Characterizing clan-algebraic categories

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Overview

Context

- In talks at HoTT/UF 2020 and at CT 2021 I presented a conjecture concerning categories of models of a *clan*.
- In this talk I will give/outline a proof of this conjecture.

Three Parts

- Recall functorial semantics of (essentially) algebraic theories
- Clans as generalized algebraic theories
- If there’s time: Examples and models in higher (homotopy) types
Part I
**Algebraic Theories**

**Definition**

A **single-sorted algebraic theory** (SSAT) is a pair \((\Sigma, E)\) consisting of

- a family \(\Sigma = (\Sigma_n)_{n \in \mathbb{N}}\), of sets of \(n\)-ary operations
- a set of **equations** \(E\) whose elements are pairs of open terms over \(\Sigma\)

**Definition**

The **syntactic category** \(C(\Sigma, E)\) of a SSAT is given as follows:

1. For each natural number \(n \in \mathbb{N}\) there is an **object** \([n]\)
2. morphisms \(\sigma : [n] \to [m]\) are \(m\)-tuples of terms in \(n\) variables modulo \(E\)-provable equality
3. **identities** are lists of variables, **composition** is given by substitution

**Proposition**

Given a SSAT \((\Sigma, E)\):

1. \(C(\Sigma, E)\) has finite products given by \([n] \times [m] = [n + m]\)
2. \(\text{Set-Mod}(\Sigma, E) \simeq \text{FP}(C(\Sigma, E), \text{Set})\)
Finite-product theories

Definition

- A **FP-theory** is just a small FP-category $\mathcal{C}$.
- **Models** of $\mathcal{C}$ are FP-functors $A : \mathcal{C} \to \text{Set}$ (or into another FP-category).

Denote the category of models by

$$\text{Mod}(\mathcal{C}) := \text{FP}(\mathcal{C}, \text{Set}) \subseteq [\mathcal{C}, \text{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an FP-theory, the co-representable functor

$$\mathcal{C}(\Gamma, -) : \mathcal{C} \to \text{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to $\text{Mod}(\mathcal{C})$. 
**Finite-limit theories**

**Definition**
- A **FL-theory** is a small finite-limit category \( \mathcal{L} \).
- A **model** of \( \mathcal{L} \) is a finite-limit preserving functor \( A : \mathcal{L} \rightarrow \text{Set} \).

FL-theories are more expressive than FP-theories – structures definable by finite-limit theories include categories, posets, 2-categories, monoidal categories, categories with families . . .

Again \( \mathcal{L}(\Gamma, -) \) is a model for every \( \Gamma \in \mathcal{L} \) and we get an embedding

\[
Z : \mathcal{L}^{\text{op}} \rightarrow \text{Mod}(\mathcal{L}) := \text{FL}(\mathcal{L}, \text{Set})^{\text{full}} \subseteq [\mathcal{L}, \text{Set}].
\]

Moreover, we can characterize the essential image of \( Z \) in \( \text{Mod}(\mathcal{L}) \).
Locally finitely presentable categories

**Definition**

- An object $C$ of a cocomplete locally small category $\mathcal{X}$ is called **compact**\(^a\), if

  $$\mathcal{X}(C, -) : \mathcal{X} \to \text{Set}$$

  preserves filtered colimits.

- A category $\mathcal{X}$ is called **locally finitely presentable**, if
  - $\mathcal{X}$ is locally small and cocomplete
  - the full subcategory $\text{comp}(\mathcal{X}) \subseteq \mathcal{X}$ on compact objects is essentially small and dense.

