Unifying Cubical Models of Homotopy Type Theory

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Unifying cubical models of HoTT

There is by now quite a zoo of cubical models:

BCH, CCHM, CHM, AFH, ABCFHL, Dedekind cubes, Orton-Pitts cubes, cubical assemblies, equivariant cubes...

How are these models related?

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How are these models related?

Evan Cavallo, Andrew Swan and I have found a new cubical model that generalizes (most of) the existing cubical models

`https://github.com/mortberg/gen-cart/blob/master/conference-paper.pdf`

(To appear in *Computer Science Logic* 2020)

Univalent and Homotopy Type Theory

In this talk:

$$\text{Univalent Type Theory} = \text{MLTT} + \text{Univalence}$$
$$\text{Homotopy Type Theory} = \text{UTT} + \text{Higher Inductive Types}$$

Theorem (Voevodsky, Kapulkin-Lumsdaine)

Univalent Type Theory has a model in Kan simplicial sets

Problem: inherently classical, how to make this constructive?

This problem motivated the use of cubical methods in HoTT

Cubical methods in HoTT

The cubical models can be developed in a constructive metatheory and have led to:

- cubical type theories,
- proof assistants with native support for HoTT,
- (homotopy) canonicity results,
- proof theoretic strength of the univalence axiom,
- independence results,
- new proofs of results in synthetic homotopy theory,
- ...

Cubical methods in HoTT

This talk:

1. Overview of cubical models of HoTT
2. Our generalization
3. A model structure constructed from the model

Our generalization is expressed in the internal language of a LCCC extended with axioms and has been (mostly) formalized in Agda

Model structures and models of HoTT

An interesting difference in how the simplicial and cubical models have been developed is that we reverse the direction of:

$$\text{Model structure on simplicial sets} \longrightarrow \text{Model of HoTT}$$

to

$$\text{Cubical model of HoTT} \longrightarrow \text{Model structure}$$

Furthermore, the obtained model structure is constructive

Part I: Cubical models of HoTT

Cubical Methods: BCH

The first breakthrough in finding constructive justifications to UTT was:

Theorem (Bezem-Coquand-Huber, 2013)

Univalent Type Theory has a constructive model in "substructural" Kan cubical sets ("BCH model").

This led to development of a variety of cubical set models

$$\widehat{\square} = [\square^{\mathrm{op}}, \mathbf{Set}]$$

Cubical Methods: CCHM

Inspired by BCH we constructed a model based on "structural" cubical sets with connections and reversals:

Theorem (Cohen-Coquand-Huber-M., 2015)

Univalent Type Theory has a constructive model in De Morgan Kan cubical sets ("CCHM model").

We also developed a **cubical type theory** in which we can prove and compute with the **univalence theorem**

$$\mathsf{ua} : (A\ B : \mathcal{U}) \to (\mathsf{Path}_{\mathcal{U}}\ A\ B) \simeq (A \simeq B)$$

Cubical Methods: cartesian models

In parallel with our developments in Sweden many people at CMU were working on models based on *cartesian* cubical sets

These have nice properties compared to CCHM cubes (Awodey, 2016)

The crucial idea for constructing univalent universes in cartesian cubical sets was found by Angiuli, Favonia, and Harper (AFH, 2017) when working on computational cartesian cubical type theory. This then led to:

Theorem (Angiuli-Brunerie-Coquand-Favonia-Harper-Licata, 2017)

Univalent Type Theory has a constructive model in cartesian Kan cubical sets ("ABCFHL model").

Cubical Methods: cubical assemblies

Building on CCHM and the work of Orton-Pitts, Taichi Uemura has constructed yet another cubical model:

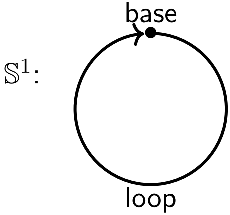
Theorem (Uemura, 2018)

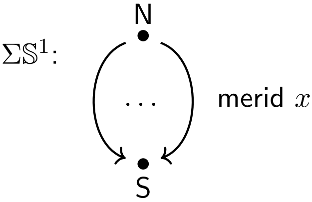
Cubical type theory extended with an impredicative univalent universe has a model in cubical assemblies

Uemura used this to prove independence of a form of propositional resizing. This model has also been extended to prove the independence of Church's thesis (Swan-Uemura, 2019)

Higher inductive types (HITs)

Types generated by point and path constructors:





These types are added axiomatically to HoTT and justified semantically[1] in "sufficiently nice model categories", e.g. Kan simplicial sets (Lumsdaine-Shulman, 2017)

[1]Modulo issues with universes...

