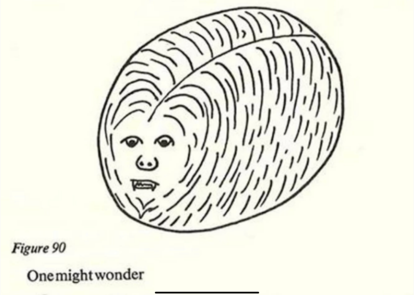
Tangent Bundles of Spheres



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- This is a synthetic homotopy theory talk.
  - Homotopy theory, but just w/ universal properties.

- We will use Book HoTT throughout.
  - pair types $(a:A)\times B(a)$ "Σ-types"
  - function types $(a:A)\to B(a)$ "Π-types"
  - natural numbers $\mathbb{N}$ and induction
  - identity types $(a=b)$ and path induction
  - Univalent universes Type
  - pushouts

$$\mathsf{gl} : (a:A) \to (\mathsf{inl}(fa) = \mathsf{inr}(g(a)))$$

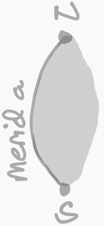
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\mathsf{gl}}{\Nearrow}"] & B \arrow[d, "\mathsf{inr}"] \\ C \arrow[r, "\mathsf{inl}"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

! – Banded types and Cohomology.

- **Pushouts** specialize to a number of important constructions

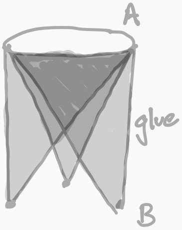
- **Suspension**

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "!"'] & * \arrow[d, "S"] \\ * \arrow[r, "N"'] \arrow[ur, phantom, "\Uparrow \text{merid}"] & \underline{\Sigma A} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



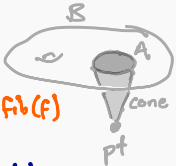
- **Join**

$$\begin{tikzcd} A \times B \arrow[r, "\text{snd}"] \arrow[d, "\text{fst}"'] & B \arrow[d, "\text{inr}"] \\ A \arrow[r, "\text{inl}"'] \arrow[ur, phantom, "\Uparrow \text{glue}"] & \underline{A * B} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



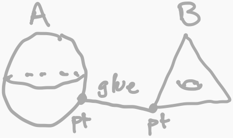
- **Cofiber**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] & B \arrow[d, "\text{cofib}(f)"] \\ * \arrow[r, "\text{pt}"'] \arrow[ur, phantom, "\Uparrow \text{cone}"] & \underline{\text{Cofib}(f)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



- **Wedge**

$$\begin{tikzcd} * \arrow[r, "\text{pt}"] \arrow[d, "\text{pt}"'] & B \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\Uparrow \text{glue}"] & \underline{A \vee B} \end{tikzcd}$$



