Directed univalence in simplicial sets

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Motivation

Motivation

∞-categories are a semantic interpretation of Directed Type Theory.

Motivation

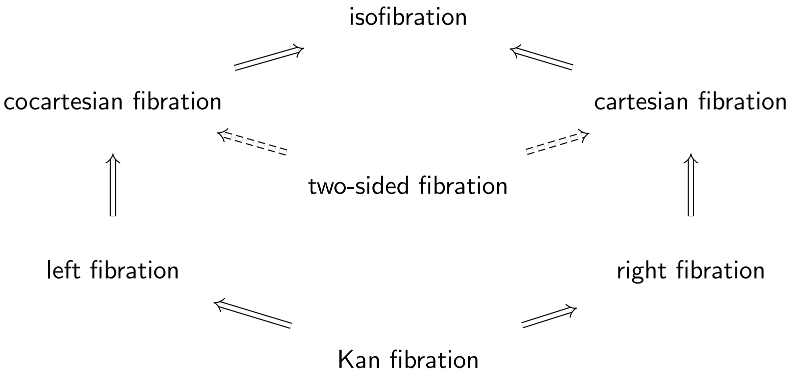
HoTT	Semantics	DTT	Semantics
Types	∞-groupoids	Types	∞-categories
Identity type $\mathrm{Id}_A(a,b)$	path space $P_{a,b}(A)$	Hom type $\mathrm{Hom}_A(a,b)$	mapping space $\mathrm{Map}_A(a,b)$
Universe	$S^{\simeq}$	Universe	Q
Univalence	$P_{X,Y}(S^{\simeq}) \xrightarrow{\simeq} \mathrm{Eq}_S(X,Y)$	Directed Univalence	?

Goal for today

Directed univalence holds in simplicial sets.

Fibrations

Fibrations in simplicial sets



Examples

Let C be an ∞-category then

- The target projection $C^{\Delta^1} \xrightarrow{t} C$ is a cocartesian fibration
- The source projection $C^{\Delta^1} \xrightarrow{s} C$ is a cartesian fibration
- The source-target projection $C^{\Delta^1} \xrightarrow{(s,t)} C \times C$ is a two-sided fibration.

Universes

Universes

There are several universes we can define in simplicial sets:

- The universe classifying small Kan fibrations $S^{\simeq}$
- The universe classifying small left fibrations S
- The universe classifying small cocartesian fibrations Q

The n-simplices are given by small fibrations (with some extra coherence data)

$$\begin{array}{c} X \\ \downarrow \\ \Delta^n \end{array}$$

In particular, the 0-simplices of $S^{\simeq}$ and S are ∞-groupoids and the 0-simplices of Q are given by ∞-categories.

Theorem (Kapulkin-Lumsdaine-Voevodsky,Cisinski, N.)

The simplicial set $S^{\simeq}$ is an ∞-groupoid and the simplicial sets S and Q are ∞-categories.

Universes

We have the following pullback diagram whose vertical maps are the universal Kan, left and cocartesian fibration respectively

$$\begin{array}{ccccc}
S_\bullet^{\simeq} & \longrightarrow & S_\bullet & \longrightarrow & Q_\bullet \\
{\scriptstyle p^{\simeq}}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\
S^{\simeq} & \longrightarrow & S & \longrightarrow & Q
\end{array}$$

The map $S^{\simeq} \to S$ is the inclusion of the maximal ∞-groupoid and the map $S \to Q$ is the inclusion of the full subcategory of spaces into the ∞-category of ∞-categories.

∞-gpds

Univalence

Univalence

The univalence axiom asserts an equivalence between the identity type of the universe and the type of equivalences between types, i.e. that the map

$$Id_{\mathfrak{U}}(A, B) \to A \simeq B$$

induced by path induction is an equivalence.

Univalence

Convention

The term fibration will mean either Kan, left or cocartesian fibration. A map of fibrations is assumed to preserve the appropriate structure and an equivalence of fibrations is the appropriate notion of equivalence.

Univalence

In simplicial sets this translates as follows: Given a fibration $p : X \to A$ there is an object of equivalences $Eq_A(X)$ whose n-simplices are given by maps $f, g : \Delta^n \to A$ and an equivalence of fibrations

$$\begin{array}{ccc} f^*X & \xrightarrow{\simeq} & g^*X \\ & \searrow \quad \swarrow & \\ & \Delta^n & \end{array}$$

Univalence

The object $Eq_A(X)$ comes with canonical maps

$$A \to Eq_A(X) \to A \times A$$

factorizing the diagonal.

Definition

The fibration p is univalent if this is a path object in the Kan-Quillen model structure (Kan fibrations), respectively in the Joyal model structure (left fibrations and cocartesian fibrations).