Higher inductive types

The cubical set models also support HITs:[2]

- De Morgan cubes: CCHM (2015), Coquand-Huber-M. (CHM, 2018)
- Cartesian cubes: Cavallo-Harper (2018)
- BCH: as far as I know not known even for $\mathbb{S}^1$, problems related to $\mathsf{Path}(A) := \mathbb{I} \multimap A$

The CHM construction has been analyzed and generalized so that it applies to e.g. cubical assemblies (Swan-Uemura, 2019)

[2] Without universe issues.

Cubical Type Theory

The cubical models hence model HoTT and there are multiple cubical type theories inspired by these models, but what makes a type theory *cubical*?

Cubical Type Theory

The cubical models hence model HoTT and there are multiple cubical type theories inspired by these models, but what makes a type theory *cubical*?

Add a formal interval $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i$$

Extend the contexts to include interval variables:

$$\Gamma \quad ::= \quad \bullet \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I}$$

Proof theory	Semantics
$$\frac{\Gamma, i : \mathbb{I} \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}(\epsilon/i)} \text{ FACE}$$	$$\Gamma \xrightarrow{d^i_\epsilon} \Gamma, i : \mathbb{I}$$
$$\frac{\Gamma \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}} \text{ WEAKENING}$$	$$\Gamma, i : \mathbb{I} \xrightarrow{\sigma_i} \Gamma$$
$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \mathcal{J}} \text{ EXCHANGE}$$	$$\Gamma, j : \mathbb{I}, i : \mathbb{I} \xrightarrow{\tau_{i,j}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$
$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}(j/i)} \text{ CONTRACTION}$$	$$\Gamma, i : \mathbb{I} \xrightarrow{\delta_{i,j}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$

Cubical Type Theory

All cubical set models have face maps, degeneracies and symmetries

BCH does not have contraction/diagonals, making it substructural

The cartesian models have contraction/diagonals, making them a simpler basis for cubical type theory

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We can also consider additional structure on $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid r \wedge s \mid r \vee s \mid \neg r$$

Axioms: connection algebra (Orton-Pitts model), distributive lattice (Dedekind model), De Morgan algebra (CCHM model), Boolean algebra...

Varieties of Cubical Sets - Buchholtz, Morehouse (2017)

Kan operations / fibrations

To get a model of HoTT we also need to equip all types with **Kan operations**: any open box can be filled

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Given a specified subset (r, s) of $\mathbb{I} \times \mathbb{I}$ we add operations:

$$\frac{\begin{array}{c}\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \Phi \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]\end{array}}{\Gamma \vdash \mathsf{com}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 : A(s/i)[\varphi \mapsto u(s/i), (r = s) \mapsto u_0]}$$

Semantically this corresponds to fibration structures

The choice of which (r, s) to include varies between the different models

Cube shapes / generating cofibrations

Another parameter: which shapes of open boxes are allowed (Φ)

Semantically this corresponds to specifying the generating cofibrations, typically these are classified by maps into Φ where Φ is taken to be a subobject of $\Omega : \widehat{\square}$

The crucial idea for supporting univalent universes in AFH was to include "*diagonal cofibrations*" – semantically this corresponds to including $\Delta_{\mathbb{I}} : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ as a generating cofibration

Cubical set models of HoTT

	Structural	$\mathbb{I}$ operations	Kan operations	Diag. cofib.
BCH			$0 \to r$, $1 \to r$	
CCHM	✓	$\wedge$, $\vee$, $\neg$ (DM alg.)	$0 \to 1$	
Dedekind	✓	$\wedge$, $\vee$ (dist. lattice)	$0 \to 1$, $1 \to 0$	
Orton-Pitts	✓	$\wedge$, $\vee$ (conn. alg.)	$0 \to 1$, $1 \to 0$	
AFH, ABCFHL	✓		$r \to s$	✓
Cubical assemblies	✓	$\wedge$, $\vee$ (conn. alg.)	$0 \to 1$, $1 \to 0$	

