Indexed Type Theories

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Basic theory

(Co)limits

Examples

Dependent version

Initial object theorem

Motivation

- Can we use HoTT to reason about ∞-categories which are not toposes?
- We need Hom-spaces to define certain constructions and properties.
- Even if we have internal Hom-spaces, sometimes the internal version of some construction is too strong.
- We might also want to reason about non-elementary constructions such as infinite (co)limits.

Indexed unary type theory

- An indexed category is a functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$.
- Indexed unary type theory has four judgements:

$$\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \mid \cdot \vdash B \qquad \Gamma \mid x : B \vdash b' : B'$$

- The first two judgements correspond to the base theory and we can have all ordinary constructions at this level.
- We have the following substitution rules:
  - of base terms in base terms and types
  - of indexed terms in indexed terms
  - of base terms in indexed terms and types

Local smallness

- We will always assume that the following rules are derivable:

$$\frac{\Gamma \mid \cdot \vdash A \qquad \Gamma \mid \cdot \vdash B}{\Gamma \vdash \mathrm{Hom}(A, B)} \qquad \frac{\Gamma \mid x : A \vdash b : B}{\Gamma \vdash \lambda x.\, b : \mathrm{Hom}(A, B)}$$

$$\frac{\Gamma \vdash f : \mathrm{Hom}(A, B) \qquad \Gamma \mid \Delta \vdash a : A}{\Gamma \mid \Delta \vdash f\, a : B}$$

$$(\lambda x.\, b)\, a \equiv b[a/x]$$
$$\lambda x.\, f\, x \equiv f$$

- This corresponds to the condition that the indexed category is locally small.

Constructions on maps

- A homotopy between maps $f, g : \mathrm{Hom}(A, B)$ is an element of $\mathrm{Id}_{\mathrm{Hom}(A,B)}(f, g)$.
- Vertical and horizontal composition of homotopies can be defined as usual by path induction.
- The type of equivalences can be defined as either the type of bi-equivalences or half-adjoint equivalences.

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(Co)limits

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Initial and terminal types

- An indexed type T is *terminal* if the base type $\mathrm{Hom}(X, T)$ is contractible for every indexed type X.
- It is *initial* if $\mathrm{Hom}(T, X)$ is contractible for every X.
- It is *zero* if it is both initial and terminal.

Pullbacks and pushouts

- A pullback of maps $f : \mathrm{Hom}(A, C)$ and $g : \mathrm{Hom}(B, C)$ consists of an object D, maps $\pi_1 : \mathrm{Hom}(D, A)$ and $\pi_2 : \mathrm{Hom}(D, B)$ and a homotopy $\pi_3 : f \circ \pi_1 = g \circ \pi_2$ satisfying the usual universal property.
- All standard properties of pullbacks hold in this setting.
- In particular, various kinds of finite limits can be defined in terms of pullbacks and terminal objects.
- Pushouts can be defined dually.

Products

The product of a family of indexed types is defined as follows:

$$\frac{\Gamma, i : I \mid \cdot \vdash B_i}{\Gamma \mid \cdot \vdash \prod_{i:I} B_i}$$

$$\frac{\Gamma, i : I \mid \cdot \vdash B_i}{\Gamma, i : I \vdash \pi_i : \mathrm{Hom}(\prod_{i:I} B_i, B_i)}$$

$$\frac{\Gamma \mid \cdot \vdash P \qquad \Gamma, i : I \vdash f : \mathrm{Hom}(P, B_i)}{\Gamma \vdash \langle f \rangle_{i:I} : \mathrm{Hom}(P, \prod_{i:I} B_i)}$$

together with homotopies

$$\begin{aligned} \Gamma, i : I \vdash \beta(f) &: \pi_i \circ \langle f \rangle_{i:I} = f, \\ \Gamma \vdash \eta(g) &: \langle \pi_i \circ g \rangle_{i:I} = g. \end{aligned}$$

