A comparison between

the Minimalist Foundation and Homotopy Type Theory

Maria Emilia Maietti

University of Padova

Homotopy Type Theory Electronic Seminar Talks

April 20th. 2023

Abstract of our talk

- on **Existential quantifiers** in **dependent type theory**
- peculiarities of the **Minimalist Foundation MF**
  in comparison with **HoTT**
- compatibility of **BOTH levels** of **MF** with **HoTT**
- compatibility of the classical **MF** wtih **Weyl's classical predicativism**
- open problems.

Answer this question...

Within **dependent type theory**

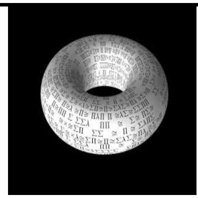
how many different

Existential quantifiers 

do you know

??

Existential quantifiers in HoTT

Within Homotopy type theory 

at least **two** different Existential quantifiers:

1. the Strong Existential Quantifier in Martin-Löf's type theory

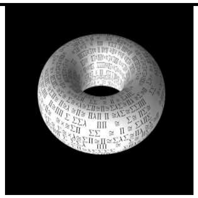
   identified with the indexed sum type

   $\Sigma x \in A\ \phi$

2. the h-propositional existential quantifier

   identified with the truncated Martin-L"of's existential quantifier

   $\exists x \in A\ \phi = ||\Sigma x \in A\ \phi||$ as an h-prop

Within **Homotopy type theory** 

at least **two different Existential quantifiers**:

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

$\Rightarrow$ **Axiom/Rule of choice holds**

2. the **h-propositional existential quantifier**

identified with the **truncated Martin-L"of's existential quantifier**

$\exists x \in A\ \phi = ||\Sigma x \in A\ \phi||$ as an **h-prop**

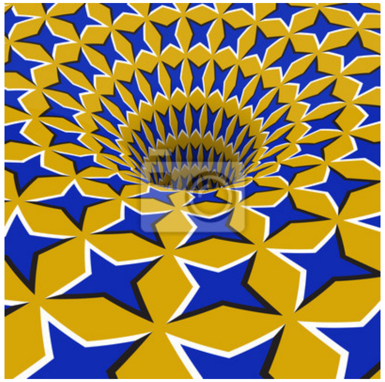
$\Rightarrow$**Axiom/Rule of UNIQUE choice holds**

because $\exists ! x \in A\ \phi\ =\ \Sigma x \in A\ \phi$

Axiom of choice

$$\forall x \in A\ \exists y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

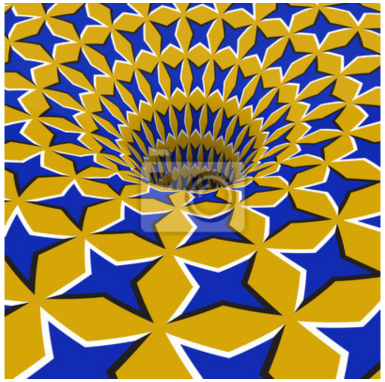
a total relation contains the graph of a type-theoretic function.



Axiom of unique choice

$$\forall x \in A\ \exists! y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, f(x))$$

turns a functional relation into a type-theoretic function.



$\Rightarrow$ identifies the two distinct notions...

Rule of choice

in a theory $\mathbf{T}$

if

$$\exists y \in B\ R(x,y)\ [x \in \Gamma]$$

is true in $\mathbf{T}$

$$\Downarrow$$

there exists a function term

$$f(x) \in B\ [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x))\ [x \in \Gamma]$$

is true in $\mathbf{T}$.

Rule of unique choice

in a theory $\mathbf{T}$

if

$$\exists! y \in B \ R(x,y) \ [x \in \Gamma]$$

is true in $\mathbf{T}$

$$\Downarrow$$

there exists a function term

$$f(x) \in B \ [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x)) \ [x \in \Gamma]$$

is true in $\mathbf{T}$.