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This work: cartesian cubical set model without diagonal cofibrations

Key idea: don't require the $(r = s)$ condition in com strictly, but only up to a path

Part II: generalizing cubical models

Orton-Pitts internal language model

We present our generalization in the internal language of $\widehat{\square}$ following

Axioms for Modelling Cubical Type Theory in a Topos
Orton, Pitts (2017)

We also formalize it in AGDA and for univalent universes we rely on[3]

Internal Universes in Models of Homotopy Type Theory
Licata, Orton, Pitts, Spitters (2018)

[3] Disclaimer: only on paper so far, not yet formalized.

Orton-Pitts style internal language model

In fact, none of the constructions rely on the subobject classifier $\Omega : \widehat{\square}$, so we work in the internal language of a LCCC $\mathcal{C}$ and do the following:

1. Add an interval $\mathbb{I}$
2. Add a type of cofibrant propositions Φ
3. Define fibration structures
4. Prove that fibration structures are closed under Π, Σ and Path
5. Define univalent fibrant universes of fibrant types
6. Prove that this gives rise to a Quillen model structure

(Disclaimer: parts of the last two steps are not (yet) internal)

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The interval $\mathbb{I}$

The axiomatization begins with an interval type

$$\mathbb{I} : \mathcal{U} \qquad\qquad 0 : \mathbb{I} \qquad\qquad 1 : \mathbb{I}$$

satisfying

$$\begin{aligned} \mathbf{ax}_1 &: (P : \mathbb{I} \to \mathcal{U}) \to ((i : \mathbb{I}) \to P\ i \uplus \neg(P\ i)) \to \\ &\qquad ((i : \mathbb{I}) \to P\ i) \uplus ((i : \mathbb{I}) \to \neg(P\ i)) \\ \mathbf{ax}_2 &: \neg(0 = 1) \end{aligned}$$

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Cofibrant propositions

We also assume a universe à la Tarski of generating cofibrant propositions

$$\Phi : \mathcal{U} \qquad\qquad [\_] : \Phi \to \texttt{Prop}$$

with operations

$$(\_ \approx 0) : \mathbb{I} \to \Phi \qquad\qquad \vee : \Phi \to \Phi \to \Phi$$
$$(\_ \approx 1) : \mathbb{I} \to \Phi \qquad\qquad \forall : (\mathbb{I} \to \Phi) \to \Phi$$

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satisfying

$$\mathbf{ax}_3 : (i : \mathbb{I}) \to [\,(i \approx 0)\,] = (i = 0)$$
$$\mathbf{ax}_4 : (i : \mathbb{I}) \to [\,(i \approx 1)\,] = (i = 1)$$
$$\mathbf{ax}_5 : (\varphi\,\psi : \Phi) \to [\,\varphi \vee \psi\,] = [\,\varphi\,] \vee [\,\psi\,]$$
$$\mathbf{ax}_6 : (\varphi : \Phi)\,(A : [\,\varphi\,] \to \mathcal{U})\,(B : \mathcal{U})\,(s : (u : [\,\varphi\,]) \to A\ u \cong B) \to \Sigma(B' : \mathcal{U}), \Sigma(s' : B' \cong B), (u : [\,\varphi\,]) \to (A\ u, s\ u) = (B', s')$$
$$\mathbf{ax}_7 : (\varphi : \mathbb{I} \to \Phi) \to [\,\forall\varphi\,] = (i : \mathbb{I}) \to [\,\varphi\, i\,]$$