- Spheres and Cubes are iterated pushouts

<u>Spheres</u>

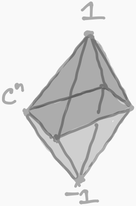
$S^{-1} :\equiv \emptyset$

$S^{n+1} :\equiv \Sigma S^n$

<u>Cubes</u>

$C^0 :\equiv \{-1, 1\}$

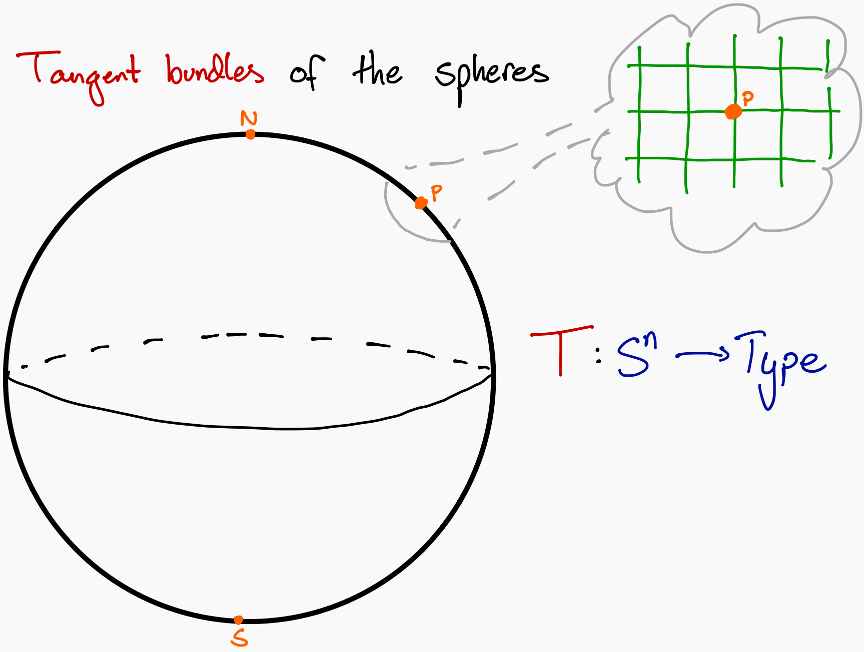
$C^{n+1} :\equiv C^0 * C^n$



- Lemma (Book):
  For any type A, $\Sigma A \simeq C^0 * A$

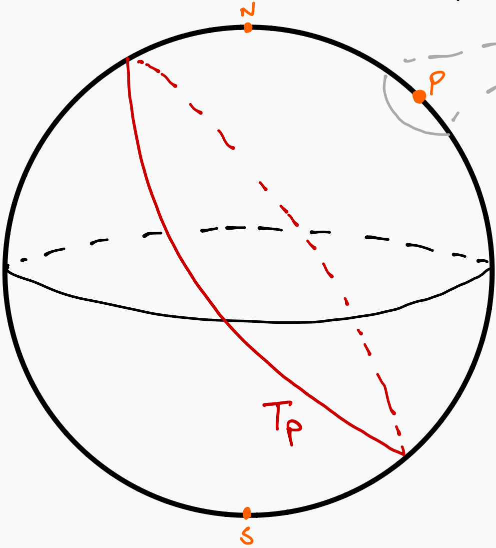
- Corollary: $S^n \simeq C^n$

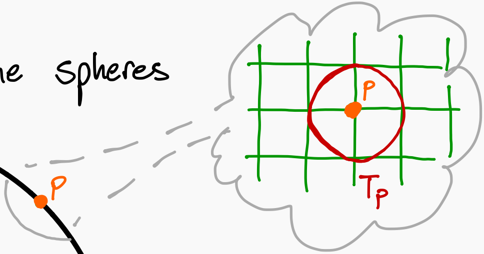
- Tangent bundles of the spheres



$T : S^n \to \mathsf{Type}$

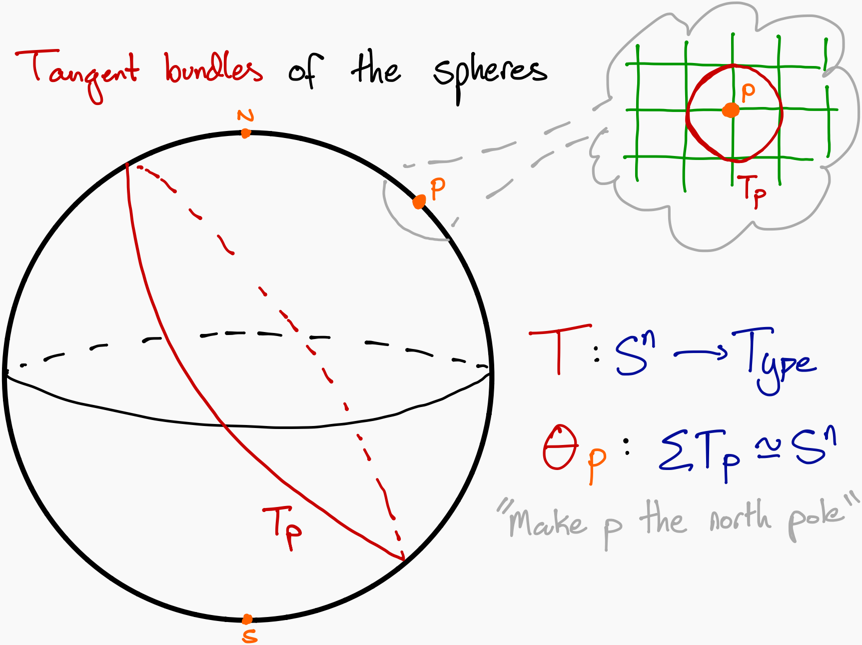
- Tangent bundles of the spheres





$T : S^n \to$ ~~Type~~ Sphere

- Tangent bundles of the spheres



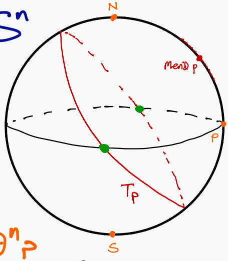
$T : S^n \to \mathrm{Type}$

$\Theta_p : \Sigma T_p \simeq S^n$

"Make p the north pole"

- **Tangent bundles** of the spheres

- Inductively define $T^n : S^n \to \mathrm{Type}$ and
  $\Theta^n : (p : S^n) \to \Sigma T^n_p \simeq S^n$



- Case $n \equiv -1$ is trivial.