\(^a\)More traditionally: ‘finitely presentable’

**Theorem**

- $\text{Mod}(\mathcal{L})$ is locally finitely presentable for all finite-limit theories $\mathcal{L}$.
- The essential image of $Z : \mathcal{L}^{\text{op}} \to \text{Mod}(\mathcal{L})$ comprises precisely the compact objects.
Gabriel-Ulmer duality

**Theorem**

There is a bi-equivalence of 2-categories

$$FL \xleftarrow{\text{comp}(\mathcal{X})^\text{op}} \mathcal{X} \xrightarrow{\mathcal{L} \mapsto \text{Mod}(\mathcal{L})} LFP^\text{op}$$

where

- **FL** is the 2-category of small FL-categories and FL-functors
- **LFP** is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits (‘forgetful functors’).

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Duality for finite-product theories

There’s a ‘restriction’ of G–U duality to finite-product theories:

\[ \begin{align*}
\mathbf{FP}_{cc} & \leftrightarrow \mathbf{FP}(\mathcal{C}, \text{Set}) \\
\mathbf{FL} & \leftrightarrow \mathbf{FL}(\mathcal{L}, \text{Set})
\end{align*} \]

- \( \mathbf{FP}_{cc} \) is the 2-category of Cauchy-complete finite-product categories
- \( \mathbf{ALG} \) is the 2-category of algebraic categories and algebraic functors
  - An algebraic category is an l.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An algebraic functor is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There’s also a formulation in terms of sifted colimits, but we don’t need it.

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Part II
Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd’s essentially algebraic theories\(^3\)
  - Cartmell’s generalized algebraic theories\(^4\) (or ‘dependent algebraic theories’)
  - Johnstone’s cartesian theories\(^5\)
  - Palmgren and Vickers’ quasi-equational theories\(^6\)
  - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They’re as expressive as FL-theories, but ‘finer’, i.e. closer to syntax

**Definition**

A **clan** is a small category $\mathcal{T}$ with terminal object $1$, equipped with a class $\mathcal{T}^+ \subseteq \text{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\rightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps

\[
\begin{array}{ccc}
\Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\
q \downarrow & \searrow & \downarrow p \\
\Delta & \xrightarrow{s} & \Gamma
\end{array}
\]

2. display maps are closed under composition, and

3. isomorphisms and terminal projections $\Gamma \rightarrow 1$ are display maps.

- Definition due to Taylor\(^7\), name due to Joyal\(^8\) (‘a clan is a collection of families’)
- Relation to semantics of dependent type theory: display maps represent **type families**.
- Observation: clans have finite products (as pullbacks over 1).

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Examples

- Finite-product categories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}^\dagger = \{\text{product projections}\}$
- Finite-limit categories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}^\dagger = \text{mor}(\mathcal{L})$

We call such clans **FP-clans**, and **FL-clans**, respectively.

- The syntactic category of every Cartmell-style **generalized algebraic theory** is a clan.
- Clan for categories:

  $$\mathcal{K} = \{\text{categories free on finite graphs}\}^{\text{op}} \subseteq \text{Cat}^{\text{op}}$$

  $$\mathcal{K}^\dagger = \{\text{functors induced by graph inclusions}\}^{\text{op}}$$

$\mathcal{K}$ can be viewed as syntactic category of a generalized algebraic theory of categories with a sort $O$ of objects, and a dependent sort $x, y: O \vdash M(x, y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.
Models

Definition
A model of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \text{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\text{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \text{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\text{op}}$.
- For FP-clans $(\mathcal{C}, \mathcal{C}^\dagger)$ we have $\text{Mod}(\mathcal{C}, \mathcal{C}^\dagger) = \text{FP}(\mathcal{C}, \text{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}^\dagger)$ we have $\text{Mod}(\mathcal{L}, \mathcal{L}^\dagger) = \text{FL}(\mathcal{L}, \text{Set})$.
- $\text{Mod}(\mathcal{K}, \mathcal{K}^\dagger) = \text{Cat}$.

