Higher Schreier Theory

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Group Extensions

Definition

An extension of a group G by a group K is a short exact sequence:

$$1 \to K \xrightarrow{i} E \xrightarrow{p} G \to 1.$$

That is, i is injective, p is surjective, and the kernel of p is the image of i.

Example (Carrying Numbers)

$$1 \to \mathbb{Z}_{/10} \xrightarrow{i \mapsto (i,0)} \mathbb{Z}_{/10} \times \mathbb{Z}_{/10} \xrightarrow{(i,j) \mapsto j} \mathbb{Z}_{/10} \to 1$$

$$1 \to \mathbb{Z}_{/10} \xrightarrow{i \mapsto 10 \cdot i} \mathbb{Z}_{/100} \xrightarrow{n \mapsto n \mod 10} \mathbb{Z}_{/10} \to 1$$

Group Extensions

Definition

An extension is *split* if

$$1 \longrightarrow K \xrightarrow{\ i\ } E \underset{p}{\overset{s}{\leftrightarrows}} G \longrightarrow 1$$

there is a section s of p.

Definition

An extension is *central* if

$$\begin{array}{ccccccccc} & & & \nearrow & ZE & & & & \\ & & & & \big\downarrow & & & & \\ 1 & \longrightarrow & K & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & G & \longrightarrow & 1 \end{array}$$

the map i lands in the center of E.

Classifying Group Extensions

Let G and K be groups.

- Split extensions of G by K are classified by homomorphisms $G \to \mathrm{Aut}(K)$ by forming the semi-direct product.
- (Iso classes of) Central extensions of G by K (for K abelian) are classified by $H^2(G; K)$.

Can we classify **all** extensions?

Yes!

- Otto Schreier did in 1926 by giving explicit cocycle conditions. This is known as **Schreier Theory**.

2-Groups and Butterflies

Definition (Axiomatic 2-Group)

A *(axiomatic) 2-group* is a monoidal groupoid G where every object has an inverse up to isomorphism.

Example

The groupoid of 1-dimensional normed complex vector spaces becomes a 2-group under tensor product, with

$$\mathcal{L}^{-1} := [\mathcal{L}, \mathbb{C}]$$

being the linear dual, since

$$\mathsf{ev} : \mathcal{L} \otimes [\mathcal{L}, \mathbb{C}] \cong \mathbb{C}$$

for all $\mathcal{L}$.

2-Groups and Butterflies

Definition (Axiomatic 2-Group)

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Definition (Crossed Module)

A 2-group may be presented by a *crossed module*. A crossed module consists of an action $\alpha : G \to \mathrm{Aut}(H)$ and a G-equivariant homomorphisms $t : H \to G$ satisfying the *Peiffer identity*:

$$\alpha(t(h_1), h_2) = h_1 h_2 h_1^{-1}.$$

2-Groups and Butterflies

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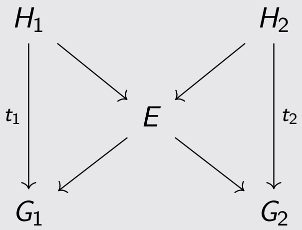
Definition (Butterfly)

A morphism between 2-groups presented by crossed modules is given by a diagram:



in which both diagonal sequences are complexes, and the sequence $H_2 \to E \to G_1$ is short exact, and where the bottom two diagonal morphisms are compatible with the actions α_1 and α_2 in the following

2-Group Reformulations of Schreier Theory

- Schreier theory is expanded and reformulated by Eilenberg and Mac Lane, Dedecker, Grothendieck, Giraud, Breen, and others. In particular, Breen showed that

$$\mathrm{EXT}(G; K)_{/\cong} = H^2(G; K \to \mathrm{Aut}(K))$$

  the (reduced) second cohomology of G valued in the crossed module $K \to \mathrm{Aut}(K)$.
- This crossed module $K \to \mathrm{Aut}(K)$ represents the 2-group of automorphisms of K. This is the groupoid of automorphisms of K and conjugating elements between them.

So, extensions of G by K are classified by
2-group homomorphisms from G to the automorphism 2-group of K

Classifying Extensions of Higher Groups

- In his paper *Théorie de Schreier supérieure*, Breen extends this theory to stacks of 2-groups.
- We will extend it to stacks of higher groups, using homotopy type theory.