- $T^{n+1} : \Sigma S^n \to \mathrm{Type}$

  $N \longmapsto S^n$ the equator

  $S \longmapsto S^n$

  $p : S^n \longmapsto S^n \overset{(\Theta^n_p)^{-1}}{=\!=} \Sigma T^n \overset{\mathrm{flip}}{=\!=} \Sigma T^n \overset{\Theta^n_p}{=\!=} S^n$

- $\Theta^{n+1} : (p : \Sigma S^n) \to \Sigma T^{n+1} \simeq \Sigma S^n$

  $N \longmapsto \Sigma S^n \xrightarrow{\mathrm{id}} \Sigma S^n$

  $S \longmapsto \Sigma S^n \xrightarrow{\mathrm{flip}} \Sigma S^n$

  $p : S^n \longmapsto$

$$
\begin{tikzcd}
\Sigma\Sigma T^n \arrow[d, "\Sigma\mathrm{flip}"'] \arrow[d, bend left, "\mathrm{flip}"] \arrow[d, phantom, "\mathrm{Lem.}"] & \Sigma S^n \arrow[l, "\Sigma(\Theta^n_p)^{-1}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "\mathrm{nat.}"] & \\
\Sigma\Sigma T^n \arrow[r, "\Sigma\Theta^n_p"'] & \Sigma S^n \arrow[r, "\mathrm{flip}"'] & \Sigma S^n
\end{tikzcd}
$$

- Crucial lemmas about $\mathrm{flip} : \Sigma A \xrightarrow{\sim} \Sigma A$

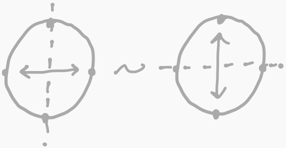
$$\begin{aligned} N &\longmapsto S \\ S &\longmapsto N \\ \mathrm{merid}\, p &\longmapsto (\mathrm{merid}\, p)^{-1} \end{aligned}$$

- Naturality:

$$\begin{tikzcd} \Sigma A \arrow[r, "\Sigma f"] \arrow[d, "\mathrm{flip}"'] \arrow[dr, phantom, "//"] & \Sigma B \arrow[d, "\mathrm{flip}"] \\ \Sigma A \arrow[r, "\Sigma f"'] & \Sigma B \end{tikzcd}$$

- Lem:

$$\begin{tikzcd} \Sigma\Sigma A \arrow[r, bend left, "\Sigma \mathrm{flip}"] \arrow[r, bend right, "\mathrm{flip}"'] \arrow[r, phantom, "\|"] & \Sigma\Sigma A \end{tikzcd}$$



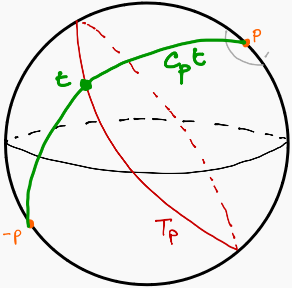
- involution: $\mathrm{flip} \circ \mathrm{flip} = \mathrm{id}$

The antipode $a^n : S^n \xrightarrow{\sim} S^n$ is defined inductively by

$$a^{n+1} \equiv \mathrm{flip} \circ \Sigma a^n$$

So:

$$a^n = \mathrm{flip} \circ \Sigma \mathrm{flip} \circ \Sigma^2 \mathrm{flip} \cdots \Sigma^n \mathrm{flip} = \begin{cases} \mathrm{flip} & \text{if } n \text{ even} \\ \mathrm{id} & \text{if } n \text{ odd} \end{cases}$$



We can show that

$\Theta_p N = p$ and $\Theta_p S = a\,p$

Therefore, we have a map:

$$\begin{tikzcd} T_p \arrow[r, "\text{merid}"] \arrow[rrr, bend left, "c_p"] & (N \underset{\Sigma T_p}{=} S) \arrow[r, "\text{ap}\,\Theta_p"] & (\Theta_p N = \Theta_p S) \simeq & (p = a\,p) \end{tikzcd}$$

So, from a section $s : (p : S^n) \to T_p$, we get a homotopy $\hat{s} : id = a$

But this can only happen if n is odd!