Observation
The same category of models may be represented by different clans. For example, SSATs can be represented by FP-clans as well as FL-clans.
The weak factorization system

- Would like duality between clans and their categories of models.
- Since the same l.f.p. category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

**Definition**

Let $\mathcal{T}$ be a clan. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\text{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \text{RLP}(\{Z(p) \mid p \in \mathcal{T}_\uparrow\})$ class of full maps
- $\mathcal{E} := \text{LLP}(\mathcal{F})$ class of extensions

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of $\mathcal{T}_\uparrow$ under $Z : \mathcal{T}^{\text{op}} \to \text{Mod}(\mathcal{T})$.

- Call $A \in \text{Mod}(\mathcal{T})$ a 0-extension, if $(0 \to A) \in \mathcal{E}$

- E.g. corepresentables $Z(\Gamma)$ are 0-extensions since terminal projections $\Gamma \to 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry in a HoTTEST talk\textsuperscript{9}, see also\textsuperscript{10}.

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\textsuperscript{9}S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, [https://youtu.be/7_X0qbSX1fk](https://youtu.be/7_X0qbSX1fk)

**Full maps**

- \( f : A \to B \) in \( \mathbf{Mod}(\mathcal{T}) \) is full iff it has the RLP with respect to all \( Z(p) \) for display maps \( p : \Delta \to \Gamma \).

\[
\begin{align*}
\mathcal{T}(\Gamma, -) & \quad \xrightarrow{\mathcal{T}(p,-)} \quad A \\
\downarrow & \quad \downarrow^f \\
\mathcal{T}(\Delta, -) & \quad \xrightarrow{\mathcal{T}(p,-)} \quad B
\end{align*}
\]

\[
\begin{align*}
A(\Delta) & \xrightarrow{f_\Delta} B(\Delta) \\
A(\Gamma) & \xrightarrow{f_\Gamma} B(\Gamma)
\end{align*}
\]

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering \( p : \Delta \to 1 \) we see that full maps are surjective and hence regular epis.

\[
\begin{align*}
A(\Delta) & \xrightarrow{f_\Delta} B(\Delta) \\
A(\Delta) & \xrightarrow{f_\Delta} B(\Delta)
\end{align*}
\]

\[
\begin{align*}
\downarrow & \quad \downarrow \\
1 & \quad 1
\end{align*}
\]

\[
\begin{align*}
A(\Delta) & \times A(\Delta) \xrightarrow{f_\Delta \times f_\Delta} B(\Delta) \times B(\Delta)
\end{align*}
\]

- For FL-clans, only isos are full (consider naturality square for diagonal \( \Delta \to \Delta \times \Delta \)).
- For FP-clans we have

\[
\begin{align*}
\text{full map} & = \text{regular epimorphism} \\
\text{extension} & = \text{coproduct inclusion } A \hookrightarrow P + A \text{ with } P \text{ projective} \\
\text{0-extension} & = \text{projective object}
\end{align*}
\]
The fat small object argument

Motivation: subcategories of models for FP-theory $\mathcal{C}$ and clan $\mathcal{T}$.

- Flat algebras are filtered colimits of corepresentables, computed \textit{freely} in the functor categories.
- For SSATs we have $\{\text{projective}\} \subseteq \{\text{flat}\}$ since
  - arbitrary free objects are filtered colimits of free objects over finite sets
  - projective objects are retracts of free objects
- In the general clan case, $\{0\text{-extension}\} \subseteq \{\text{flat}\}$ by the \textbf{fat small object argument}\textsuperscript{11}.

\textsuperscript{11}M. Makkai, J. Rosicky, and L. Vokrinek. “On a fat small object argument”. In: \textit{Advances in Mathematics} (2014).
Reconstructing the clan

Definition
Given a clan $\mathcal{T}$, let $\mathcal{C} \subseteq \text{Mod}(\mathcal{T})$ be the full subcategory on compact 0-extensions.