Within **Dependent type theory** in general 

at least **THREE different Existential quantifiers:**

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

2. the **h-propositional existential quantifier**

identified with **the truncated Martin-L"of's existential quantifier**

$\exists x \in A\ \phi = ||\Sigma x \in A\ \phi||$ as an **h-prop**

3. **Intuitionistic existential quantifier** defined

as in **Coq** or as in the **Minimalist Foundation**

Elimination of Martin-Löf's Existential Quantifier 1.

$$\frac{M(z)\ type\ [z \in \Sigma_{x \in B} C(x)] \qquad d \in \Sigma_{x \in B} C(x) \quad m(x,y) \in M(\langle x,y \rangle)\ [x \in B, y \in C(x)]}{El_{\Sigma}(d,m) \in M(d)}$$

existential quantifier elimination

towards all types!

Elimination of the Intuitionistic existential quantifier 3.

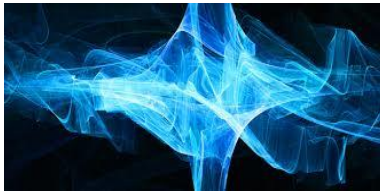
$\phi \; prop$

$$\frac{d \in \exists_{x \in B} \alpha(x) \quad m(x,y) \in \phi \, [x \in B, y \in \alpha(x)]}{El_{\exists}(d,m) \in \phi}$$

proof-relevant version of **usual intuitionistic** **existential quantifier elimination**

RESTRICTED to propositions only (NOT dependent on $\exists$) and **NOT towards all types**!

two notions of function in Coq



a *primitive notion* of type-theoretic function

$$f(x) \in B \; [x \in A]$$

$\neq$ (syntactically)

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

$\Rightarrow$ NO axiom of unique choice in **Coq**

Within **Dependent type theory** in general 

at least **THREE different Existential quantifiers:**

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

$\Rightarrow$ **Axiom/Rule of choice holds**

2. the **h-propositional existential quantifier**

identified with **the truncated Martin-L"of's existential quantifier**

$\exists x \in A\ \phi = |\Sigma x \in A\ \phi|$ as an **h-prop**

$\Rightarrow$ **Axiom/Rule of UNIQUE choice holds**

because $\exists! x \in A\ \phi \ = \ \Sigma x \in A\ \phi$

3. **Intuitionistic existential quantifier** defined

as in **Coq** or as in the **Minimalist Foundation**

$\Rightarrow$ **NO Axiom/Rule of UNIQUE choice holds**

3 different notions of existential quantifiers in categorical logic

from the more specifc to the more general

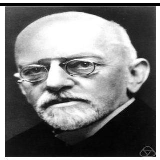
1. The **Weak subobject** doctrine $\Phi : \mathcal{C}^{OP} \longrightarrow \mathsf{InfSL}$ of any finite product category $\mathcal{C}$ with weak equalizers $\mathbf{\Phi}(A) \equiv$ **Posetal reflection**$(\mathcal{C}/A)$
2. The **Subobject doctrine** $\mathrm{Sub} : \mathcal{C}^{OP} \longrightarrow \mathsf{InfSL}$ of any **regular category** $\mathcal{C}$
3. Any **existential doctrine** $P : \mathcal{C}^{OP} \longrightarrow \mathsf{InfSL}$ of any **finite product category** $\mathcal{C}$ examples **not generally in 2. and 3.**:   **Strong subobject doctrine** $\mathrm{Stsub} : \mathcal{C}^{OP} \longrightarrow \mathsf{InfSL}$ of a genuine **quasi-topos** $\mathcal{C}$ **not a topos**

Plurality of foundations ⇒ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory **HoTT** and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the **Minimalist Foundation MF**

our **foundational** approach

as a **revised Hilbert program**: 

we need of a **trustable** foundation for **mathematics**

compatible with **most relevant foundations**

$\Downarrow$

predicative à la **Weyl** 

constructive à la **Bishop** 

open-ended to further extensions according to **Martin-Löf** 

for **computed-aided formalization of its proofs** as advocated by **V. Voevodsky** 

Notion of compatibility between theories

a theory T_1 is compatible with a theory T_2

iff

there is a translation $i\colon T_1 \longrightarrow T_2$

preserving the meaning of logical and set-theoretic operators

Notion of compatibility between theories

a theory T_1 is **compatible** with a theory T_2

iff

there is a translation $i\colon T_1 \longrightarrow T_2$

preserving the meaning of logical and set-theoretic operators

Examples:

Intuitionistic logic is **compatible** with **Classical logic**

Classical logic is **NOT compatible** with **Intuitionistic logic**

Our TWO-LEVEL Minimalist Foundation

from [Maietti'09] in agreement with [M. Sambin2005]

its intensional level **mTT**

Minimalist Type Theory

= a PREDICATIVE VERSION of Coquand's Calculus of Constructions (Coq).