Define $BAut^+(X) :\equiv (Y : \mathrm{Type}) \times \| Y = X \|_0$

We have $T : S^n \to BAut^+(S^{n-1})$.

oriented spheres

Define the Euler class:

$$e : BAut^+(S^n) \longrightarrow BAut^+(K(\mathbb{Z}, n))$$
$$(S, \ldots) \longmapsto (\|S\|_n, \ldots)$$

The theory of banded types (Taxeras, et al.) proves

$$BAut^+(K(\mathbb{Z}, n)) \xrightarrow{\sim} K(\mathbb{Z}, n+1) \quad \text{delooping}$$
$$Aut^+(K(\mathbb{Z}, n)) \xrightarrow{ev_{pt}} K(\mathbb{Z}, n)$$

So $|e \circ T|_0$ is a class in $H^n(S^n)$.

To compute $e \circ T^{\mathrm{adj}}$, transpose

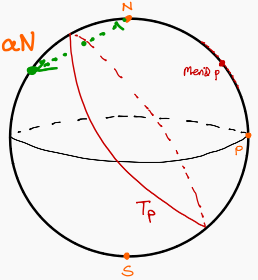
$$\Sigma S^n \xrightarrow{T} BAut^+(S^n) \xrightarrow{e} BAut^+(K(\mathbb{Z},n))$$

to:

$$p \mapsto \left(S^n \xrightarrow{\theta_p^{-1}} \Sigma T_p \xrightarrow{\mathrm{flip}} \Sigma T_p \xrightarrow{\theta_p} S^n\right)$$

$$\begin{tikzcd}
S^n \arrow[r, "\mathrm{merid}"] \arrow[rrd, dashed, bend right, "\alpha"'] & (N =_{S^{n+1}} S) \arrow[r, "T_*"] & Aut^+(S^n) \arrow[r, "\|-\|_n"] \arrow[d, "ev_N"] & Aut^+(K(\mathbb{Z},n)) \arrow[d, "ev_{pt}"] \\
 & & S^n \arrow[r, "|\cdot|_n"] & K(\mathbb{Z},n)
\end{tikzcd}$$

i.e.: $e \circ T = \deg \alpha$.



We then compute that

$$\alpha_*(\mathrm{merid}\, x) = \mathrm{merid}(ax)^{\wedge} \cdot \mathrm{merid}(x)$$

Therefore, $e \circ T = a + id$ in $H^n(S^n)$

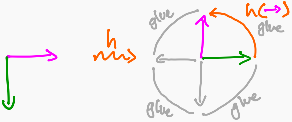
We have a more general approach using joins:

Def: A **reflection** $r : A \simeq A$ satisfies

① **reflect** : $\begin{tikzcd} A * A \arrow[r, bend left, "r * A"] \arrow[r, bend right, "\mathrm{swap}"'] \arrow[r, phantom, "\|"] & A * A \end{tikzcd}$

$t \overset{h}{\mapsto} \begin{bmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

② **coh** : $\mathrm{reflect}(\mathrm{inl}(a)) = \mathrm{glue}(a, a)$



Def: An **A-orthorsor** is

① $\tau : E \to \mathrm{Type}$ ← "orthogonal complement"

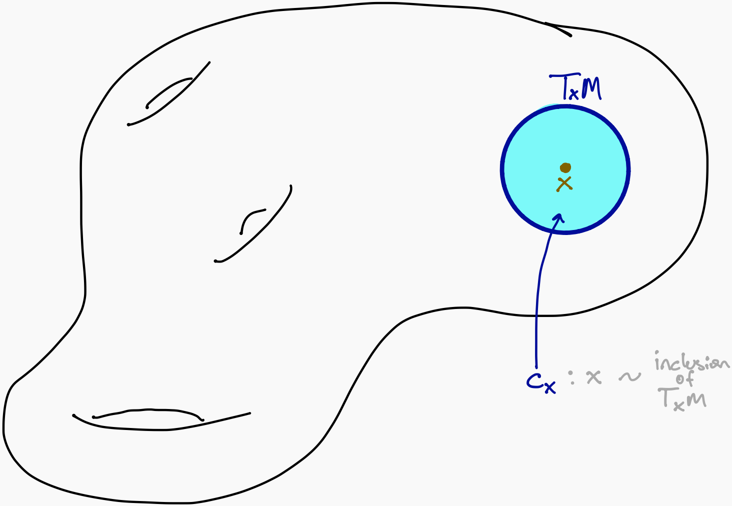
② $\theta : (e : E) \to (A * \tau(e) \simeq E)$

