∞-OPERADS AS POLYNOMIAL MONADS

joint with David Gepner & Rune Haugseng

	over <u>Set</u>	over $\mathcal{S}$ spaces
one in one out	<u>Cat</u>	∞ Cat
many in one out	<u>Opd</u>	∞ <u>Opd</u>

- quasicats
- Segal spaces

- Lurie
- . . .
-
-

operads

Kelly 72

monoids in <u>Sym Seq</u>

$$\left(\left(\overset{\circlearrowleft}{A_n} \mid n \in \mathbb{N} \right), \circ, 1 \right)$$

$$(B \circ A)_n = \sum_{n \to k} A_{n_1} \times \cdots \times A_{n_k} \times B_k \Big/ \mathrm{aut}\, p$$

species Joyal 81

$$F : \mathbb{B} \longrightarrow \mathrm{Set}$$
$$S \longmapsto F[S]$$

$$g_F(z) = \sum_{n \in \mathbb{N}} |F[n]| \times \frac{z^n}{n!}$$

$$g_{G \circ F} = g_G \circ g_F$$

operads are monoids in species

$$\begin{tikzcd} \mathrm{Set} \arrow[r, dotted, "\bar{F} = \mathrm{lan}\, F"] & \mathrm{Set} \\ \mathbb{B} \arrow[u, "J"', "\text{not full}"] \arrow[ur, "F"'] & \end{tikzcd}$$

$$\bar{F}(X) = \sum_{n \in \mathbb{N}} F[n] \times_{\mathfrak{S}_n} X^n$$

analytic functor

Joyal

$$\text{Species} \simeq \text{AnEnd}$$

$$\text{operads} = \text{monoid in } (\text{AnEnd}, \circ)$$

$$= \text{analytic monads.}$$

Joyal $\quad P : \mathrm{Set} \to \mathrm{Set}$

analytic

$\Updownarrow$

preserves filtered colims
preserves wide pullbacks <u>WEAKLY</u>

because of problems
with colims in <u>Set</u>

$$\bullet / G = \bullet \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \bullet \end{tikzcd}$$

polynomial functors

$$\begin{array}{ccc} & E \xrightarrow{p} B & \\ {}^{s}\swarrow & & \searrow{}^{t} \\ 1 & & 1 \end{array} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{Set} &\to \mathrm{Set} \\ X &\longmapsto \sum_{b \in B} X^{E_b} \end{aligned}$$

$$= t_! \, p_* \, s^* X$$

finitary = pres. filt. colims

$\Leftrightarrow$ p finite fibres

special case of analytic:

- those for which $\mathfrak{S}_n$ actions free
- pres. wide pbks strictly
- flat species

polynomial monad = sigma cofibrant operads many var's

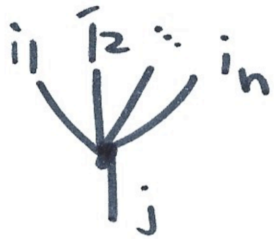
$$\begin{tikzcd} X \arrow[dr] & & E \arrow[dl, "s"'] \arrow[r, "p"] & B \arrow[dr, "t"] & \\ & I & & & I \end{tikzcd}
\qquad
\begin{array}{ccc} \mathbf{Set}/I & \longrightarrow & \mathbf{Set}/I \\ \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} & \longmapsto & t_! p_* s^* \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \end{array}$$

p_* dependent product

$t_!$ dep sum

$$(X_i \mid i \in I) \longmapsto \left(\sum_{b \in B_j} \prod_{e \in E_b} X_{se} \,\middle|\, j \in I \right)$$

coloured operads colour set I



ex small category = coloured operad with only unary operations

∞-categories = "weak cat objects in $\mathcal{S}$" spaces

enriched in $\mathcal{S}$

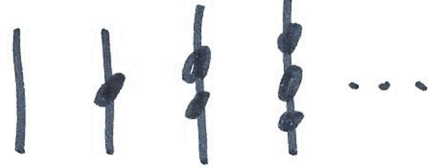
best model : quasicategories (Joyal) see [HTT]

another model : Segal spaces

$X : \Delta^{op} \to \mathcal{S}$ + Segal conditions

$$X_n \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

note Δ = linear trees



∞-operads

(Lurie $\quad \begin{array}{c}\mathcal{C}\\ \downarrow \\ \Gamma^{op}\end{array} \quad \ldots$)

other models ...

