Polynomial comonads + comodules

Three ways to think about internal catys in type theory.

Always start with a type of objects C_0. Then for arrows:

(1) "Internal": type of arrows C_1 and $s, t : C_1 \rightrightarrows C_0$

(2) "Enriched": dep. type of arrows $(x, y \in C_0)\ C_1(x, y)$

(3) ?? : dep-type of arrows $(x \in C_0)\ C_1(x)$ and $(x \in C_0, f \in C_1(x))\ t(x, f) \in C_0$

1) The multicategory $\mathrm{Poly}(\mathcal{E})$

Let $\mathcal{E}$ be a caty with a class of display maps $\mathcal{D}$.

Define a caty $\underline{\mathrm{Poly}}(\mathcal{E})$ with:

- objects are $\mathcal{D}$-maps $p : E \to B$ $\qquad (b \in B)\ E(b)$ type
- maps $p \to p'$ are

$$\begin{tikzcd}
E \arrow[d, "p"'] & \bullet \arrow[l] \arrow[r, bend left, "g"] \arrow[dr, "\iota"] & E' \arrow[d, "p'"] \\
B & & B' \arrow[ll, "f"]
\end{tikzcd}$$

$(b' \in B')\ f(b') \in B$

$(b' \in B', e \in E(f(b')))\ g(b', e) \in E'(b')$

- composite of above map with $(h, k) : p' \to p''$ is $(fh, k \circ h^* g)$

$$\begin{tikzcd}
E \arrow[ddd, "p"'] & \bullet \arrow[l] \arrow[ddd, "\iota"'] \arrow[dr, "g"] & & \bullet \arrow[ll] \arrow[dr, "h^*(g)"] & & \\
& & E' \arrow[ddl] & & \bullet \arrow[ll] \arrow[ddl, "\iota"'] \arrow[dr, "k"] & \\
& & & & & E'' \arrow[dll] \\
B & B' \arrow[l, "f"] & & B'' \arrow[ll, "h"] & &
\end{tikzcd}$$

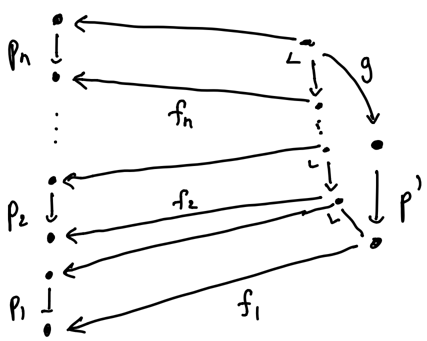
$(b'' \in B'')\ f(h(b'')) \in B$

$(b'' \in B'', e \in E(fhb''))\ k(b'', g(hb'', e))$

Idea: $p : E \to B$ is a game: opponent chooses $b \in B$; player chooses a response in $e \in E(b)$. A O-strategy is a choice of $b \in B$; a P-strategy is a section of p. A map $p \to p'$ in $\mathrm{Poly}(\mathcal{E})$ allows a O-strategy for p' to be played against a P-strategy for p.

In fact, $\mathrm{Poly}(\mathcal{E})$ can be made into a <u>multicategory</u>.

- objects as before
- unary maps $(p) \longrightarrow p'$ as before
- nullary maps $() \longrightarrow p'$ are sections of p'
- n-ary maps $(p_1, \ldots, p_n) \longrightarrow p'$ $(n \geq 2)$ are:



M. Weber – Polynomials in a caty with pullbacks.

<u>Idea:</u> n-ary map allows an O-strategy for p' to be played against P-strategies for each of $p_1, \ldots, p_n$

When $\mathbb{D}$-maps are exponentiable, this multicaty is actually a monoidal caty with tensor being polynomial composition.

2) <u>Monoids in $\mathrm{Poly}(\mathcal{E})$</u>

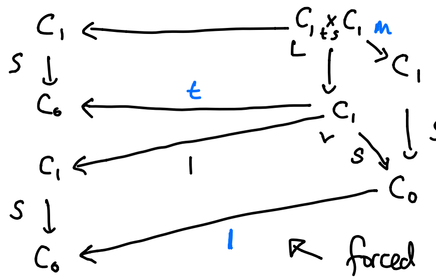
In any multicaty $\mathcal{M}$, a <u>monoid</u> is an object X + two multimaps
$() \xrightarrow[\eta]{} X$, $(X, X) \xrightarrow[\mu]{} X$ + axioms.

<u>What's a monoid in $\mathrm{Poly}(\mathcal{E})$?</u>

(1) An object: ie a $\mathbb{D}$-map $C_1 \xrightarrow{s} C_0$ $\quad (x \in C_0)\ C_1(x)$

(2) Nullary map $() \xrightarrow{i} s$, ie a section i of s $\quad (x \in C_0)\ i(x) \in C_1(x)$

(3) Binary map $(s,s) \xrightarrow{m} s$ looks like:



$(x \in C_0, f \in C_1(x))$ $t(x,f) \in C_0$

$(x \in C_0, f \in C_1(x), g \in C_1(t(x,f)))$

$m(x,f,g) \in C_1(x)$

So we have the data of a category (expressed in style of (3)) and the monoid axioms = category axioms.

