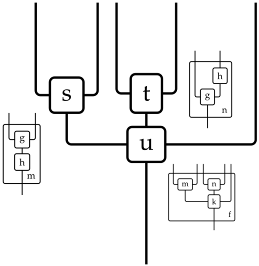

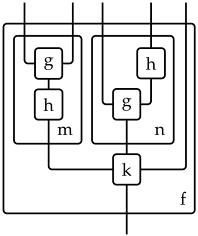

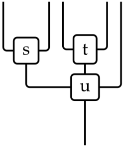

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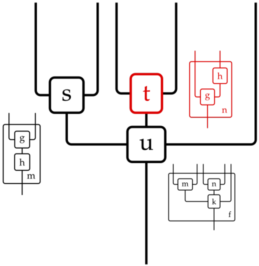

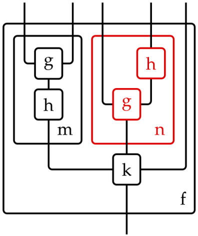

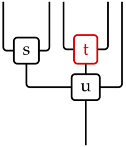

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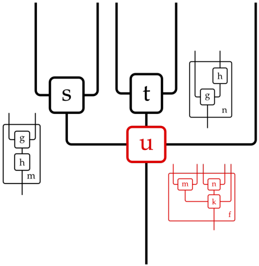

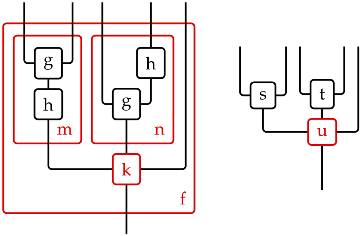

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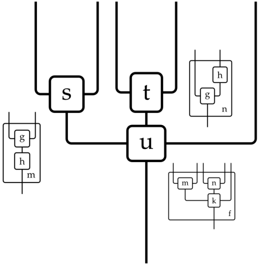

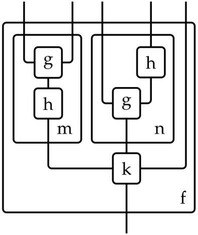

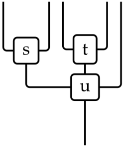

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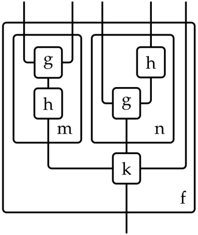

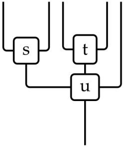

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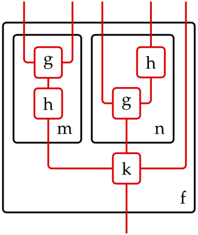

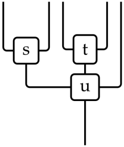

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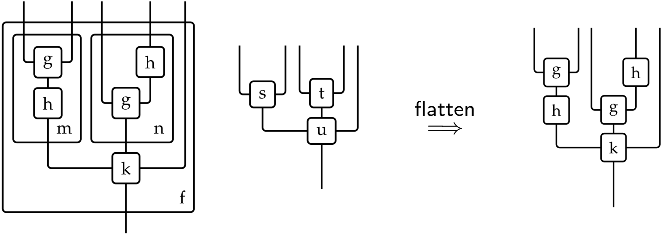

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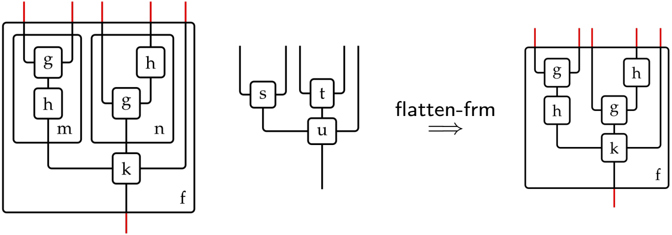

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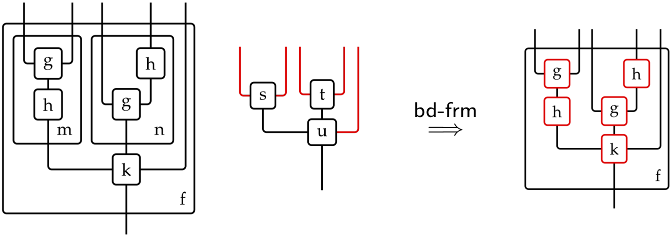

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Polynomial Magmas

Polynomials serve as our notion of higher signature. Following ideas from the categorical approach to universal algebra, we are going to encode the *relations* or *axioms* of our structure using a *monadic multiplication* on P.