Cisinski-Moerdijk dendroidal Segal spaces

Ω cat of trees

$X : \Omega^{op} \to \mathcal{S}$ + "Segal condition"

$X($  $) \simeq X($  $) \times_{X(|)} X($  $) \times_{X(|)} X($  $)$

Theorems to compare all these models
(Cisinski - Moerdijk , Heuts , Haugseng ,
Chu , Hinich , Barwick)

Note : none of these has the feature
"monoid is SymSeq"

New model ~~(~~ (GMK)

∞-operad := analytic monad

Thm (GMK) $\quad \mathrm{AnMnd} \simeq \mathrm{Pr}_{seg}(\Omega)$

dendroidal Segal sp.

polynomial functors / $\mathcal{S}$

theory develops as expected

$$\begin{tikzcd} & E \arrow[r] \arrow[dl] & B \arrow[dr] & \\ I & & & I \end{tikzcd} \qquad \mathcal{S}/_I \to \mathcal{S}/_I$$

new features : colims are computed in

$Pr(\to \leftarrow \leftarrow)$

$$\begin{tikzcd} I' \arrow[d] & E' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r] \arrow[d] & I' \arrow[d] \\ I & E \arrow[l] \arrow[r] & B \arrow[r] & I \end{tikzcd}$$

PolyEnd/P $\rightsquigarrow$ topos

<u>Obs</u> all analytic functors are polynomial!

because all group actions are like free!

<u>Proof</u> ingredients

- <u>construction of free monads</u>
- tree description
- nerve theorem (Weber theory)

initial algebras free monads

$P : \mathcal{C} \to \mathcal{C}$

P-algebra : (A, a) $a : PA \to A$

P-coalgebra : (C, c) $c : C \to PC$

$$\begin{tikzcd}
\mathrm{coalg}_P(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d] \\
\mathcal{C} \arrow[r, "(\mathrm{Id}, P)"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

twisting morphism

(C, c) (A, a)

$$\begin{tikzcd} & C \arrow[dl, "c"'] \arrow[dd, "f"] \\ PC \arrow[d, "Pf"'] & \\ PA \arrow[r, "a"'] & A \end{tikzcd}$$

$$Tw_P(C, A) \longrightarrow Map(C, A) \underset{f \mapsto aPfc}{\overset{Id}{\rightrightarrows}} Map(C, A)$$

obs if (C, c) coals

$\quad\quad\downarrow c \qquad\qquad$ homo.

then (PC, Pc) coals

<u>Lemma 1</u>

$$\begin{tikzcd} & C \arrow[dl, "c"'] \arrow[ddd, "f"] \\ PC \arrow[d] & \\ PA \arrow[dr] & \\ & A \end{tikzcd}$$

$$\mathrm{Twp}(PC, A) \xrightarrow{\sim} \mathrm{Twp}(C, A)$$

$$g \longmapsto gc$$

$$a \circ Pf \longleftarrow\!\shortmid\; f$$

assume $\mathcal{C}$ has filtered colimits, P pres. them

(C, c)

$$C \xrightarrow{c} PC \xrightarrow{Pc} PPC \to \cdots \qquad I := \operatorname{colim}_n P^n C$$

$$\begin{tikzcd} I = \operatorname{colim} P^{n+1} C \arrow[r, "\sim", "\boxed{P \text{ pres filt col}}"'] & P(\operatorname{colim} P^n C) = PI \arrow[l, bend left, dashed, "u"] \end{tikzcd}$$

$\Omega C := (I, u)$ P-algebra

cobar construction

Lemma 2 If (U,u) coalg

with u equiv. with inverse v

(A,a) alg

$$\mathrm{Map}_{\mathrm{alg}_p}((U,v),(A,a)) \simeq \mathrm{Tw}_p(U,A)$$

Prop $\boxed{\mathrm{Map}_{\mathrm{alg}}(\Omega C, A) \simeq \mathrm{Tw}_p(C,A)}$

later B bar constr.