= first order Martin-Löf's intensional type theory + primitive propositions

+ **one UNIVERSE of small propositions**

its extensional level **emTT**

extensionalMinimalist Type Theory

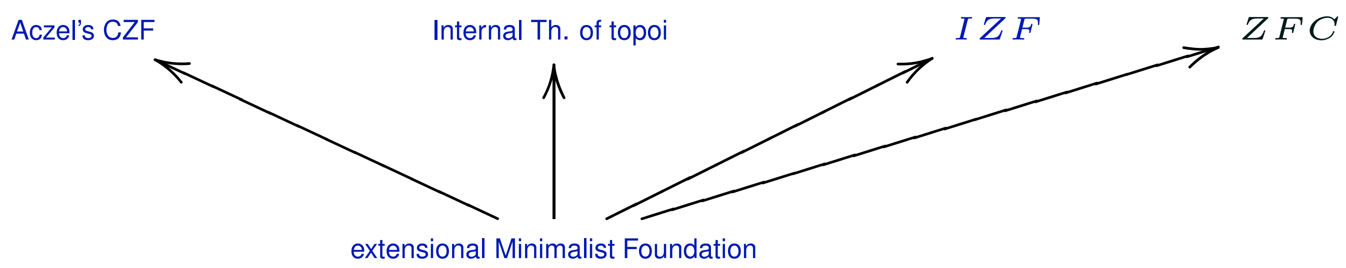
has a PREDICATIVE LOCAL set theory

(NO choice principles)

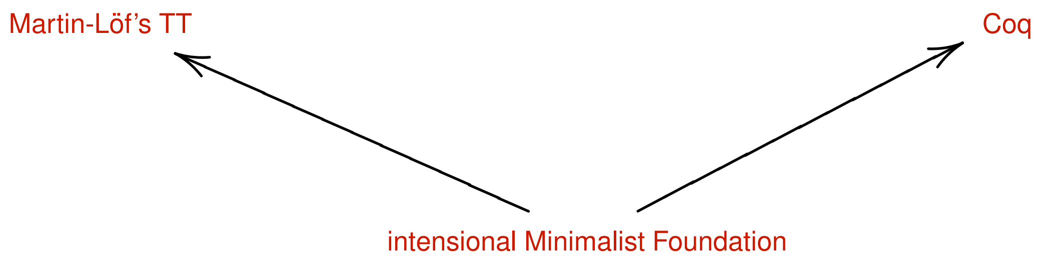
Why **two-levels** *in* **MF** *? for compatibility!*

COMPARE

extensional theories with our extensional level **emTT**



intensional theories with our intensional level **mTT**



crucial use of category theory

to interpret the extensional level in the intensional one

need of a **quotient model** over the intensional level

as a **elementary QUOTIENT COMPLETION** of a **Lawvere**'s elementary doctrine

expressed in the language of **CATEGORY THEORY**

[M.-Rosolini'12] "Quotient completion for the foundation of constructive mathematics", Logica Universalis

[M.-Rosolini'13] ""Elementary quotient completion", TAC

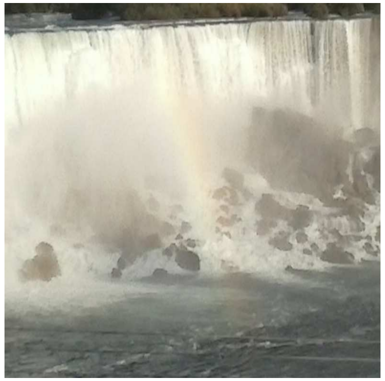
+ cfr. other papers with F. Pasquali, D. Trotta