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1. A function $\mu : \{i : I\} \to \mathsf{Tr}\, P\, i \to \mathsf{Op}\, P\, i$
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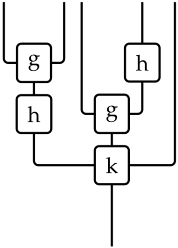
Notice that a magma M determines a polynomial relation on P by using the identity type:

$$\mathrm{MgmRel} : \mathrm{PolyMagma}\, P \to \mathrm{PolyRel}\, P$$
$$\mathrm{MgmRel}\, M\, f\, w\, \alpha := (\mu\, w, \mu_{frm}\, w) = (f, \alpha)$$

Polynomial Magmas (cont'd)

Using the graphical notation we have developed, we can "picture" the multiplication μ as follows:



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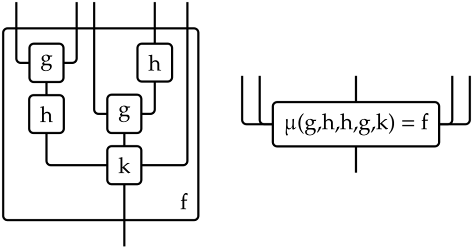

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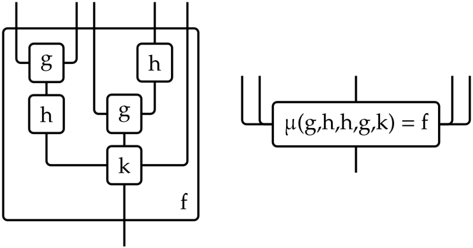
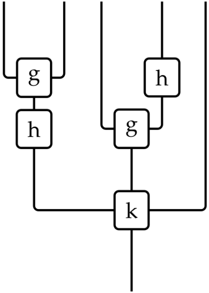


In algebraic notation, this corresponds to the relation

$$k(h(g(x,y)), g(u,h(v)), w) = f(x,y,u,v,w)$$

Coherent Relations

Furthermore, we can now interpret a pasting diagram $pd : \mathrm{Tr}(P//M)\,(i, f)$ as a sequence of multiplications applied to subterms of flatten pd:



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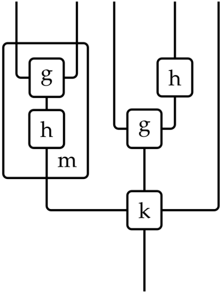

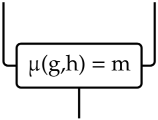

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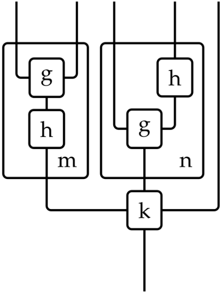

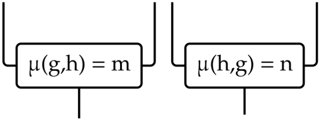

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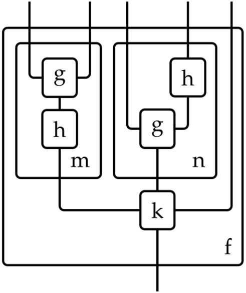

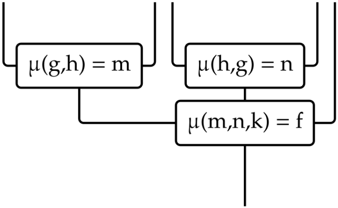

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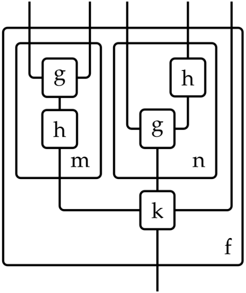