Ahman-Uustalu observed this fact.

Warning! If $\mathbb{C}$, $\mathbb{D}$ are internal catys, seen as monoids in $Poly(\mathcal{E})$, then monoid maps between them involves — these are in fact cofunctors (Higgins-Mackenzie, Aguiar PhD thesis)

$$\begin{tikzcd} C_1 & \cdot \arrow[l] \arrow[r, "f_1"] \arrow[dr] & D_1 \arrow[d, "s"] \\ C_0 \arrow[from=u, "s"'] & & D_0 \arrow[ll, "f_0"] \end{tikzcd}$$

$$\begin{tikzcd} D_0 \arrow[r, "D_1", "|" marking] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Uparrow f_1"] & D_0 \arrow[d, "f_0"] \\ C_0 \arrow[r, "C_1"', "|" marking] & C_0 \end{tikzcd}$$

3) Bimodules in $Poly(\mathcal{E})$

A <u>bimodule</u> in a multicaty $\mathcal{M}$ between monoids X and Y is:

- an object M
- multimaps $(X, M) \xrightarrow{\lambda} M$ $\quad (M, Y) \xrightarrow{\rho} M$
- + axioms

What is a bimodule in $Poly(\mathcal{E})$ between $\mathbb{C}$ and $\mathbb{D}$?

- First, $\pi : E \to B$ a $\mathcal{D}$-map $\quad (x \in B)$ $E(x)$

- Left action by $\mathbb{C}$ involves:

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] & & \cdot \arrow[ll] \arrow[d] \arrow[dr, "\beta"] & \\
B & & \cdot \arrow[ll, "\alpha"'] \arrow[dll] \arrow[dr] & E \arrow[d, "\pi"] \\
C_1 \arrow[d, "s"'] & & & B \arrow[dlll, "p"] \\
C_0 & & &
\end{tikzcd}
$$

... turns out to be the same as:

$$
\mathbb{E} \xrightarrow[\pi]{\text{discrete fibration}} \mathbb{B} \xrightarrow[p]{\text{discrete opfibration}} \mathbb{C}
$$

- Right action by $\mathbb{D}$ involves:

$$
\begin{tikzcd}
D_1 \arrow[d, "s"'] & & \cdot \arrow[ll] \arrow[d] \arrow[dr, "\gamma"] & \\
D_0 & & E \arrow[ll, "q"'] \arrow[dll] \arrow[dr, "\pi"] & E \arrow[d, "\pi"] \\
E \arrow[d, "\pi"'] & & & B \arrow[dlll, "1"] \\
B & & &
\end{tikzcd}
$$

... turns out to be same as

$$
\begin{tikzcd}
& \mathbb{E} \arrow[dl, "q"'] \arrow[dr, "\pi"] & \\
\mathbb{D} & & \mathrm{disc}(\mathbb{B})
\end{tikzcd}
$$

fibrewise a discrete fibration

- Bimodule condition forces compatibility, so we get

$$
(*) \qquad \mathbb{D} \xleftarrow{q} \mathbb{E} \xrightarrow{\pi} \mathbb{B} \xrightarrow{p} \mathbb{C}
$$

two-sided discrete fibration   discrete opfibration

4) PARAMETRIC RIGHT ADJOINT FUNCTORS

If $\mathcal{E}$, $\mathbb{F}$ are catys with terminal objs, a functor $F : \mathcal{E} \to \mathbb{F}$ is parametric right adjoint if $F : \mathcal{E} \longrightarrow \mathbb{F}/_{F1}$ is a right adjoint.

Now: pra functors $[\mathbb{D}, Set] \longrightarrow [\mathbb{C}, Set]$

right adjoints $[\mathbb{D}, Set] \longrightarrow [\mathbb{C}, Set]/_X$ (some $X \in [\mathbb{C}, Set]$)

left adjoints $[\mathbb{C}, Set]/_X \longrightarrow [\mathbb{D}, Set]$

functors $\qquad (\mathrm{el}\, X)^{op} \longrightarrow [\mathbb{D}, \mathrm{Set}]$

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two-sided disc fibrs. $\qquad \mathbb{D} \longleftarrow \mathbb{E} \longrightarrow \mathrm{el}\, X$

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diagrams $\qquad \underbrace{\mathbb{D} \longleftarrow \mathbb{E} \longrightarrow}_{\text{2-sided disc fibr.}} \mathbb{B} \underbrace{\longrightarrow}_{\text{d.o.f.}} \mathbb{C}$

ie: bimodules $\mathbb{C} \nrightarrow \mathbb{D}$ in $\mathrm{Poly}(\mathrm{Set})$ $\equiv$ pra adjoints functors $[\mathbb{D}, \mathrm{Set}] \longrightarrow [\mathbb{C}, \mathrm{Set}]$