Calculating a Brunerie Number

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In his PhD thesis, Brunerie constructed a number $n$ s.t.
\[ \pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z} \]

He then proved that $n = \pm 2$, thereby also showing that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

Proving that $n = \pm 2$ should not be necessary – everything is constructive, so we should be able to simply compute $n$ by plugging it into our favourite proof assistant.

But $n$ is still constructively defined. Maybe if we unfold its definition enough, we should be able to deduce $n = \pm 2$ by simply staring at it.

In this talk, I will present such a proof.
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Introduction

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Definition 1 (Suspensions)
The suspension of a type $A$, denoted $\Sigma A$, is given by the following HIT

- north, south : $\Sigma A$
- merid : $A \rightarrow$ north $=$ south
Definition 2 (The circle)

We define the circle $S^1$ by the HIT

- base : $S^1$
- loop : base $= \text{base}$

Definition 3 (Spheres)

For $n \geq 1$, we define the $n$-sphere by $(n - 1)$-fold suspension of $S^1$, i.e.

$$S^n := \Sigma^{n-1} S^1$$
Suspension maps

For a pointed type $A$, there is a canonical map

$$\sigma : A \to \Omega(\Sigma A)$$

where $\Omega(\Sigma A) := \{(\text{north} = \text{north})\}$

given by

$$\sigma(a) = \text{merid}(a) \cdot \text{merid}(\ast_A)^{-1}$$

In particular, when $A = S^n$, we get

$$\sigma : S^n \to \Omega S^{n+1}$$
**Definition 4 (Joins)**

The join of two types $A$ and $B$, denoted $A \ast B$, is given by

- $\text{inl} : A \rightarrow A \ast B$
- $\text{inr} : B \rightarrow A \ast B$
- $\text{push} : ((a, b) : A \times B) \rightarrow \text{inl}(a) = \text{inr}(b)$
Joins

- There is a very useful way to construct maps $A \ast B \to C$ out of maps $A \times B \to \Omega C$.

**Definition 5**

Let $f : A \times B \to \Omega C$. Define $\iota_f : A \ast B \to C$ by

$$
\iota_f(\text{inl}(a)) = \star C
$$

$$
\iota_f(\text{inr}(b)) = \star C
$$

$$
\text{ap}_{\iota_f}(\text{push}(a, b)) = f(a, b)
$$

- We note that functions $f, g : A \times B \to \Omega C$ can be ‘composed’:

$$
(f \cdot g)(a, b) = f(a, b) \cdot g(a, b)
$$

- Q: is there a way of saying that $\iota$ is a ‘homomorphism’ i.e.

$$
\iota_{f \cdot g} = \iota_f + \iota_g?
$$
An ad hoc construction

- A: yes, if $A$ and $B$ are reasonable.
- Recall, $\pi_n(A) := \|S^n \to \ast A\|_0$

**Definition 6**
For a pointed type $A$, define $\pi_{n+m+1}^*(A) = \|S^n \ast S^m \to \ast A\|_0$
An ad hoc construction

- A: yes, if $A$ and $B$ are reasonable.
- Recall, $\pi_n(A) := \|\mathbb{S}^n \to \ast A\|_0$

**Definition 6**
For a pointed type $A$, define $\pi^{*}_{n+m+1}(A) = \|\mathbb{S}^n \ast \mathbb{S}^m \to \ast A\|_0$

**Theorem 7**
There is a group structure on $\pi^{*}_{n+m+1}(A)$ such that
- $\pi^{*}_{n+m+1}(A) \simeq \pi_{n+m+1}(A)$
- For $f, g : \mathbb{S}^n \times \mathbb{S}^m \to \Omega A$, we have $\iota_{f \cdot g} = \iota_f + \iota_g$
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There is a group structure on $\pi^*_{n+m+1}(A)$ such that

• $\pi^*_{n+m+1}(A) \cong \pi_{n+m+1}(A)$

• For $f, g : S^n \times S^m \to \Omega A$, we have $\imath_{f \cdot g} = \imath_f + \imath_g$

• Disclaimer: Formalisation only for $n = m = 1$ and $A$ 1-connected. (only case we’ll use)
Here is a particularly important example of the $\iota$-construction.

There is a canonical map $\iota : S^1 \times S^1 \to S^2$.