Construction: A **join** for orthorsors:

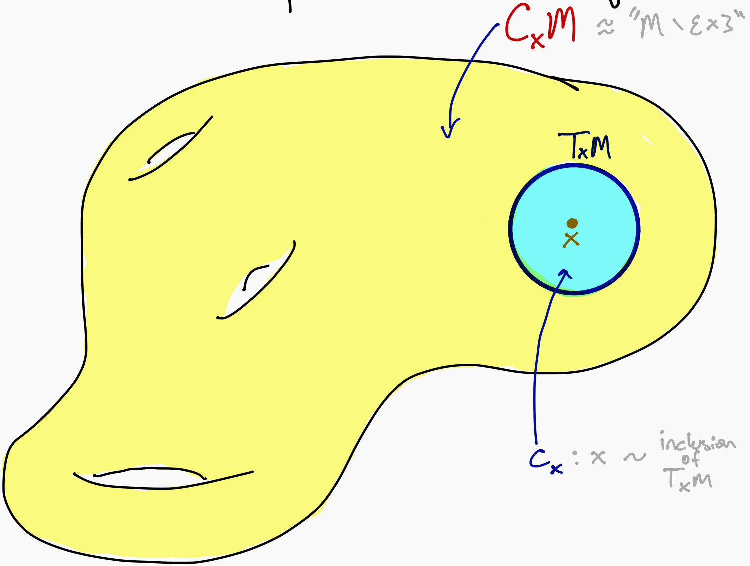
$(E, \tau^E, \theta^E) * (F, \tau^F, \theta^F) :\equiv (E * F, \text{“}\tau^E * F \rightsquigarrow E * \tau^F\text{”}, \dots)$

Tangents arise as $(A, \emptyset, \mathrm{const}\ \mathrm{id})^{*n} : \mathrm{Orthorsor}(A)$

What makes a sphere bundle a "tangent bundle"?



What makes a sphere bundle a "tangent bundle"?



Def: A homotopy pre-manifold structure on a type M consists of a pushout complement

$$\begin{tikzcd}
T_pM \arrow[r, "\partial_p"] \arrow[d, "!"'] & C_pM \arrow[d, "\iota_p"] \\
* \arrow[r, "p"'] \arrow[ur, phantom, "\Nearrow c_p"] & M \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

≈ Lashoff '72, classically

of every point $p : M$, where T_pM is merely a sphere.

Thm:

$$\begin{tikzcd}
T_p \arrow[r, "!"] \arrow[d, "!"'] & * \arrow[d, "\mathrm{ap}"] \\
* \arrow[r, "p"'] \arrow[ur, phantom, "\Nearrow c_p"] & S^n \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

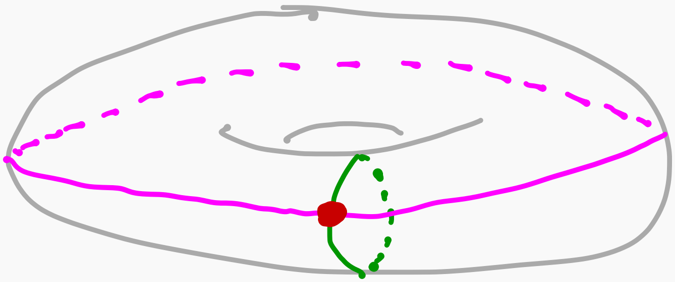
endows S^n with the structure of a homotopy pre-manifold.

Suppose we have homotopy pre-manifolds M and N

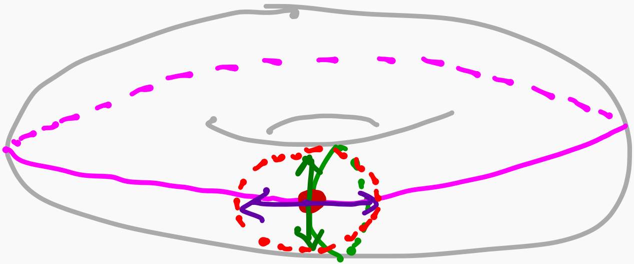
$$
\begin{tikzcd}
T_pM \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & C_pM \arrow[d] \\
* \arrow[r, "p"'] & M \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & C_qN \arrow[d] \\
* \arrow[r, "q"'] & N \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

How can we define a pre-manifold structure on $M \times N$?