Univalence

There is a canonical choice of path object in these model structures. Let J be the nerve of the free walking isomorphism

$$J = N\left(\bullet \rightleftarrows \bullet \right)$$

This induces the path object

$$A \to A^J \to A \times A$$

Path induction defines a lift in the commutative square

$$\begin{array}{ccc} A & \longrightarrow & Eq_A(X) \\ \downarrow & \nearrow & \downarrow \\ A^J & \longrightarrow & A \times A \end{array}$$

and the fibration $p : X \to A$ is univalent if and only if this lift is an equivalence.

Univalence

Theorem(K.-L.-V.,C., C.-N.)

The universal fibrations

$$\begin{array}{ccccc} S^{\simeq}_{\bullet} & & S_{\bullet} & & Q_{\bullet} \\ \downarrow{\scriptstyle p^{\simeq}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ S^{\simeq}, & & S, & & Q \end{array}$$

are univalent.

Univalence

In homotopy type theory this is all we can express. The universe of ∞-groupoids is itself an ∞-groupoid, so it only sees equivalences between types.

On the other hand, the universes S and Q, being ∞-categories, also see non-invertible functions between types.

Directed univalence

Directed univalence

Since S and Q are ∞-categories, we want to express a directed univalence axiom.

The identity type should then be replaced by a hom type. Directed univalence then should assert an equivalence between hom types and function types.

The hom type

Given an ∞-category C, we have a factorization of the diagonal

$$C \to C^{\Delta^1} \xrightarrow{(s,t)} C \times C$$

The source-target map is a two-sided fibration and in particular an isofibration.

However, this does not define a path object in the Joyal model structure.

The hom type

Taking fibers at a point (a, b) of $C \times C$ defines the mapping space of C

$$\begin{array}{ccc} \mathrm{Map}_C(a, b) & \longrightarrow & C^{\Delta^1} \\ \downarrow & & \downarrow {\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow{(a,b)} & C \times C \end{array}$$

The function type

Given a fibration $p : X \to A$ there is an object of morphisms $\underline{\mathrm{Hom}}_{A \times A}(X^0, X^1)$ whose n-simplices are given by maps $f, g : \Delta^n \to A$ and a map of fibrations

$$\begin{array}{ccc} f^*X & \longrightarrow & g^*X \\ & \searrow \qquad \swarrow & \\ & \Delta^n & \end{array}$$

We obtain a canonical factorization of the diagonal

$$A \to \underline{\mathrm{Hom}}_{A \times A}(X^0, X^1) \to A \times A$$

The function type

Definition

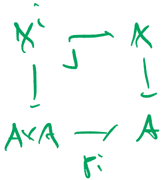
We call

$$\begin{array}{c} \underline{\mathrm{Hom}}_{Q\times Q}(Q^0_\bullet, Q^1_\bullet) \\ \big\downarrow \\ Q \times Q \end{array}$$

the universal morphism classifier.

Indeed, a map $K \to \underline{\mathrm{Hom}}_{Q\times Q}(Q^0_\bullet, Q^1_\bullet)$ is determined by cocartesian fibrations

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$



and a map preserving cocartesian edges.

The function type

The function type should be a dependent type. It is easy to see that we always get an isofibration, but more is true:

Theorem

The isofibration

$$\begin{array}{c} \underline{\mathrm{Hom}}_{A\times A}(X^0, X^1) \\ \downarrow \\ A \times A \end{array}$$

is a two-sided fibration.

Directed univalence

We always have a lift in the square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_{A\times A}(X^0, X^1) \\ \downarrow & \nearrow & \downarrow \\ A^{\Delta^1} & \longrightarrow & A \times A \end{array}$$

Definition

A fibration is called directed univalent if this lift is an equivalence of ∞-categories.

Directed Univalence

Remark

The lift

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_{A\times A}(X^0, X^1) \\ \downarrow & \nearrow & \downarrow \\ A^{\Delta^1} & \longrightarrow & A \times A \end{array}$$

can be constructed directly from the lifting properties of fibrations. It can also be viewed as an instance of directed path induction.

Directed univalence

Let's unpack this for the universal cocartesian fibration.

$$\begin{array}{ccc} Q & \longrightarrow & \underline{\mathrm{Hom}}_{Q\times Q}(Q^0_\bullet, Q^1_\bullet) \\ \downarrow & \nearrow & \downarrow \\ Q^{\Delta^1} & \longrightarrow & Q\times Q \end{array}$$

A point in $Q \times Q$ is given by a pair of ∞-categories (C, D). The fiber of the universal morphism classifier at (C, D) is given by $\mathrm{Fun}(C, D)^{\simeq}$. Thus the induced map on fibers is

$$\mathrm{Map}_Q(C, D) \to \mathrm{Fun}(C, D)^{\simeq}$$

Theorem(Cisinski-N.)