+ PhD thesis by C. Cioffo

our notion of Constructive Foundation combines different languages

language of **LOCAL** **AXIOMATIC SET THEORY**	for extensional level
language of **CATEGORY THEORY**	**algebraic structure** to link intensional/extensional levels via a **quotient completion**
language of TYPE THEORY	for intensional level
a **computational** language	for **a realizability model**- extra auxiliary level for **programs-extractions** from **proofs**

Why two-levels in MF? to distinguish various forms Axiom of Choice



EXTENSIONAL level **emTT**: Zermelo axiom of choice

formulated as AC

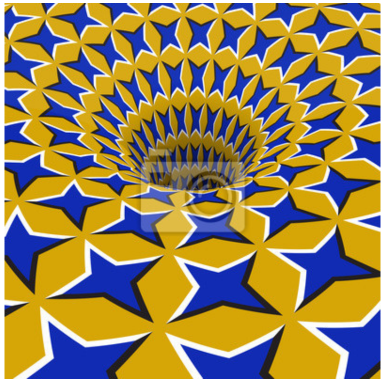
$\Downarrow$

INTENSIONAL level **mTT** : Martin-Löf's extensional axiom of choice

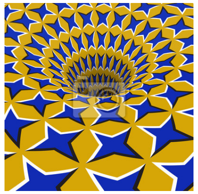
Axiom of choice

$$\forall x \in A\, \exists y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

a total relation contains the graph of a type-theoretic function.



What corresponds to Martin-Löf's Axiom of Choice

Extensional level of **MF**-theory	**Axiom of unique choice**
	$\Updownarrow$ 
Intensional level of **MF**:	**Martin-Löf**'s **axiom of choice**

Motivation:

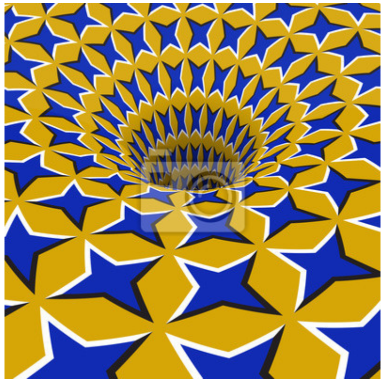
the validity of the rule of **unique choice** *characterizes* **exact completions**

among the **elementary quotient completions** of a *Lawvere's elementary doctrine*

and this holds

iff

the starting **Lawvere doctrine** satisfies a **rule of choice**



in [Maietti-Rosolini 2016] "Relating quotient completions...."

key application of HoTT

Compatibility of both levels of **MF** with **HoTT**

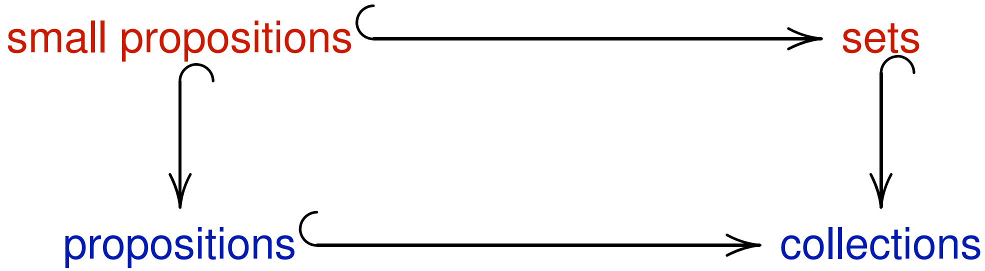
emTT → HoTT

mTT → HoTT

within

[M. Contente, M.E. Maietti 23] The Compatibility of the Minimalist Foundation with Homotopy Type Theory. Arxiv March 2023

ENTITIES in the Minimalist Foundation



Compatibility of the intensional level mTT with HoTT

mTT is compatible with **HoTT**

by interpreting each **mTT**-type as a **h-set** with a chosen proof for IsProp or IsSet

mTT-propositions	$\mapsto$	**HoTT**-**propositions**
mTT-small propositions	$\mapsto$	**h-propositions** in the first universe U_0
mTT-sets	$\mapsto$	**HoTT**-sets in the first universe U_0
mTT-collections	$\mapsto$	**HoTT**–sets
mTT universe of small propositions $\mathsf{prop_s}$	$\mapsto$	**HoTT**-universe $Prop_0$ of **h-propositions** in the first universe U_0