$$\simeq \mathrm{Map}_{\mathrm{coalg}}(C, BA)$$

$$\Omega \dashv B$$

Thm $\mathrm{Map}(\Omega C, A) \simeq \mathrm{Tw}_p(C, A)$

<u>Pf</u> $$\mathrm{Map}(\Omega C, A) \underset{\boxed{\text{Lemma 2}}}{\simeq} \mathrm{Tw}_p(\operatorname{colim}_n P^n C, A)$$

remains to show

$$\simeq \mathrm{Tw}_p(C, A)$$

$$\mathrm{Tw}_p(\operatorname{colim} P^n C, A) \to \mathrm{Map}(\operatorname{colim} P^n C, A) \rightrightarrows \mathrm{Map}(\text{''})$$

$$\underline{\lim \mathrm{Tw}_p(P^n C, A)} \to \lim \mathrm{Map}(P^n C, A) \rightrightarrows \lim \mathrm{Map}(P^n C, A$$

$$
\begin{tikzcd}
\lim Tw_p(P^n C, A) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \lim Map(P^n C, A) \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, dotted] & \lim Map(P^n C, A) \arrow[dd, dotted] \\
Tw_p(C, A) \arrow[d, dotted] & & \\
Tw_p(PC, A) \arrow[r] \arrow[d, phantom, "\cong\ \text{Lemma 1}" rotate=90] & Map(PC, A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Map(PC, A) \arrow[d] \\
Tw_p(C, A) \arrow[r] & Map(C, A) \arrow[r, shift left] \arrow[r, shift right] & Map(C, A)
\end{tikzcd}
$$

Cor $\emptyset \in \mathcal{C}$ then $\Omega\emptyset$ is initial P-alg.

Pf $\mathrm{Map}_{\mathrm{alg}}(\Omega\emptyset, A) \simeq \mathrm{Tw}_P(\emptyset, A) \simeq *$

$P : \mathcal{C} \to \mathcal{C}$ $\qquad x \in \mathcal{C}$

$P_X : \mathcal{C}_{X/} \longrightarrow \mathcal{C}_{X/}$

$$\mathcal{C}_{X/} \to \mathcal{C} \xrightarrow{P} \mathcal{C} \longrightarrow \mathcal{C}_{X/}$$
$$A \longmapsto X + A$$

$$\begin{array}{c} X \\ \downarrow \\ A \end{array} \longmapsto \begin{array}{c} X \\ \downarrow \\ X + PA \end{array}$$

Thm

$$\begin{array}{c} alg_P(\mathcal{C}) \\ \big\downarrow \\ \mathcal{C} \end{array}$$

is monadic.

Pf

Left adj. show that each coslice $alg_P(\mathcal{C})_{X/}$ has initial obj.

$\simeq alg_{P_X}(\mathcal{C}_{X/})$

$\Omega(id_X)$

The functor
$$\begin{tikzcd} \mathcal{C} \arrow[loop above, "\Omega"] \end{tikzcd}$$
is free monad on P.

completeness only refers to

unary part $X\left(\phi\right)$

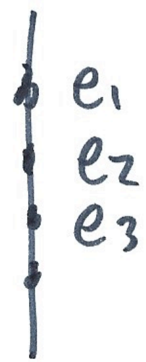
$\bar{P}$ free monad on P

if P polynomial

$$\begin{tikzcd} I & E \arrow[l] \arrow[r] & B \arrow[r] & I \end{tikzcd}$$

then $\bar{P}$ polynomial

$$\begin{tikzcd} & tr'(P) \arrow[dl] \arrow[r] & tr(P) \arrow[dr] & \\ I & & & I \end{tikzcd}$$



$$\begin{tikzcd}
 & F_{01} \arrow[rr] \arrow[dl, "r"'] \arrow[dr] & & F_1 \arrow[dl, "v?"] \arrow[ddr] & \\
F_0 \arrow[rr] \arrow[ddr, "?"] & & F \arrow[ddr, dotted] & & \\
 & & U_{01} \arrow[rr] \arrow[dl] & & U_1 \arrow[dl] \\
 & U_0 \arrow[rr] & & U &
\end{tikzcd}$$

$$\begin{tikzcd}
I' \arrow[d] & E' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r] \arrow[d] & I' \arrow[d] \\
I & E \arrow[l] \arrow[r] & B \arrow[r] & I
\end{tikzcd}$$