Theorem (Higher Schreier Theory)

For higher groups G and K,

$$\{\text{Extensions of } G \text{ by } K\} = \mathrm{Hom}_{\infty\mathrm{Grp}}(\mathrm{G}, \mathrm{Aut}(\mathrm{BK}))$$

Outline:

1. The HoTT approach to higher groups.
2. Proof of the main theorem.
3. Using Higher Schreier Theory.
4. Central Extensions: Subtleties and Directions.

The HoTT perspective on (higher) groups

A (higher) group is the type of self-identifications of a given object $x : X$.

- Instead of axiomatizing the *algebra* of self-identifications, we work with the type of all objects *identifiable* with x:

$$\mathrm{BAut}_X(x) :\equiv (y : X) \times \|y = x\|.$$

We consider this as a *pointed type* with point

$$\mathrm{pt}_{\mathrm{BAut}_X(x)} :\equiv (x, |\mathrm{refl}|).$$

- If G is any axiomatic 1-group, we may take

$$\mathrm{B}G :\equiv \mathrm{BAut}_{G-\mathrm{Act}}(G) = \mathrm{Tors}_G.$$

See *Buchholtz, van Doorn, Rijke* and the *Symmetry Book*.

The HoTT perspective on (higher) groups

A (higher) group is the type of self-identifications of a given object $x : X$.

Definition

A *higher group* is a type G identified with the loop space of a pointed, 0-connected type BG (called its *delooping*):

$$G = (\mathrm{pt}_{BG} = \mathrm{pt}_{BG}).$$

Definition

An $(n+1)$-group is a higher group of symmetries of an n-type object; that is, its delooping is an $(n+1)$-type.

E.g. a 1-group is the group of symmetries of a set-level object, whose delooping is a groupoid.

The HoTT perspective on (higher) groups

Definition

A *homomorphism* $G \to H$ of higher groups is a pointed map $BG \cdot\!\!\to BH$.

Example

The determinant $\mathsf{det} : GL_n \to GL_1$ is given by the n^{th} exterior power $V \mapsto \Lambda^n V : BGL_n \to BGL_1$.

Example

An action of a higher group $\mathsf{Aut}_X(x)$ on a type A is a homomorphism

$$C : B\mathsf{Aut}_X(x) \cdot\!\!\to B\mathsf{Aut}(A).$$

Explictly, this is a construction C which takes a $y : X$ identifiable with x to a type $C(y)$ together with an identification of $C(x)$ with A. E.g:

- The action of GL_n on $\mathbb{R}^n$ is given by $C(V) :\equiv V$, with $\mathsf{refl} : C(\mathbb{R}^n) = \mathbb{R}^n$.

HoTT Perspective versus the Axiomatic Approach

Definition

An *axiomatic* 1-group is a set G equipped with a unit $1 : G$ and binary operation $\mu : G \times G \to G$ which is associative and under which every element has an inverse.

An *axiomatic* 2-group is a monoidal groupoid G where every object has an inverse.

Theorem (Symmetry Book)

The theory of 1-groups (homomorphisms, actions, torsors) is equivalent to the theory of axiomatic 1-groups.

Proof.

Long... □

- We can't prove an analogue generally (in HoTT) until we have a definition of *axiomatic* ∞*-group*.

Extensions of Higher Groups

Example

Minkowski Spacetime

- Lorentz space is a vector space with a $(+,+,+,-)$-norm; that is, it is identifiable with $\mathbb{R}^{3+1}$ with the norm $x^2 + y^2 + z^2 - t^2$. The type of Lorentz spaces is
$$\mathrm{B}\,\mathcal{L} :\equiv \mathrm{BAut}_{\mathrm{Vect}_q}(\mathbb{R}^{3+1}).$$
- Minkowski spacetime is an affine space over a Lorentz space. The type of Minkowski spacetimes is
$$\mathrm{B}\,\mathcal{P} :\equiv (V : \mathrm{B}\,\mathcal{L}) \times \mathrm{B}V.$$
- So, we have a fiber sequence:
$$\mathrm{B}\,\mathbb{R}^{3+1} \cdot\!\!\to \mathrm{B}\,\mathcal{P} \cdot\!\!\to \mathrm{B}\,\mathcal{L}$$