- $Z : \mathcal{T}^{\text{op}} \to \text{Mod}(\mathcal{T})$ factors through $\mathcal{C}$ since corepresentables $Z(\Gamma)$ are compact and 0-extensions.

\[
\begin{array}{ccc}
\mathcal{T}^{\text{op}} & \xrightarrow{Z} & \text{Mod}(\mathcal{T}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{E} & \\
\end{array}
\]

- $0 \in \mathcal{C}$ and if $\begin{array}{ccc}
C & \rightarrow & D \\
\downarrow e & & \downarrow f \\
E & \rightarrow & F
\end{array}$ is a pushout with $F \in \mathcal{C}$ and $e \in \mathcal{E}$ then $F \in \mathcal{C}$.

- Therefore $\mathcal{C}$ is a coclan with extensions as "co-display maps".
Reconstructing the clan

**Theorem**

The full inclusion $E : \mathcal{T}^{\text{op}} \hookrightarrow \mathcal{C}$ exhibits $\mathcal{C}$ as *Cauchy-completion* of $\mathcal{T}^{\text{op}}$, i.e. every compact 0-extension is a retract of a corepresentable.

**Proof.**

- Let $C \in \mathcal{C}$.
- Since 0-extensions are flat, $\int C$ is filtered, thus $C$ is a filtered colimit of corepresentables.
- Since $C$ is compact, $\text{id}_C$ factors through a colimit inclusion map.

![Diagram](https://via.placeholder.com/150)
Clan-algebraic categories

**Definition**

A **clan-algebraic category** is a category $\mathcal{X}$ with a w.f.s. $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\Clan_{cc} \leftrightarrow \begin{array}{c}
\text{comp}(\mathcal{X})^{op} \leftrightarrow \mathcal{X} \\
\mathcal{T} \leftrightarrow \text{Mod}(\mathcal{T})
\end{array} \rightarrow \Alg^{op}$$

between

- the 2-category $\Clan_{cc}$ of Cauchy-complete clans and functors preserving $1$, display maps, and pullbacks of display maps, and
- the 2-category $\Alg$ of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?
Characterizing clan-algebraic categories

Assume $\mathfrak{X}$ is clan-algebraic with w.f.s. $(\mathcal{E}, \mathcal{F})$. Then

1. $\mathfrak{X}$ is cocomplete,
2. $\mathfrak{X}$ has a small dense family of compact 0-extensions, and
3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions.

Now assume we have a category $\mathfrak{X}$ with w.f.s. $(\mathcal{E}, \mathcal{F})$ satisfying 1–3.

Then the subcategory $\mathbb{C} \subseteq \mathfrak{X}$ of compact 0-extensions is a coclan.

We get a nerve/realization adjunction

$$
\begin{array}{ccc}
\mathbb{C} & \overset{J}{\longrightarrow} & \mathfrak{X} \\
\downarrow{N} & \searrow{L} & \\
\text{Mod}(\mathbb{C}^{\text{op}}) & \underset{\text{colim}}{\longleftarrow} & \mathfrak{X}
\end{array}
$$

$$
L(A) = \text{colim}(\int A \to \mathbb{C} \overset{J}{\to} \mathfrak{X})
$$

$$
N(X) = \mathfrak{X}(J(-), X)
$$

However, this adjunction is not an equivalence in general:
Characterizing clan-algebraic categories

Counterexample

Consider

- $\mathcal{X} \subseteq [2^{\text{op}}, \text{Set}]$ full subcategory on injections
- $(\mathcal{E}, \mathcal{F})$ w.f.s. on $\mathcal{X}$ cofib. generated by $\{(0 \to Y_0), (0 \to Y_1)\}$

Then $\text{Mod}(\{\text{compact 0-extensions}\}^{\text{op}}) \simeq [2^{\text{op}}, \text{Set}]$ and $N$ is the subcategory inclusion.

Conclusion: We’re missing an ‘exactness condition’ analogous to ‘Barr-exactness’ in the characterization of algebraic categories!
Recall that a FL-category $\mathcal{L}$ is called Barr-exact, if all equivalence relations in $\mathcal{L}$ have stable effective quotients.