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Partial elements

A *partial element* of A is a term $f : [\,\varphi\,] \to A$

Given such a partial element f and an element $x : A$, we define the *extension* relation

$$f \nearrow x \triangleq (u : [\,\varphi\,]) \to f\ u = x$$

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We write

$$A[\varphi \mapsto f] \triangleq \Sigma(x : A), f \nearrow x$$

Given $f : [\,\varphi\,] \to \mathsf{Path}(A)$ and $r : \mathbb{I}$ we write

$$f \cdot r \triangleq \lambda u.f\ u\ r : [\,\varphi\,] \to A\ r$$

Weak fibration structures

Given $r : \mathbb{I}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\,\varphi\,] \to \mathsf{Path}(A)$ and $x_0 : (A\ r)[\varphi \mapsto f \cdot i]$, a *weak composition structure* is given by two operations

$$\mathsf{wcom} : (s : \mathbb{I}) \to (A\ s)[\varphi \mapsto f \cdot s]$$

$$\underline{\mathsf{wcom}} : \mathsf{fst}\,(\mathsf{wcom}\ r) \sim \mathsf{fst}\,x_0$$

satisfying $(i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}}\ i$.

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\end{aligned}$$

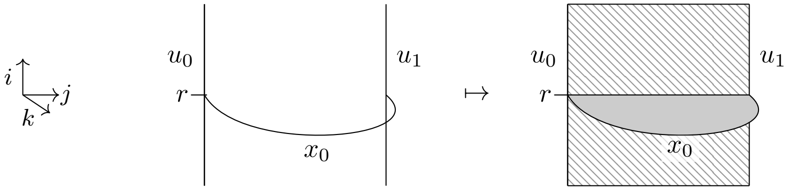
satisfying $(i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}}\ i$.

A *weak fibration* (A, α) over $\Gamma : \mathcal{U}$ is a family $A : \Gamma \to \mathcal{U}$ equipped with

$$\begin{aligned}
\mathsf{isFib}\ A \triangleq (r : \mathbb{I})\ (p : \mathbb{I} \to \Gamma)\ (\varphi : \Phi)\ (f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i)) \\
(x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \to \mathsf{WComp}\ r\ (A\ \circ\ p)\ \varphi\ f\ x_0
\end{aligned}$$

Example: weak composition

Given u_0 and u_1 at $(j \approx 0)$ and $(j \approx 1)$ together with $x_0 : A\ r$ at $(i \approx r)$, the weak composition and path from r to i is



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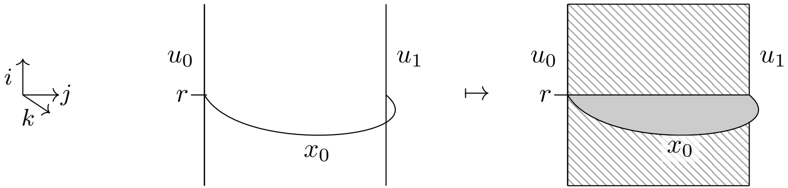


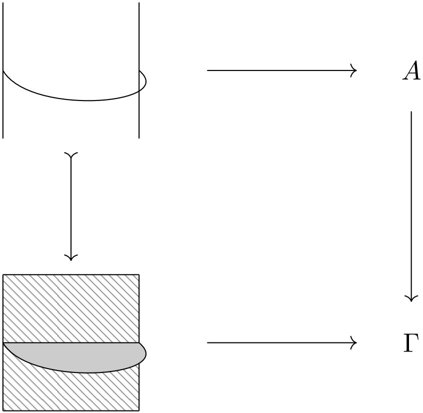
With suitable notations:

$$\mathsf{wcom}_A^{r \to i}\ [(j \approx 0) \mapsto u_0, (j \approx 1) \mapsto u_1]\, x_0 : A\ i$$

$$\underline{\mathsf{wcom}}_A^{r;i}\ [\varphi \mapsto f]\, x_0 : \mathsf{wcom}_A^{r \to i}\ [(j \approx 0) \mapsto u_0, (j \approx 1) \mapsto u_1]\, x_0 \sim x_0$$

Weak fibration structures diagrammatically

We can also see these operations as a lifting diagram:



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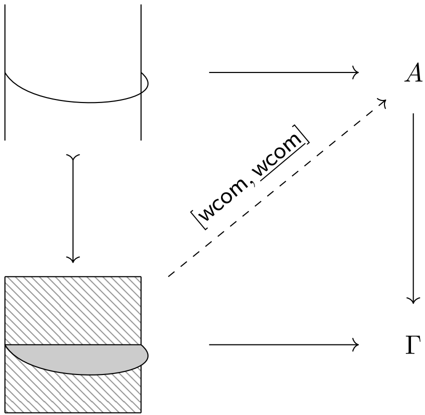