Extensions of Higher Groups

Definition

An extension of a higher group G by a higher group K is a fiber sequence

$$\mathsf{BK} \cdot\!\!\to \mathsf{BE} \cdot\!\!\to \mathsf{BG}.$$

Example

The fiber sequence

$$\mathsf{B}\,\mathbb{R}^{3+1} \cdot\!\!\to \mathsf{B}\,\mathcal{P} \cdot\!\!\to \mathsf{B}\,\mathcal{L}$$

witnesses that the Poincaré group $\mathcal{P}$ is an extension of the Lorentz group $\mathcal{L}$ by the group of translations $\mathbb{R}^{3+1}$.
Note that this fiber sequence is split by the assignment $\mathsf{V} \mapsto (\mathsf{V}, \mathsf{V})$, pointed by reflexivity; this witnesses that the Poincaré group is a semi-direct product.

Pointed families

Definition

A **pointed family** of types (E, pt_E) on a pointed type B is a family of types $E : B \to \mathbf{Type}$ with a point $pt_E : E_{pt_B}$ in the fiber over pt_B.
The type of pointed sections of a pointed family is

$$(b : B) \cdot\!\!\to E_b :\equiv (f : (b : B) \to E_b) \times (f\,pt_B = pt_E).$$

Warning

A pointed family is *not* a family of pointed types.

Proposition

The equivalence $B \to \mathbf{Type} = (E : \mathbf{Type}) \times (E \to B)$ extends to an equivalence

$$\mathrm{PtdFam}(B) = (E : \mathbf{Type}_*) \times (E \cdot\!\!\to B).$$

Higher Schreier Theory

Theorem (Higher Schreier Theory)

For higher groups G and K,

$$\{\text{Extensions of } G \text{ by } K\} = \mathsf{BG} \cdot\!\!\to \mathsf{BAut}(\mathsf{BK}).$$

Proof Sketch.

We use the fact that in an extension

$$\mathsf{BK} \cdot\!\!\xrightarrow{\mathsf{Bi}} \mathsf{BE} \cdot\!\!\xrightarrow{\mathsf{Bp}} \mathsf{BG},$$

The pointed map $\mathsf{Bp} : \mathsf{BE} \cdot\!\!\to \mathsf{BG}$ corresponds to a pointed family over BG. Since BG is 0-connected and the fiber over the base point is BK, every fiber is identifiable with BK.

Going the other direction, we take the dependent sum of the type family $\mathsf{c} : \mathsf{BG} \cdot\!\!\to \mathsf{BAut}(\mathsf{BK})$. □

Where did all the work go?

Theorem (Higher Schreier Theory)

For higher groups G and K,

$$\{\text{Extensions of } G \text{ by } K\} = \mathrm{BG} \cdot\!\to \mathrm{BAut}(\mathrm{BK}).$$

In other words:

Extensions of G by K are given by actions of G on the type of K-torsors.

- That was a lot easier to prove (and for all stacks of higher groups!) than traditional Schreier theory.
- Where are all the cocycle conditions? Where is all the work?

All the cocycle conditions are hidden in the proof that the HoTT approach to higher groups is equivalent to the axiomatic approach.

Corollary: Classifying split extensions

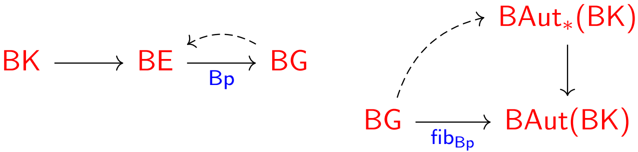
Since homomorphisms between higher groups are pointed maps between their deloopings,

$$\mathrm{Aut}_{\infty\mathrm{Grp}}(K) = \mathrm{Aut}_*(BK).$$

We have a forgetful function $\mathrm{BAut}_*(BK) \cdot\!\!\to \mathrm{BAut}(BK)$, forgetting the base point.

Corollary

Split extensions of G by K are classified by homomorphisms $G \to \mathrm{Aut}_{\infty\mathrm{Grp}}(K)$.



Corollary: Extensions by crisply discrete groups in cohesion

We work in Shulman's *Cohesive HoTT*.