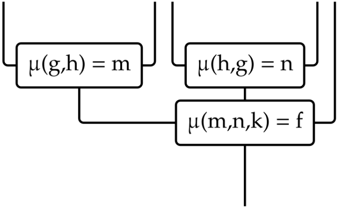

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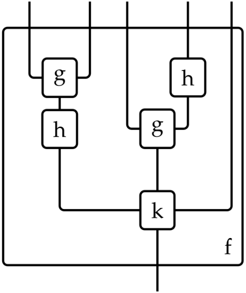

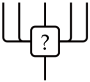

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Subdivision Invariance

Definition

Let P be a polynomial and R a relation on P. We say that R is *subdivision invariant* if we are given a function.

$$\Psi : \{i : I\}\{f : \mathrm{Op}\, P\, i\}(pd : \mathrm{Tr}(P//R)\,(i, f)) \\ \to R\, f\,(\mathrm{flatten}\, pd)\,(\text{flatten-frm}\, pd)$$

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We write SubInvar for the associated predicate on polynomial relations.

$$\begin{aligned}\mathrm{SubInvar} &: \mathrm{PolyRel}\, P \to \mathit{Type} \\ \mathrm{SubInvar}\ R &:= \{i : I\}\{f : \mathrm{Op}\, P\, i\}(pd : \mathrm{Tr}(P//R)\,(i,f)) \\ &\quad \to R\, f\,(\mathrm{flatten}\, pd)\,(\text{flatten-frm}\, pd)\end{aligned}$$

The Slice Magma

Observation

Let P be a polynomial and R a relation on P. Given a witness Ψ that R is subdivision invariant, the slice polynomial $P//R$ admits a magma structure given by

$$\mu(\mathsf{SlcMgm}\, R)\, pd := ((\mathsf{flatten}\, pd, \mathsf{flatten\text{-}frm}\, pd), \Psi\, pd)$$
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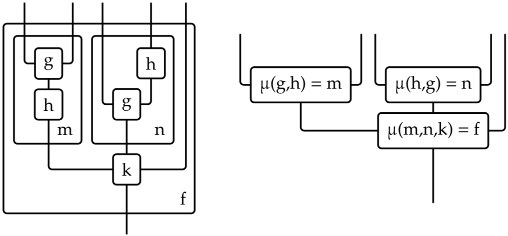

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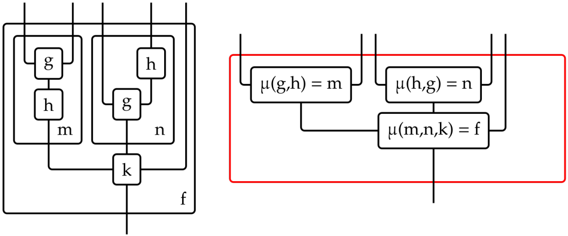

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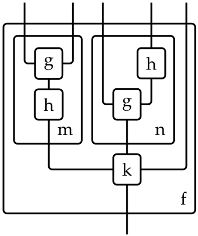

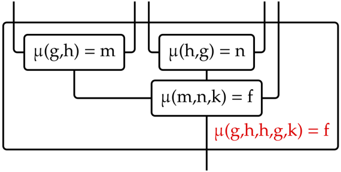

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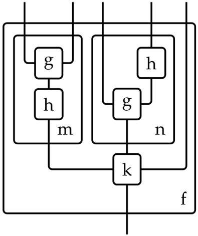

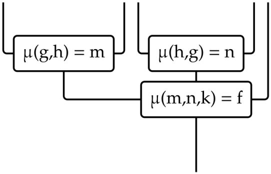

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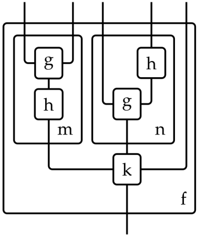

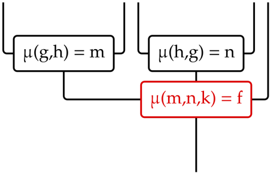

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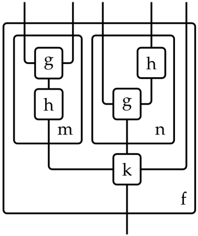

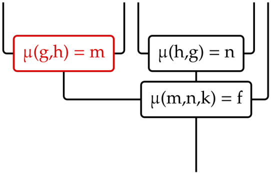

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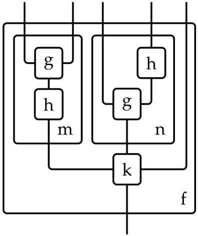