This can’t be the case for clan algebraic categories in general. However, we have:

**Lemma**

For any clan $\mathcal{T}$, $\text{Mod}(\mathcal{T})$ has full and effective quotients of componentwise-full equivalence relations.

**Proof.**

Given equivalence relation $r : R \leftrightarrow A \times A$ with $r_0, r_1 : R \rightarrow A$ full, show that component-wise quotient is a model again.
Characterizing clan-algebraic categories

**Definition**

An **adequate category** is a category $\mathcal{X}$ with a w.f.s. $(\mathcal{E}, \mathcal{F})$ (whose maps we call extensions and full, respectively), s.th.

1. $\mathcal{X}$ is cocomplete,
2. $\mathcal{X}$ has a small dense family of compact 0-extensions (in particular $\mathcal{X}$ is l.f.p.),
3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions, and
4. $\mathcal{X}$ has full and effective quotients of componentwise-full equivalence relations.

**Lemma**

Assume $\mathcal{X}$ is adequate and $F : \mathcal{X} \to \text{Set}$ preserves finite limits and sends full maps to surjections. Then $F$ preserves quotients of componentwise-full equivalence relations.

**Proof.**

Let $\begin{array}{c} R \xrightarrow{r_0} A \xrightarrow{f} B \end{array}$ be a **full exact sequence** in $\mathcal{X}$, i.e. all arrows are full, $f$ is the coequalizer of $r_0$, $r_1$, and $r_0, r_1$ is the kernel pair of $f$. Then $Ff$ is a surjection with kernel pair $Fr_0, Fr_1$. But surjections are always coequalizers of their kernel pair.
Idea of proof

- Assume that $\mathcal{X}$ is adequate.
- To show that it is clan-algebraic, we want to show that its nerve/realization adjunction

$$
\text{Mod}(\mathcal{C}^{\text{op}}) \xrightarrow{N} \mathcal{X} \xleftarrow{L} \text{colim}(\int A \to \mathcal{C} \xrightarrow{J} \mathcal{X}) \quad \text{for all } A \in \text{Mod}(\mathcal{C}^{\text{op}}) \text{ and } C \in \mathcal{C}.
$$

is an equivalence.

- By density the right adjoint $N$ is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$
A(C) \xrightarrow{\cong} \mathcal{X}(C, \text{colim}(\int A \to \mathcal{C} \xrightarrow{J} \mathcal{X})).
$$

for all $A \in \text{Mod}(\mathcal{C}^{\text{op}})$ and $C \in \mathcal{C}$.

- We know that $\mathcal{X}(C, -)$ preserves filtered colimits and quotients of componentwise-full equivalence relations, so we’d like to decompose $\text{colim}(\int A \to \mathcal{C} \xrightarrow{J} \mathcal{X})$ in terms of these constructions.
- This is essentially what we’re doing in the following.
Jointly full cones

- Let $D : \mathcal{I} \to \mathcal{X}$ be a diagram in an adequate category.
- A cone $(A, \phi)$ over $D$ is called **jointly full**, if for every cone $(C, \gamma)$, extension $e : B \to C$ and map $g : B \to A$ constituting a cone morphism $g : (B, \gamma \circ e) \to (A, \phi)$, there exists a map $h : C \to A$ such that

\[
\begin{array}{c}
B & \xrightarrow{g} & A \\
\downarrow{e} & & \downarrow{\phi_i} \\
C & \xrightarrow{\gamma_i} & D_i
\end{array}
\]

commutes for all $i \in \mathcal{I}$.
- **Observation:** The cone $(A, \phi)$ is jointly full iff the canonical map to the limit is full.
Definition

A nice diagram in an adequate category $\mathcal{X}$ is a truncated simplicial diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{d_0} & A_1 & \xleftarrow{s_0} & A_0 \\
\downarrow d_1 & & \downarrow s_1 & & \\
A_1 & \xrightarrow{d_1} & A_0
\end{array}
\]

where

1. $A_0$, $A_1$, and $A_2$ are 0-extensions,
2. the maps $d_0, d_1 : A_1 \to A_0$ are full,
3. in the square the span constitutes a jointly full diagram over the cospan,

\[
\begin{array}{ccc}
A_2 & \xrightarrow{d_0} & A_1 \\
\downarrow d_2 & & \downarrow d_1 \\
A_1 & \xrightarrow{d_0} & A_0
\end{array}
\]