Strict products

There is a strict version of products:

$$\frac{\Gamma, i : I \mid \cdot \vdash B}{\Gamma \mid \cdot \vdash \prod_{i:I} B} \qquad \frac{\Gamma, i : I \mid \Delta \vdash b : B}{\Gamma \mid \Delta \vdash \lambda i.\, b : \prod_{i:I} B}, \; i \notin \mathrm{FV}(\Delta)$$

$$\frac{\Gamma \mid \Delta \vdash f : \prod_{i:I} B \qquad \Gamma \vdash j : I}{\Gamma \mid \Delta \vdash f\, j : B[j/i]}$$

$$(\lambda i.\, b)\, j \equiv b[j/i]$$
$$\lambda i.\, f\, i \equiv f$$

Coproducts

The coproduct of a family of indexed types is defined dually:

$$\frac{\Gamma, i : I \mid \cdot \vdash B_i}{\Gamma \mid \cdot \vdash \coprod_{i:I} B_i}$$

$$\frac{\Gamma, i : I \mid \cdot \vdash B_i}{\Gamma, i : I \vdash \mathrm{in}_i : \mathrm{Hom}(B_i, \coprod_{i:I} B_i)}$$

$$\frac{\Gamma \mid \cdot \vdash C \qquad \Gamma, i : I \vdash f : \mathrm{Hom}(B_i, C)}{\Gamma \vdash [f]_{i:I} : \mathrm{Hom}(\coprod_{i:I} B_i, C)}$$

together with homotopies

$$\Gamma, i : I \vdash \beta(f) : [f]_{i:I} \circ \mathrm{in}_i = f,$$
$$\Gamma \vdash \eta(g) : [g \circ \mathrm{in}_i]_{i:I} = g.$$

Copowers

Copowers are coproducts of constant families ($X \cdot A = \coprod_{x:X} A$).
We have the following identities:

$$\bot \cdot X \simeq 0$$
$$(I \amalg_K J) \cdot X \simeq I \cdot X \amalg_{K \cdot X} J \cdot X$$
$$\top \cdot X \simeq X$$
$$\left(\sum_{i:I} J\right) \cdot X \simeq \coprod_{i:I} J \cdot X$$
$$(\Sigma I) \cdot 1 \simeq \Sigma(I \cdot 1)$$
$$S^n \cdot 1 \simeq S^n$$

Basic theory

(Co)limits

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Self-indexing

- Every finitely complete (∞-)category gives rise to an indexed (∞)-category over itself.
- This can be reformulated as follows: indexed type theory can be interpreted in (homotopy) type theory.
- Hom-types correspond to function types, products correspond to Π-types, coproducts correspond to Σ-types.
- This means that indexed type theory is weaker than type theory.

Quasicategories

- We can interpret base contexts as Kan complexes and indexed types over Γ as categorical fibrations $E \to B$ together with a map $\Gamma \to B$.
- To prove that this model is Cartesian closed, we need to show that the exponent of a pair of fibrations in $\mathbf{sSet}/B$ is a fibration, but this is false.
- To fix this, we can replace B with its "core" before taking the exponent.
- These "cores" have several nice properties of Kan complexes: evevy categorical fibration with such a codomain is Cartesian fibration and the "core" functor maps categorical fibrations to Kan fibrations,
- These facts imply that the theory is Cartesian closed and locally small.

Localizations of presheaves

- For every small category $\mathcal{J}$ and every nice enough model of the base theory $\mathcal{C}$, the functor category $\mathcal{C}^{\mathcal{J}}$ is a model of the indexed type theory.
- In particular, we have a model in which base types are simplicial sets / Kan fibrations and indexed types are bisimplicial sets / injective fibrations.
- Localization often preserves models.
- In particular, we have a model in which base types are the same, but indexed types are bisimplicial sets satisfying either Segal (or Rezk) condition.

Pointed types

- Suppose that we have zero types and a fixed indexed type S^0.
- Then every indexed type A has underlying base type $\mathrm{Hom}(S^0, A)$. This type is based with base point $S^0 \to 0 \to A$.
- Every map of indxed types A and B induces a map of pointed base types $\mathrm{Hom}(S^0, A)$ and $\mathrm{Hom}(S^0, B)$.
- We can add an axiom assering that

$$\mathrm{Hom}(A, B) \to (\mathrm{Hom}(S^0, A) \to_* \mathrm{Hom}(S^0, B))$$

  is an equivalence.
- Moreover, we can add an indexed type $R(X, *)$ for every pointed base type $(X, *)$ together with a pointed equivalence between $\mathrm{Hom}(S^0, R(X, *))$ and $(X, *)$.
- This theory is "complete" in some sense.