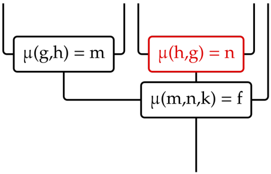

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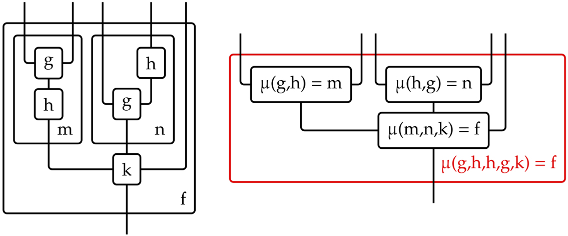

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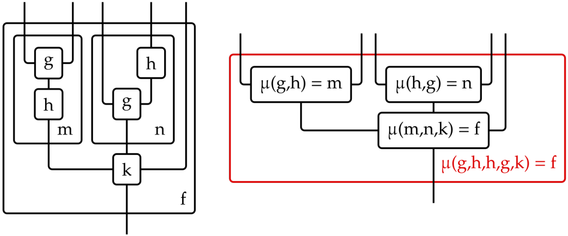

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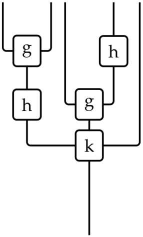

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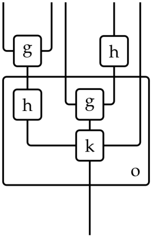

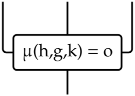

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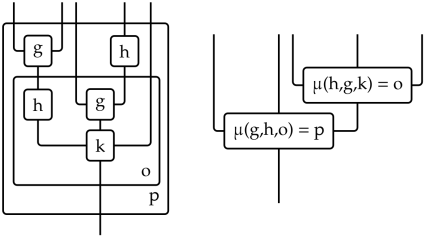

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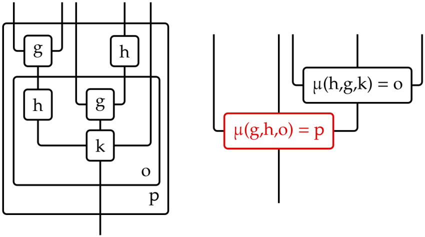

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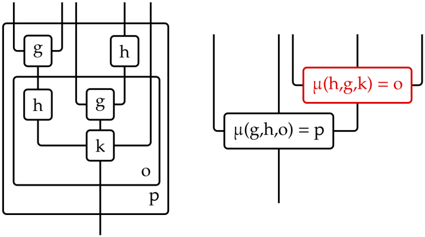

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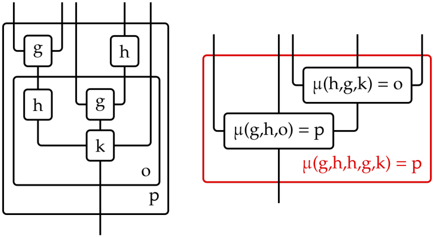

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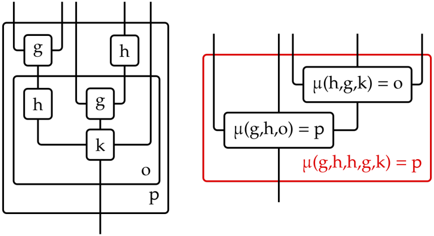

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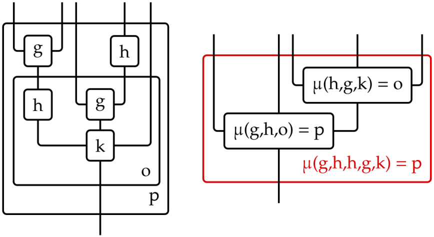

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- The multiplication μ can now be used to define a composition operation

$$\_ \circ \_ : \mathsf{UnaryOp} \times \mathsf{UnaryOp} \to \mathsf{UnaryOp}$$

Univalence for Monads (cont'd)

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Let M be a polynomial monad. A unary operation $f : j \to i$ is said to be an *isomorphism* if satisfies the bi-inverse property:

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M is said to be *univalent* if the above map is an equivalence.

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