Orton-Pitts style internal language model

1. Add an interval $\mathbb{I}$
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3. Define fibration structures
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A model of HoTT based on weak fibrations

Using $\mathbf{ax}_1 - \mathbf{ax}_5$ we can prove that isFib is closed under Σ, Π, Path and that natural numbers are fibrant if $\mathcal{C}$ has a NNO

The proofs are straightforward adaptations of the AFH/ABCFHL proofs, but extra care has to be taken to compensate for the weakness

Semantically closure of isFib under Π corresponds to the "Frobenius property" (pullback along fibrations preserve trivial cofibrations)

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A model of HoTT based on weak fibrations

Following Orton-Pitts we can use $\mathbf{ax}_6$ to define Glue types and using $\mathbf{ax}_7$ we can prove that they are also fibrant (by far the most complicated part)

Semantically this corresponds to the "Equivalence Extension Property": equivalences between fibrations extend along cofibrations

Theorem (Universe construction, LOPS)

If $\mathbb{I}$ is tiny, then we can construct a universe U *with a fibration* El *that is classifying in the sense of LOPS Theorem 5.2*

A model of HoTT based on weak fibrations

The model also supports identity types (3 different constructions in the formalization) and higher inductive types

We hence get a class of models of HoTT based on cartesian cubical sets with weak fibrations, without using diagonal cofibrations

What is the relationship to the other models?

AFH fibrations

As in AFH and ABCFHL we can define

$$\text{isAFHFib } A \triangleq (r : \mathbb{I})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))$$
$$(x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \to \text{AFHComp } r\ (A \circ p)\ \varphi\ f\ x_0$$

If we assume diagonal cofibrations

$$(\_ \approx \_) : \mathbb{I} \to \mathbb{I} \to \Phi$$
$$\mathbf{ax}_{\Delta} : (r\ s : \mathbb{I}) \to [\,(r \approx s)\,] = (r = s)$$

then we can prove

Theorem

Given $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$*, we have* isAFHFib A *iff we have* isFib A*.*

CCHM fibrations

Inspired by Orton-Pitts we can define:

$$\text{isCCHMFib}\ A \triangleq (\varepsilon : \{0,1\})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))$$
$$(x_0 : A(p\ \varepsilon)[\varphi \mapsto f \cdot r]) \to \text{CCHMComp}\ \varepsilon\ (A\ \circ\ p)\ \varphi\ f\ x_0$$

If we assume a *connection algebra*

$$\begin{aligned}
&\sqcap, \sqcup : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \\
&\mathbf{ax}_{\sqcap} : (r : \mathbb{I}) \to (0 \sqcap r = 0 = r \sqcap 0) \wedge (1 \sqcap r = r = r \sqcap 1) \\
&\mathbf{ax}_{\sqcup} : (r : \mathbb{I}) \to (0 \sqcup r = r = r \sqcup 0) \wedge (1 \sqcup r = 1 = r \sqcup 1)
\end{aligned}$$

then we can prove

Theorem

Given $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$, *we have* isCCHMFib A *iff we have* isFib A.

Cubical set models of HoTT

This hence generalizes the structural models in:

	Structural	$\mathbb{I}$ operations	Kan operations	Diag. cofib.
BCH			$0 \to r$, $1 \to r$	
CCHM	✓	$\wedge$, $\vee$, $\neg$ (DM alg.)	$0 \to 1$	
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Orton-Pitts	✓	$\wedge$, $\vee$ (conn. alg.)	$0 \to 1$, $1 \to 0$	
AFH, ABCFHL	✓		$r \to s$	✓
Cubical assemblies	✓	$\wedge$, $\vee$ (conn. alg.)	$0 \to 1$, $1 \to 0$	
Cavallo-M.-Swan	✓		$r \to s$, weak	

Bonus model: cubical assemblies without connections and diagonal cofibrations

Part III: A model structure

Cubical set models of HoTT

Which of these cubical set models give rise to model structures where the fibrations correspond to the Kan operations?