$(-)^J$: Raw-syntax $(mTT) \longrightarrow$ Raw-syntax $(HoTT)$

$(A\ set\ [\Gamma])^J$	is defined as	$A^J : \mathcal{U}_0\ [\Gamma^I]$ such that $\mathsf{pr}_\mathsf{S}(A^J) : \mathsf{IsSet}(A^J)$ is derivable
$(A\ col\ [\Gamma])^{J\mathsf{pr}_\mathsf{P}}$	is defined as	$A^J : \mathcal{U}_1\ [\Gamma^I]$ such that $\mathsf{pr}_\mathsf{S}(A^J) : \mathsf{IsSet}(A^J)$ is derivable
$(P\ \mathit{prop}_s\ [\Gamma])^J$	is defined as	$P^J : \mathcal{U}_0\ [\Gamma^I]$ such that $\mathsf{pr}_\mathsf{P}(P^J) : \mathsf{IsProp}(P^J)$ is derivable
$(P\ \mathit{prop}\ [\Gamma])^J$	is defined as	$P^J : \mathsf{U}_1\ [\Gamma^I]$ such that $\mathsf{pr}_\mathsf{P}(P^J) : \mathsf{IsProp}(P^J)$ is derivable
$(A = B\ set\ [\Gamma])^J$	is defined as	$(A^J, \mathsf{pr}_\mathsf{S}(A^J)) \equiv (B^J, \mathsf{pr}_\mathsf{S}(B^J)) : \mathsf{Set}_{\mathcal{U}_0}\ [\Gamma^I]$
$(A = B\ col\ [\Gamma])^J$	is defined as	$(A^J, \mathsf{pr}_\mathsf{S}(A^J)) \equiv (B^J, \mathsf{pr}_\mathsf{S}(B^J)) : \mathsf{Set}_{\mathcal{U}_1}\ [\Gamma^I]$
$(P = Q\ \mathit{prop}_s\ [\Gamma])^J$	is defined as	$(P^J, \mathsf{pr}_\mathsf{P}(P^J)) \equiv (Q^J, \mathsf{pr}_\mathsf{P}(Q^J)) : \mathsf{Prop}_{\mathcal{U}_0}\ [\Gamma^I]$
$(P = Q\ \mathit{prop}\ [\Gamma])^J$	is defined as	$(P^J, \mathsf{pr}_\mathsf{P}(P^J)) \equiv (Q^J, \mathsf{pr}_\mathsf{P}(Q^J)) : \mathsf{Prop}_{\mathcal{U}_1}\ [\Gamma^I]$
$(a \in A\ [\Gamma])^J$	is defined as	$a^J : A^J\ [\Gamma^I]$
$(a = b \in A\ [\Gamma])^J$	is defined as	$a^J \equiv b^J : A^J\ [\Gamma^I]$

Compatibility of the extensional level emTT of MF with HoTT

emTT with equality reflection is compatible with **HoTT**

by interpreting **up to canonical isomorphisms**

emTT-propositions	$\mapsto$	**HoTT**-*propositions*
emTT-small propositions	$\mapsto$	**HoTT**-*propositions* in the first universe U_0
emTT-sets	$\mapsto$	**HoTT**-sets in the first universe U_0
emTT-quotients	$\mapsto$	**HoTT**-quotient sets in the first universe U_0
emTT-collections	$\mapsto$	**HoTT**-sets
emTT extensional universe $\mathcal{P}(1)$ of small propositions	$\mapsto$	**HoTT**-universe $Prop_0$ of *propositions* in the first universe U_0
definitional equality of **emTT**-types	$\mapsto$	propositional equality of **HoTT**-sets
definitional equality of **emTT**-terms	$\mapsto$	propositional equality of **HoTT**-terms

Canonical isomorphisms

as an **HoTTinductive type** among those h-sets interpreting **MF**-types such that

Canonical isomorphisms

- **preserve canonical elements of type constructors involved**

-are **closed under compositions** and **identities**

(but NOT all identities of **HoTT**-types are included to avoid to include all isos by univalence!)