Corollary

Let K be a crisply discrete n-group, and G a crisp higher group (e.g. a Lie group). Then then extensions of G by K are determined by the homotopy n-type of G:

$$\mathrm{EXT}(G; K) = \mathrm{EXT}(\textstyle\int_n G; K).$$

In particular, if K is a 1-group and G is cohesively simply connected ($\int_1 G = *$), then

$$\mathrm{EXT}(G; K) = \{K \to G \times K \to G\}.$$

Corollary: Extensions of $\mathbb{Z}$

Corollary

For any higher group K,

$$\mathrm{EXT}(\mathbb{Z}; K) = \mathrm{Aut}(BK).$$

Proof.

$$\mathrm{EXT}(\mathbb{Z}; K) = B\,\mathbb{Z} \cdot\!\!\to BAut(BK) = \mathrm{Aut}(BK). \quad \square$$

In particular, since

$$\mathrm{BOut}(K) :\equiv \|\mathrm{BAut}(BK)\|_1,$$

We have that

$$\mathrm{EXT}(\mathbb{Z}; K)_{/\cong} = \mathrm{Out}(K).$$

Abstract Kernels and Central Extensions

Definition (Eilenberg and Mac Lane)

The *abstract kernel* of an extension

$$1 \to K \to E \xrightarrow{p} G \to 1$$

is the homomorphism $\varphi : G \to \mathrm{Out}(K)$ defined by

$$\varphi(g) :\equiv [k \mapsto eke^{-1}] \quad \text{for any } e \overset{p}{\mapsto} g.$$

Proposition (E-ML)

An extension is central if and only if its abstract kernel is trivial.

Definition

The *abstract kernel* of an extension of higher group G by K classified by $c : BG \cdot\!\!\to BAut(BK)$ is the composite

$$BG \cdot\!\xrightarrow{c} BAut(BK) \cdot\!\xrightarrow{|\cdot|_1} BOut(K) :\equiv \|BAut(BK)\|_1.$$

Centers of higher groups

Definition (Buchholtz)

A higher group K acts on itself by conjugation:

$$t : BK \mapsto (t = t).$$

The *center* of K is the type of fixed points of this action:

$$\begin{aligned} ZK &:\equiv (t : BK) \to (t = t) = (\mathrm{id}_{BK} = \mathrm{id}_{BK}). \\ BZK &:\equiv \mathrm{BAut}_{\mathrm{Aut}(BK)}(\mathrm{id}). \\ B^2ZK &:\equiv (X : \textbf{Type}) \times \|X = BK\|_0. \end{aligned}$$

We have a fiber sequence:

$$B^2ZK \cdot\!\!\to \mathrm{BAut}(BK) \cdot\!\!\to \mathrm{BOut}(K) :\equiv \|\mathrm{BAut}(BK)\|_1$$

The fiber sequence

$$\mathrm{B}^2\mathrm{ZK} \cdot\!\!\to \mathrm{BAut(BK)} \cdot\!\!\xrightarrow{|\cdot|_1} \mathrm{BOut(K)}$$

gives us the following corollary with a cohomological flair:

The *abstract kernel* of an extension of higher group G by K classified by c : BG $\cdot\!\!\to$ BAut(BK) is the composite

$$\mathrm{BG} \cdot\!\!\xrightarrow{c} \mathrm{BAut(BK)} \cdot\!\!\xrightarrow{|\cdot|_1} \mathrm{BOut(K)} :\equiv \|\mathrm{BAut(BK)}\|_1.$$

Corollary

Let G and K be higher groups. Then

$$\{\mathrm{E} : \mathrm{EXT(G; K)} \mid E\text{'s abstract kernel is trivial}\} = \mathrm{BG} \cdot\!\!\to \mathrm{B}^2\mathrm{ZK}$$

In particular, for 1-groups, we have

$$\{\text{Central extensions of } G \text{ by K}\}_{/\cong} = \mathrm{H}^2(\mathrm{G}; \mathrm{K}).$$

References

Thanks!

- *Higher Groups in Homotopy Type Theory,* Buchholtz, van Doorn, Rijke
- *Symmetry Book*, UniMath
- *Théorie de Schreier supérieure*, Breen
- *Cohomology of Groups II: Groups Extensions with non-Abelian Kernel*, Eilenberg, Mac Lane
- *Higher Dimensional Analysis V: 2-Groups*, Baez, Lauda