Composing it with $\sigma$ gives us $(\sigma \circ \iota) : S^1 \times S^1 \to \Omega S^3$

Define $F = \iota(\sigma \circ \iota) : S^1 \ast S^1 \to S^3$

**Proposition 8**

$F$ is an equivalence, and $(\_ \circ F^{-1}) : \pi_3^*(A) \cong \pi_3(A)$
The Hopf Map and the Brunerie Map

• Define \( h, \beta : S^1 \times S^1 \to \Omega S^2 \) by

\[
\begin{align*}
  h(x, y) &= \sigma(y - x) \\
  \beta(x, y) &= \sigma(y) \cdot \sigma(x)
\end{align*}
\]

• Above, the subtraction comes from the group structure on \( S^1 \)
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$$h(x, y) = \sigma(y - x)$$
$$\beta(x, y) = \sigma(y) \cdot \sigma(x)$$

• Above, the subtraction comes from the group structure on $S^1$
• The induced maps $\iota_h, \iota_\beta : S^1 \ast S^1 \to S^2$ are called the Hopf map and the Brunerie Map respectively.
Brunerie’s First Theorem

- By precomposition with $F^{-1} : S^3 \to S^2$, we get two corresponding elements $\hat{i}_h, \hat{i}_\beta : \pi_3(S^2)$. 
Brunerie’s First Theorem

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• **Fact:** $\pi_3(S^2) \cong \mathbb{Z}$ and is generated by $\hat{i}_h$. 
Brunerie’s First Theorem

- By precomposition with \( F^{-1} : S^3 \rightarrow S^2 \), we get two corresponding elements \( \hat{i}_h, \hat{i}_\beta : \pi_3(S^2) \).

- **Fact:** \( \pi_3(S^2) \cong \mathbb{Z} \) and is generated by \( \hat{i}_h \).

**Theorem 9 (Brunerie 16)**

\[ \pi_4(S^3) \cong \mathbb{Z} / n\mathbb{Z} \] where \( n \) is the integer s.t.

\[ n \cdot \hat{i}_h = \hat{i}_\beta \]
Brunerie’s First Theorem

• By precomposition with $\mathcal{F}^{-1} : \mathbb{S}^3 \to \mathbb{S}^2$, we get two corresponding elements $\hat{\iota}_h, \hat{\iota}_\beta : \pi_3(\mathbb{S}^2)$.

• **Fact:** $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ and is generated by $\hat{\iota}_h$.

**Theorem 9 (Brunerie 16)**

$\pi_4(\mathbb{S}^3) \cong \mathbb{Z} / n\mathbb{Z}$ where $n$ is the integer s.t.

$$n \cdot \hat{\iota}_h = \hat{\iota}_\beta$$

• We will attempt to solve this equation directly. I claim that $n = -2$ is the solution.
Proof sketch

• In order to show that $n = -2$, we would like to show that

$$\hat{\iota}_h + \hat{\iota}_h = -\hat{\iota}_\beta$$

i.e.

$$(\iota_h \circ \mathcal{F}^{-1}) + (\iota_h \circ \mathcal{F}^{-1}) = - (\iota_\beta \circ \mathcal{F}^{-1})$$

• With our $\pi_3^*$ construction, the above can be rewritten to something much nicer:

$$(\iota_h + \iota_h) \circ \mathcal{F}^{-1} = (\iota_\beta) \circ \mathcal{F}^{-1}$$
Proof sketch

- Idea for the rest of the proof: keep rewriting the above equation by passing it through the sequence of isomorphisms

\[ \pi_3(S^2) \xrightarrow{\circ F} \pi_3^*(S^2) \xrightarrow{(\iota h \circ -)^{-1}} \pi_3^*(S^1 \ast S^1) \xrightarrow{F \circ} \pi_3^*(S^3) \]

- When we reach \( \pi_3^*(S^2) \), the equation will have turned into something cute!
Applying the highlighted isomorphism above reduces our old equation (in $\pi_3(S^2)$)

$$(\iota_h + \iota_h) \circ \mathcal{F}^{-1} = (-\iota_\beta) \circ \mathcal{F}^{-1}$$

to the following equation in $\pi_3^*(S^2)$

$$\iota_h + \iota_h = -\iota_\beta$$
Step 2

\[ \pi_3(S^2) \xrightarrow{\circ F} \pi_3(S^2) \xrightarrow{(\iota_h \circ \_)^{-1}} \pi_3(S^1 \ast S^1) \xrightarrow{F \circ} \pi_3(S^3) \]