Spectra

- We can also define a theory of spectra.
- We will say that a theory is *stable* if it has zero types, pullback and pushout and canonical maps $A \to \Omega\Sigma A$ and $\Sigma\Omega A \to A$ are equivalences.
- If S is a fixed indexed type in a stable theory, then, for every indexed type A, we can define a spectrum $U_S(A)_n = \mathrm{Hom}(S, \Sigma^n(A))$.
- This is an Ω-spectrum:

$$\Omega(\mathrm{Hom}(S, \Sigma^{n+1}(A))) \simeq \mathrm{Hom}(S, \Omega\Sigma^{n+1}(A) \simeq \mathrm{Hom}(S, \Sigma^n(A)$$

- Every map between indexed types induces a map of the underlying spectra.
- We can make this theory "complete" as before.

Synthetic higher category theory

- We can use indexed type theory to work with ∞-categories synthetically.
- We need to add an indexed type Δ^1 together with terms $l, r : \Delta^1$ and require that every indexed type is a (complete) Segal type.
- This is similar to Riehl, Shulman approach, but we can define the core of an ∞-category.
- Can we make this theory complete?

Basic theory

(Co)limits

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Rules

- Indexed dependent type theory is a version of the theory in which the second level is also dependent. It has the following judgements:

$$\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \mid \Delta \vdash B \qquad \Gamma \mid \Delta \vdash b : B$$

- We add usual identity types, Σ-types, and unit types to the second level.
- We add "Hom-extensionality", which requires the following canonical function to be an equivalence:

$$\mathrm{Id}_{\mathrm{Hom}(A,B)}(f,g) \to \mathrm{Sec}(\pi_1 : \mathrm{Hom}(\Sigma_{x:A}\mathrm{Id}_B(f\,x, g\,x), A))$$

Finite (co)limits

- The pullback of maps $f : \mathrm{Hom}(A, C)$ and $g : \mathrm{Hom}(B, C)$ can be defined as usual:

$$\sum_{x:A} \sum_{y:B} \mathrm{Id}_C(f\,x, g\,y)$$

- Hom-extensionality implies that this map is a pullback in the sense we defined before.
- The unit type is a terminal type.
- We can add the usual type-theoretic definition of the empty type, coproducts, and pushouts, which will give us corresponding categorical structures as we defined them before.

Dependent products

Dependent products generalize (strict) products.

$$\frac{\Gamma, i : I \mid \Delta \vdash B}{\Gamma \mid \Delta \vdash \prod_{i:I} B}, \; i \notin \mathrm{FV}(\Delta)$$

$$\frac{\Gamma, i : I \mid \Delta \vdash b : B}{\Gamma \mid \Delta \vdash \lambda i.\, b : \prod_{i:I} B}, \; i \notin \mathrm{FV}(\Delta)$$

$$\frac{\Gamma \mid \Delta \vdash f : \prod_{i:I} B \qquad \Gamma \vdash j : I}{\Gamma \mid \Delta \vdash f\, j : B[j/i]}$$

$$(\lambda i.\, b)\, j \equiv b[j/i]$$
$$\lambda i.\, f\, i \equiv f$$

Stable dependent coproducts

$$\frac{\Gamma, i : I \mid \Delta \vdash B_i}{\Gamma \mid \Delta \vdash \coprod_{i:I} B_i}, \; i \notin \mathrm{FV}(\Delta) \qquad \frac{\Gamma \vdash j : I \qquad \Gamma \mid \Delta \vdash b : B_j}{\Gamma \mid \Delta \vdash (j, b) : \coprod_{i:I} B_i}$$

$$\frac{\begin{array}{l}\Gamma \mid \Delta, z : \coprod_{i:I} B_i \vdash D \\ \Gamma, i : I \mid \Delta, x : B_i \vdash d : D[(i,x)/z] \qquad \Gamma \mid \Delta \vdash e : \coprod_{i:I} B_i\end{array}}{\Gamma \mid \Delta \vdash \coprod\text{-elim}(z.D, ix.d, e) : D[e/z]}$$

$$\coprod\text{-elim}(z.D, ix.d, (j, b)) \equiv d[j/i, b/x]$$

Dependent coproducts

If we don't want coproducts to be stable under pullbacks, we need to modify the rules as follows:

$$\frac{\Gamma, i : I \mid \cdot \vdash B_i}{\Gamma \mid \Delta \vdash \coprod_{i:I} B_i}, \; i \notin \mathrm{FV}(\Delta) \qquad \frac{\Gamma \vdash j : I \qquad \Gamma \mid \Delta \vdash b : B_j}{\Gamma \mid \Delta \vdash (j, b) : \coprod_{i:I} B_i}$$