Theorem (Sattler, 2017)

General construction of model structures using ideas from CCHM model (in particular fibrant universes)

This gives model structures for the cubical sets with connections, it also generalizes to cartesian cubical sets with AFH/ABCFHL fibrations and diagonal cofibrations (Coquand-Sattler, Awodey)

A model of HoTT based on weak fibrations

We also use Sattler's theorem to get a model structure from our cartesian cubical set model without connections and diagonal cofibrations

There are 3 parts involved in proving this:

1. Cofibration - Trivial Fibration awfs
2. Trivial Cofibration - Fibration awfs
3. 2-out-of-3 for weak equivalences

Cofibration-trivial fibration awfs

Cofibrant propositions $[\,-\,] : \Phi \to \texttt{Prop}$ correspond to a monomorphism

$$\top \colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$$

where $\Phi_{\mathsf{true}} \triangleq \Sigma(\varphi : \Phi), [\,\varphi\,] = 1$

Definition (Generating cofibrations)

Let $m \colon A \to B$ be a map in $\mathcal{C}$. We say that m is a *generating cofibration* if it is a pullback of $\top$

Get (C, F^t) awfs by a version of the small object argument

Trivial cofibration-fibration awfs

Given $m : A \to B$ we write $A \xrightarrow{L(m)} \mathsf{Cyl}(m) \xrightarrow{R(m)} B$ for the *mapping cylinder factorization* defined by a suitable pushout

Theorem (Weak fibrations and fibrations)

f is a weak fibration iff it has the right lifting property against the map $L(\Delta) \hat{\times} \top$ in $\mathcal{C}/(\mathbb{I} \times \Phi)$ where Δ is the map $1_{\mathbb{I}\times\Phi} \to \mathbb{I}_{\mathbb{I}\times\Phi}$ defined as the diagonal map $\mathbb{I} \times \Phi \to \mathbb{I} \times \mathbb{I} \times \Phi$

Get (C^t, F) awfs by a version of the small object argument as well

Weak left lifting problems

We say that $m : A \to B$ has the *weak left lifting property* against $f : X \to Y$ if there is a diagonal map as in

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\sim", pos=0.3] & X \arrow[d, "f"] \\ B \arrow[r, "b"] \arrow[ur, dotted] & Y \end{tikzcd}$$

Theorem (Weak fibrations and weak LLP)

f is a weak fibration iff for every object B, every map $r : 1_B \to \mathbb{I}_B$ and generating cofibration $m : A \to B$ in $\mathcal{C}$, r has the weak left lifting property against $\widehat{\mathrm{hom}}_B(B^*(m), f)$.

A model structure based on weak fibrations

By adapting Sattler's theorem we obtain a full model structure

Theorem (Model structure)

Suppose that $\mathcal{C}$ satisfies axioms $\mathbf{ax}_1$–$\mathbf{ax}_5$ and that every fibration is U*-small for some universe of small fibrations where the underlying object* U *is fibrant. Let (C, F^t) and (C^t, F) be the awfs defined above, then C and F form the cofibrations and fibrations of a model structure on $\mathcal{C}$.*

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Theorem (Minimality of the model structure)

The class C^t is as small as possible subject to

1. *For every object B, the map $\delta_{B0} : B \to B \times \mathbb{I}$ belongs to C^t.*
2. *C and C^t form the cofibrations and trivial cofibrations of a model structure.*

Model structure comparison

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What is the relationship to the existing model structures constructed from cubical set models of HoTT?

As the (co)fibrations coincide with the ones in the other model structures we recover them when assuming appropriate additional structure (diagonal cofibrations for cartesian and connections for Dedekind)

We have hence not only generalized the cubical models of HoTT, but also the model structures constructed from these models

Summary

We have:

- Constructed a model of HoTT that generalizes the earlier cubical set models, except for the BCH model
- Mostly formalized in AGDA
- Adapted Sattler's model structure construction to this setting

Future work:

- Formalize the universe construction and model structure in AGDA-$\flat$
- What about BCH? Is it inherently different or does it fit into this generalization?
- Relationship between model structures and the standard one on Kan simplicial sets? Can we also incorporate the equivariant model?

Thank you for your attention!

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