4. there exists a symmetry map making the triangles commute, and

\[
\begin{array}{ccc}
A_1 & \xrightarrow{d_1} & A_0 \\
\downarrow d_0 & & \uparrow d_0 \\
A_0 & \xleftarrow{d_1} & A_1
\end{array}
\]

5. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \to A_1$ constituting a jointly full cone over the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{d_1} & A_1 \\
\downarrow d_0 & & \downarrow d_1 \\
A_0 & \xleftarrow{d_0} & A_0
\end{array}
\]
Nice diagrams

Lemma

For any nice diagram, the pairing $A_1 \langle d_0, d_1 \rangle \rightarrow A_0 \times A_0$ admits a decomposition $A_1 \rightarrow R \langle r_0, r_1 \rangle \rightarrow A_0 \times A_0$ into a full map and a monomorphism, and $\langle r_0, r_1 \rangle$ is a componentwise-full equivalence relation.

Lemma

Assume $\mathcal{X}$ is adequate and $F : \mathcal{X} \rightarrow \text{Set}$ preserves finite limits and sends full maps to surjections. Then for every nice diagram, $F$ preserves coequalizers of the arrows $d_0, d_1 : A_1 \rightarrow A_0$.

Lemma

The restriction $L'$ of $L$ in the nerve.realization adjunction $C_{\text{Mod}(\mathcal{C}^{\text{op}})} \downarrow \uparrow {0\text{-ext}}$ to $0$-extensions is fully faithful and preserves full maps and nice diagrams.
Nice diagrams

**Lemma**

For every object $A$ of an adequate category $\mathcal{X}$ there exists a nice diagram

![Diagram](image)

such that $A$ is the coequalizer of $d_0, d_1 : A_1 \to A_0$.

**Proof.**

- $A_0$ is given by covering $A$ by a 0-extension, i.e. factoring $0 \to A$ as $0 \hookrightarrow A_0 \twoheadrightarrow A$.
- $A_1$ is given by covering the kernel of $A_0 \to A$ by a 0-extension.
- $A_2$ is given by covering the following pullback:
The theorem

**Theorem**

**Adequate categories are clan-algebraic.**

**Proof.**

Let \( \mathcal{X} \) be adequate and let \( C \subseteq \mathcal{X} \) be the co-clan of compact 0-extensions. It remains to show that

\[
AC \cong \mathcal{X}(C, LA).
\]

for all \( A \in \text{Mod}(\mathcal{C}^{\text{op}}) \) and \( C \in \mathcal{C} \). Let \( A \) be a nice diagram with coequalizer \( A \). We have

\[
\begin{align*}
\mathcal{X}(C, LA) &= \mathcal{X}(C, L(\text{coeq}(A_1 \Rightarrow A_0))) \\
&\cong \mathcal{X}(C, \text{coeq}(LA_1 \Rightarrow LA_0)) \\
&\cong \text{coeq}(\mathcal{X}(C, LA_1) \Rightarrow \mathcal{X}(C, LA_0)) \\
&\cong \text{coeq}(A_1C \Rightarrow A_0C) \\
&\cong \text{coeq}(\text{Mod}(ZC, A_1) \Rightarrow \text{Mod}(ZC, A_0)) \\
&\cong \text{Mod}(ZC, \text{coeq}(A_1 \Rightarrow A_0)) \\
&\cong \text{Mod}(ZC, A) \\
&\cong AC
\end{align*}
\]

since \( A = \text{coeq}(A_1 \Rightarrow A_0) \)

since \( L \) preserves colimits

since \( \mathcal{X}(C, -) \) preserves coeqs of nice diags

since \( LA_i = \text{colim}(\text{f} A_i \rightarrow C \rightarrow \mathcal{X}) \) filtered
Part III
Models in higher types

Let $\mathcal{S}$ be the $\infty$-topos of spaces/types.