$$\frac{\begin{array}{l}\Gamma \mid z : \coprod_{i:I} B_i \vdash D \\ \Gamma, i : I \mid x : B_i \vdash d : D[(i,x)/z] \qquad \Gamma \mid \Delta \vdash e : \coprod_{i:I} B_i\end{array}}{\Gamma \mid \Delta \vdash \coprod\text{-elim}(z.D, ix.d, e) : D[e/z]}$$

$$\coprod\text{-elim}(z.D, ix.d, (j, b)) \equiv d[j/i, b/x]$$

Cohesive type theory

- A cohesive ∞-topos is an ∞-topos $\mathcal{E}$ with an adjoint quadruple of functors between $\mathcal{E}$ and **sSet** (one of which is $\mathrm{Hom}(1,-) : \mathcal{E} \to$ **sSet**) satisfying some conditions.
- Shulman proposed a type theory that formalizes this setup.
- We can extend indexed type theory with additional rules corresponding to functors in the quadruple.
- The left adjoint to $\mathrm{Hom}(1,-)$ is given by the copower functor $(- \cdot 1)$. It is fully faithful if these copowers are disjoint.
- We can formalize the other two functors in a way similar to Shulman's theory.

Linear dependent type theory

- Instead of unary or dependent type theory, we can use linear type theory at the second level.
- This is useful for theories of pointed types and spectra.
- We can add smash product, linear function types, and wedge sums with nice computational rules.

Basic theory

(Co)limits

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Initial object theorem

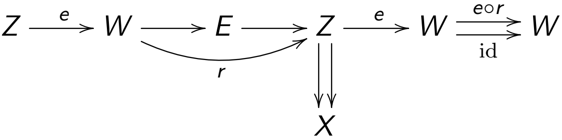
- Adjoint functor theorem holds for indexed categories.
- We cannot formulate it in the settings of indexed type theory because we cannot talk about functors, but we can formulate and prove its special case:

Theorem

If the base theory has natural numbers, the indexed theory has small limits, and there is a weakly initial family of indexed types, then there is an initial indexed type.

Proof of 1-categorical initial object theorem

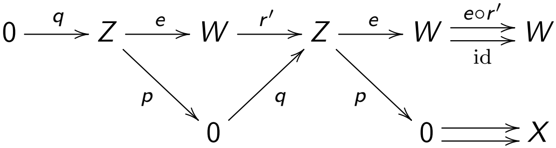
- The product W of the weakly initial family is weakly initial.
- Let $e : Z \to W$ be the equalizer of all endomaps on W.
- Consider a pair of maps $Z \rightrightarrows X$. Let $E \to Z$ be the equalizer of these maps and let r be the composite of $E \to Z$ and any map $W \to E$:



- Since e equalizes $e \circ r$ and id and e is a mono, $r \circ e = \mathrm{id}$. Since $E \to Z$ equalizes the maps $Z \rightrightarrows X$, it follows that they are equal.

Proof of ∞-categorical initial object theorem

- Let $r' : W \to Z$ be any map and assume that $r' \circ e$ is a split idempotent.
- Let $p : Z \to 0$ and $q : 0 \to Z$ be its splitting.
- Consider a pair of maps $0 \rightrightarrows X$.



- If we define r as before, we can show that $p \circ r \circ e = p \circ r' \circ e$.
- Since $p \circ r$ equalizes the maps $Z \rightrightarrows X$ and $p \circ q = \mathrm{id}$, it follows that they are equal.

Splitting of idempotents

- Lurie showed that idempotents split in any ∞-category with countable colimits.
- Shulman proved this in HoTT using exponential natural numbers.
- We can repeat his argument in indexed type theory, but using the external version of these assumptions:

Theorem

If the base theory has natural numbers and the indexed theory has finite limits and countable products, then idempotents between indexed types split.

Coherence of idempotents

- To split an idempotent h, we need to know it is coherent (i.e., there are homotopies $I : h \circ h = h$ and $J : I * h = h * I$).
- The original definition of Z gives us only I

$$Z = \sum_{x:W} \prod_{f:\mathrm{Hom}(W,W)} f\,x = x$$

- To get J, we need to modify the definition of Z:

$$\sum_{(x:W)} \sum_{(t:\prod_{f:\mathrm{Hom}(W,W)} fx=x)} \prod_{(f:\mathrm{Hom}(W,W))} \prod_{(s:f\circ f=f)} \mathrm{hap}(s,x) = \mathrm{ap}(f, t\,f)$$