- are **at most one** between two given **MF h-sets**

$\Rightarrow$ we can define an **H-category** $\mathsf{Set}_{mf} / \simeq_c$ quotiented under **Canonical isomorphisms** within **HoTT**

as in (but without setoids)

[Maietti2009] A minimalist two-level foundation for constructive mathematics. Annals of Pure and Applied Logic

[Hofmann95] M. Hofmann. Conservativity of equality reflection over intensional type theory.(**canonical isos but with AC**)

alternative approaches in: N. Oury 2005 and T. Winterhalter, M. Sozeau, and N. Tabareau 2019

(**with an heterogenous equality**)

Interpretation of extensional judgements of emTT in HoTT

an extensional dependent set	$\rightsquigarrow^I$	dependent h-set

$$(B\ set\ [\,\Gamma\,])^I \ \equiv\ B^I\ [\Gamma^I]$$

up to **canonical isomorphisms**

via a multi-function on **MF**-types and terms

extensional set equality	$\rightsquigarrow^I$	canonical isomorphism

$(B = C\ set\ [\Gamma])^I \ \equiv$ there exists $\tau_{B^I}^{C^I} : B^I \rightarrow C^I$ under Γ^I

$\Rightarrow \quad B^I =_{U_1} C^I$ **by univalence**

extensional dependent term	$\rightsquigarrow^I$	intensional term + can. iso

$$(b \in B\ [\Gamma])^I \ \equiv\ {\tau_D}^{B^I}(b^I) \in B^I\ [\,\Gamma^I\,]$$

extensional definitional term equality	$\rightsquigarrow^I$	proof of equivalence

$(b = c \in B\ [\Gamma])^I \ \equiv$ there exists $p \in {\tau_D}^{B^I}(b^I) =_{B^I} {\tau_C}^{B^I}(c^I) \quad [\,\Gamma^I\,]$

Conservativity *over* first-order logic

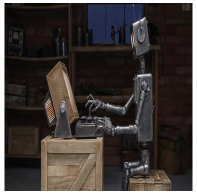
MF inherits **conservativity** over **first order intuitionistic logic**

by **its compatibility** with **the internal theory of a topos**

Question:

Is **HoTT** conservative over first order intuitionistic logic??

MF is strictly predicative a' la Feferman



the **intensional level** **mTT** of **MF** has a **realizability model** for **program-extraction**:

as an extension of Kleene realizability validating

Formal Church Thesis + **Axiom of choice**

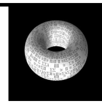
formalized in **Feferman's theory** $\widehat{ID_1}$

in

H. Ishihara, M.E.M., S. Maschio, T. Streicher

Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice , AML, 2018

HoTT + *excluded middle becomes impredicative*

Theorem: **Homotopy type theory + classical logic** 

becomes IMPREDICATIVE

because the category of h-ses **in the first universe** U_0 becomes a topos

---

Proof. the **h-set** $\Pi_{x \in Nat}$ Bool becomes **the power-set** $\mathcal{P}(Nat)$

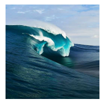
since **HoTT** validates exponentiation of **functional relations**

for impredicativity: recall that the power-object is closed

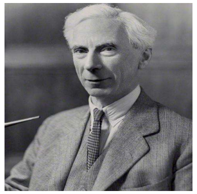
under a subset defined **impredicativily** definition

by a quantification over all subsets

(including itself!!!!)

such as any $\{x \in Nat \;\mid\; \forall\, U \in \mathcal{P}(Nat)\; \phi(x, U)\;\} \,\varepsilon\, \mathcal{P}(Nat)$ 

Characteristics of predicative definitions

in the sense of *Russell-Poincarè*  

> “Whatever involves an apparent variable
>
> must *not be among the possible values* of that variable.”

classical **predicative** *mathematics is* **viable**

according to **Hermann Weyl** 

... **the continuum**... *cannot at all be battered into a single set of elements*.