Let $\mathcal{C}_{\text{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\text{Mon}}$ be the finite-limit theory of monoids. Then

$$\text{FP}(\mathcal{C}_{\text{Mon}}, \text{Set}) \simeq \text{FL}(\mathcal{L}_{\text{Mon}}, \text{Set})$$

but $\text{FP}(\mathcal{C}_{\text{Mon}}, \mathcal{S})$ and $\text{FL}(\mathcal{L}_{\text{Mon}}, \mathcal{S})$ are different:

- $\text{FL}(\mathcal{L}_{\text{Mon}}, \mathcal{S})$ is just the category of monoids
- $\text{FP}(\mathcal{C}_{\text{Mon}}, \mathcal{S})$ is the $\infty$-category ‘$A_\infty$-algebras’, i.e. homotopy-coherent monoids.

**Moral**

By being ‘slimmer’, finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name ‘animation’ in:

Four clans for categories

**Cat** admits several clan-algebraic weak factorization systems:

- \((\mathcal{E}_1, \mathcal{F}_1)\) is cofib. generated by \(\{(0 \to 1), (2 \to 2)\}\)
- \((\mathcal{E}_2, \mathcal{F}_2)\) is cofib. generated by \(\{(0 \to 1), (2 \to 2), (2 \to 1)\}\)
- \((\mathcal{E}_3, \mathcal{F}_3)\) is cofib. generated by \(\{(0 \to 1), (2 \to 2), (\mathbb{P} \to 2)\}\)
- \((\mathcal{E}_4, \mathcal{F}_4)\) is cofib. generated by \(\{(0 \to 1), (2 \to 2), (\mathbb{P} \to 2), (2 \to 1)\}\)

where \(\mathbb{P} = (\bullet \cong \bullet)\).

The right classes are:

\[
\begin{align*}
\mathcal{F}_1 &= \{\text{full and surjective-on-objects functors}\} \\
\mathcal{F}_2 &= \{\text{full and bijective-on-objects functors}\} \\
\mathcal{F}_3 &= \{\text{fully faithful and surjective-on-objects functors}\} \\
\mathcal{F}_4 &= \{\text{isos}\}
\end{align*}
\]

Note that \(\mathcal{F}_3\) is the class of trivial fibrations for the canonical model structure on **Cat**.
Four clans for categories

These correspond to the following clans:

\[ \mathcal{T}_1 = \{ \text{free cats on fin. graphs} \}^{\text{op}} \]
\[ \mathcal{T}_1^\dagger = \{ \text{graph inclusions} \} \]
\[ \mathcal{T}_2 = \{ \text{free cats on fin. graphs} \}^{\text{op}} \]
\[ \mathcal{T}_2^\dagger = \{ \text{injective-on-edges maps} \} \]
\[ \mathcal{T}_3 = \{ \text{f.p. cats} \}^{\text{op}} \]
\[ \mathcal{T}_3^\dagger = \{ \text{injective-on-objects functors} \} \]
\[ \mathcal{T}_4 = \{ \text{f.p. cats} \}^{\text{op}} \]
\[ \mathcal{T}_4^\dagger = \{ \text{all functors} \} \]

Models in higher types:

\[ \infty\text{-Mod}(\mathcal{T}_1) = \{ \text{Segal spaces} \} \]
\[ \infty\text{-Mod}(\mathcal{T}_2) = \{ \text{Segal categories} \} \]
\[ \infty\text{-Mod}(\mathcal{T}_3) = \{ \text{pre-categories} \} \]
\[ \infty\text{-Mod}(\mathcal{T}_4) = \{ \text{discrete 1-categories} \} \]
Thanks for your attention!