- We would like to rewrite our equation to an equation in \( \pi_3^*(S^1 \ast S^1) \) via the highlighted isomorphism.
- To this end, we construct two maps in \( f, g : S^1 \ast S^1 \rightarrow S^1 \ast S^1 \) s.t.
  \[
  \begin{align*}
  \iota_h \circ f &= \iota_h + \iota_h \\
  \iota_h \circ g &= \iota_\beta
  \end{align*}
  \]
- \( f \) is given by \( \text{id} + \text{id} \)
- \( g \) has a somewhat more complicated construction
Step 2

\[ \pi_3(\mathbb{S}^2) \xrightarrow{\circ \mathcal{F}} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\iota_h \circ \_)^{-1}} \pi_3^*(\mathbb{S}^1 \ast \mathbb{S}^1) \xrightarrow{\mathcal{F} \circ} \pi_3^*(\mathbb{S}^3) \]

• Define \( g : \mathbb{S}^1 \ast \mathbb{S}^1 \rightarrow \mathbb{S}^1 \ast \mathbb{S}^1 \) by

\[
g(\text{inl}(x)) = \text{inr}(-x) \\
g(\text{inr}(y)) = \text{inr}(y) \\
ap_g(\text{push}(x, y)) = \text{push}(y - x, -x)^{-1} \cdot \text{push}(y - x, y)
\]

• It is very direct to verify that \( \iota_h \circ g = \iota_\beta \)
Step 3

\[ \pi_3(S^2) \xrightarrow{-\circ \mathcal{F}} \pi_3^*(S^2) \xrightarrow{(\iota_h \circ \_ )^{-1}} \pi_3^*(S^1 \ast S^1) \xrightarrow{\mathcal{F} \circ} \pi_3^*(S^3) \]

- So we have new equation in \( \pi_3^*(S^1 \ast S^1) \):
  \[ \text{id} + \text{id} = -g \]
- Let's apply the highlighted isomorphism to \((\text{id} + \text{id})\) and \(g\).
- For the LHS: we have, trivially,
  \[ \mathcal{F} \circ (\text{id} + \text{id}) = \mathcal{F} + \mathcal{F} \]
Proposition 10
\[ \mathcal{F} \circ g = (-\mathcal{F}) + (-\mathcal{F}) \]

Proof.
Using the fact that \( \mathcal{F} \) is just \( \iota(\sigma \circ \sim) \) and the homomorphism property of \( \iota \), the proof boils down to proving

\[-((y - x) \sim (-x)) = -(x \sim y) \]

\[(y - x) \sim y = -(x \sim y)\]

which is easy.
So we are reduced to verifying

\[ \mathcal{F} + \mathcal{F} = -((-\mathcal{F}) + (-\mathcal{F})) \]

which, of course, is trivial.

Combining all the steps, we have shown:

**Theorem 11**

*The Brunerie number (with our definition) is $-2$.*
Concluding remarks

• Paired together with chapters 1–3 in Brunerie’s thesis, the above theorem allows us to conclude

\[ \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z} \]

• Cool things about this:
  • Much shorter than Brunerie’s original proof (skips chapters 4–6)
  • Does not use (co)homology
Concluding remarks

• Ignoring chapters 1–3, we also get a short, standalone proof of the following fact

Theorem 13

If \( \pi_4(\mathbb{S}^3) \) is non-trivial, then \( \pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z} \).

• The proof only uses \( |n| = 2 \), the Freudenthal suspension theorem and Eckmann-Hilton.

• Proving \( \Sigma\mathbb{C}P^2 \not\simeq \mathbb{S}^3 \vee \mathbb{S}^5 \) can be done using Steenrod squares (WIP, joint with David Wärn)

• But a direct proof, not relying on cohomology would be amazing (suggestions?)
Concluding remarks

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**Theorem 13**

*If* $\pi_4(S^3)$ *is non-trivial, then* $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.

• The proof only uses $|n| = 2$, the Freudenthal suspension theorem and Eckmann-Hilton.

• In particular, an easy corollary is the following:

**Theorem 14**

*If* $\Sigma \mathbb{C}P^2 \not\cong S^3 \vee S^5$, *then* $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.
Concluding remarks

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Future work

- Prove $\Sigma \mathbb{C}P^2 \not\cong S^3 \vee S^5$ to complete the new proof of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$
- The Brunerie map is an example of a ‘Whitehead product’:

$$[\_, \_] : \pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$$

These play an important role in the computation of the homotopy groups of spheres. The methods used here could possibly be mimicked for other Whitehead products too.