Not *the relationship of an element to a set*,

but *of a part to a whole* ought to be taken as a basis for the analysis of a continuum.

modern confirmation: **Friedman -Simpson**'s program

"*most basic classical mathematics* can be founded **predicatively**"

*Addition of classical logic to **MF** keeps predicative features à la Weyl*

in **MF** + **classical logic**:
power-objects $\mathcal{P}(Nat)$ is NOT a set + **Dedekind reals =Cauchy real numbers** are NOT sets

As a consequence of NO choice principles in MF

$$\Downarrow$$

three distinct notions of real numbers:

Bishop reals =regular Cauchy sequences à la Bishop as **typed-terms**
$\neq$ (NO axiom of unique choice in **MF**)
Cauchy reals =regular Cauchy sequences as **functional relations**
$\neq$ (NO countable choice in **MF**)
Dedekind reals =**Dedekind cuts (lower + upper)**

As a consequence of low proof-theoretic strength of MF

$\Downarrow$

three distinct notions of real numbers:

Bishop reals =regular Cauchy sequences à la Bishop as **typed-terms** **Yes, a MF-set**
Cauchy reals =regular Cauchy sequences as **functional relations** **a NON MF-set**
Dedekind reals =Dedekind cuts (lower + upper) **a NON MF-set**

why Dedekind reals do NOT form a set in emTT + classical logic

we model emTT/mTT +**excluded middle**

in the quasi-topos of **assemblies**

within **Hyland's Effective topos**

the category of assemblies

assembly	(X, ϕ) with X set and $\phi \subseteq X \times Nat$ a *total relation* from X to Nat
assembly morphism	$(\mathbf{f}, \mathbf{m})$: $(\mathbf{X}, \phi) \rightarrow (\mathbf{Y}, \psi)$ with $\mathbf{f}$: $\mathbf{X} \rightarrow \mathbf{Y}$ and $\mathbf{m} \in \mathbf{Nat}$ such that $\mathbf{m}$ **tracks** f i.e. for all $\mathbf{x} \in \mathbf{X}$ and $\mathbf{n} \in \mathbf{Nat}$ if $\mathbf{x} \, \phi \, \mathbf{n}$ then $\mathbf{f}(\mathbf{x}) \, \psi \, \{\mathbf{m}\}(\mathbf{n})$
morphism equality $(f, \mathbf{m}) = (g, \mathbf{m}')$	iff $f = g$ as functions

the interpretation of emTT in the quasi-topos of assemblies

emTT entities	**their semantics**
emTT sets	assemblies (X, ϕ) with X countable
operations between sets	assemblies morphisms
propositions	strong monomorphisms of assemblies
proper collections (= NO sets)	assemblies (X, ϕ) with X not countable

from the model of emTT in assemblies within Eff

corollary:

- **axiom of unique choice between natural numbers is NOT valid in emTT/mTT**
- **Cauchy reals and Dedekind reals of emTT are NOT emTT-sets but only emTT-collections.**
- $\mathcal{P}(Nat)$ **is not an emTT-set but only an emTT-collection.**

Proof. The mentioned **reals** are interpreted as NOT countable assemblies!

Conclusion

HoTT has a remarkable expressive power

as a **dependent type theory**

able to interpret BOTH levels of **the Minimalist Foundation**

because of set quotients + **univalence**

but INCOMPATIBLE with **classical predicativity**

for its existential quantifier of regular logic

the Minimalist Foundation is **strictly predicative a la Weyl**

Dedekind real numbers do not form a set

even with the addition of **classical logic**!

for its intuitionistic existential quantifier

primitively defined over dependent type theory

Open issues

- Extend **compatibility** with **HoTT** + **Palmgren's superuniverse**

  to **MF** +**inductive-coinductive** **topological definitions**

  (cfr work with Maschio-Rathjen (2021-2022) and with P. Sabelli (2023))

- **Equiconsistency** of **the Minimalist Foundation** with its **classical counterpart**

- Extend **compatibility** with **HoTT** to **MF** +**classical logic**