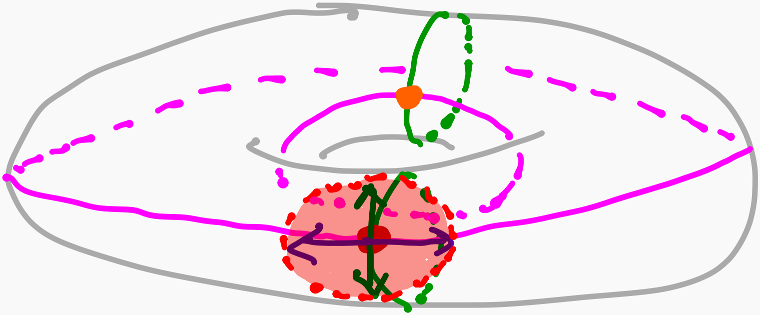
$$
\begin{tikzcd}
? \arrow[r] \arrow[d] \arrow[dr, phantom, "/\!/"] & ? \arrow[d] \\
* \arrow[r, "{(p,q)}"'] & M \times N \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$



$$* \xrightarrow[(p,q)]{} S^1 \times S^1$$



$$\begin{tikzcd} T_pS^1 \times T_qS^1 \arrow[d] & \\ * \arrow[r, "{(p,q)}"'] & S^1 \times S^1 \end{tikzcd}$$



$$\begin{tikzcd} T_pS^1 \ast T_qS^1 \arrow[r] \arrow[d] & S^1 {}_{aq}\!\vee_{ap} S^1 \arrow[d] \\ \ast \arrow[r, "(p,q)"'] & S^1 \times S^1 \end{tikzcd}$$

This is a case of the pushout product:

$$
\begin{tikzcd} A_1 \arrow[d, "f"'] \\ B_1 \end{tikzcd}
,\quad
\begin{tikzcd} A_2 \arrow[d, "g"'] \\ B_2 \end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
A_1 \times A_2 \arrow[r, "f \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times g"'] & B_1 \times A_2 \arrow[d] \arrow[ddr, bend left, "\mathrm{id} \times g"] & \\
A_1 \times B_2 \arrow[r] \arrow[drr, bend right, "f \times \mathrm{id}"'] & f \square g \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & B_1 \times B_2
\end{tikzcd}
$$

E.g.:

Join:

$$
\begin{tikzcd} A_1 \arrow[d, "!"'] \\ * \end{tikzcd}
,\quad
\begin{tikzcd} A_2 \arrow[d, "!"'] \\ * \end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd} A_1 * A_2 \arrow[d, "!"'] \\ * \end{tikzcd}
$$

Wedge:

$$
\begin{tikzcd} * \arrow[d, "b_1"'] \\ B_1 \end{tikzcd}
,\quad
\begin{tikzcd} * \arrow[d, "b_2"'] \\ B_2 \end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd} B_1 \vee B_2 \arrow[d] \\ B_1 \times B_2 \end{tikzcd}
$$

Suppose we have homotopy pre-manifolds M and N

$$
\begin{tikzcd}
T_pM \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & C_pM \arrow[d, "\partial_p^M"] \\
* \arrow[r, "p"'] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & C_qN \arrow[d, "\partial_q^N"] \\
* \arrow[r, "q"'] & N
\end{tikzcd}
$$

To define a pre-manifold structure on $M \times N$,

Take the pushout product of the constituent squares:

$$
\begin{tikzcd}
T_pM * T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & \partial_p^M \square \partial_q^N \arrow[d] \\
* \arrow[r, "{(p,q)}"'] & M \times N
\end{tikzcd}
$$

For this to work, we need the following lemma:

Given pushout squares

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "S_1"] & B_1 \arrow[d, "g_1"] \\ C_1 \arrow[r] & D_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[r] \arrow[d, "f_2"'] \arrow[dr, phantom, "S_2"] & B_2 \arrow[d, "g_2"] \\ C_2 \arrow[r] & D_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Their pushout product is a pushout too.

$$\begin{tikzcd} f_1 \square f_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "S_1 \square S_2"] & g_1 \square g_2 \arrow[d] \\ C_1 \times C_2 \arrow[r] & D_1 \times D_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Some

Future Work

- Thom class vs Euler Class
- Stable normal bundles and Atiyah Duality

Thanks!