The universal left fibration and the universal cocartesian fibration

$$\begin{array}{ccc} S_\bullet & & Q_\bullet \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ S, & & Q, \end{array}$$

are directed univalent.

Sketch of proof

Since the hom space and the universal morphism classifier are two-sided fibrations, it is enough to check fiberwise that the comparison map

$$\begin{array}{ccc} Q & \longrightarrow & \underline{\mathrm{Hom}}_{Q\times Q}(Q^0_\bullet, Q^1_\bullet) \\ \downarrow & \nearrow & \downarrow \\ Q^{\Delta^1} & \longrightarrow & Q \times Q \end{array}$$

is an equivalence. Hence we need to show that

$$\mathrm{Map}_Q(C, D) \to \mathrm{Fun}(C, D)^{\simeq}$$

is an equivalence of ∞-groupoids.

Sketch of proof

To proof this, we consider the functor

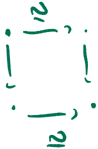
$$ho(Q) \to ho(\mathrm{sSet}^{\mathrm{Joyal}})$$

It sends an object $\Delta^0 \to Q$ to the fibration that it classifies, i.e. to a small ∞-category C. On morphisms we simply take π_0 of

$$\mathrm{Map}_Q(C, D) \to \mathrm{Fun}(C, D)^{\simeq}$$

One checks that this is compatible with composition.

Sketch of proof



The essential result needed is:

Theorem

There is an equivalence of categories for A a simpl. set.

$$ho(\mathrm{Fun}(A, Q)) \xrightarrow{\simeq} \mathrm{coCart}(A)$$

homotopy cat. of the coCart. model structure on $sSet^+/A^\#$

Sketch of proof

This theorem implies that we have for all simplicial sets A an isomorphism

$$\pi_0(\mathrm{Fun}(A, \mathrm{Map}_Q(C, D))) \to \pi_0(\mathrm{Fun}(A, \mathrm{Fun}(C, D)^{\simeq}))$$

which implies the desired equivalence of spaces

$$\mathrm{Map}_Q(C, D) \simeq \mathrm{Fun}(C, D)^{\simeq}$$

Fun fact

The proof of directed univalence uses univalence in an essential way: univalence implies that the functor

$$ho(\mathrm{Fun}(A, Q)) \to \mathrm{coCart}(A)$$

is fully faithful.

Remark

The proof shows a close connection between directed univalence and straightening/unstraightening. In fact it is easy to see that directed univalence is equivalent to the equivalence of categories

$$ho(\mathrm{Fun}(A, Q)) \simeq \mathrm{coCart}(A)$$

With some effort this can be improved to the full straightening/unstraightening equivalence, which in particular implies that Q is equivalent to the localization of simplicial sets at the Joyal equivalences.

$$\mathrm{Map}_{hoQ}(C, D) \simeq \mathrm{Fun}(C, D)^{\simeq}$$

Directed univalent fibrations

Directed univalent fibrations

Recall that a fibration $p : X \to A$ is directed univalent if the lift in the square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_{A\times A}(X^0, X^1) \\ \downarrow & \nearrow & \downarrow \\ A^{\Delta^1} & \longrightarrow & A \times A \end{array}$$

is an equivalence.

Directed univalent fibrations

Theorem (Cisinski-N.)

A small cocartesian fibration

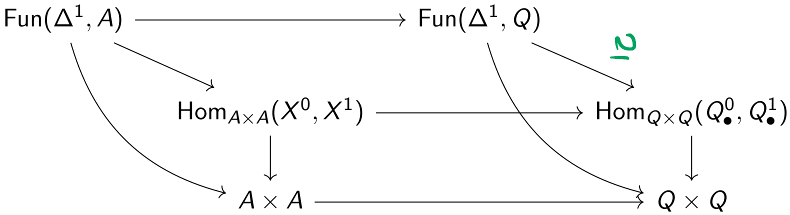
$$\begin{array}{c} X \\ \downarrow p \\ A \end{array}$$

is directed univalent if and only if its classifying map

$$F : A \to Q$$

is fully faithful.

Proof



Take fibres + 2-out-of-3

Directed univalent fibrations

Corollary

The classifying map of the universal left fibration

$$S \to Q$$

is fully faithful.

How to get to a full straightening / unstraightening equivalence

small 1-cat.

$$\mathrm{Fun}(I, sSet^+/A^\#) \xrightarrow{\sim} sSet^+/I^\# \times A^\#$$

Quillen equivalence

Get $\mathrm{Fun}(I, sSet^+/A^\#) \longrightarrow \mathrm{coCart}(I \times A)$

$\simeq$

$\mathrm{ho}\, \mathrm{Fun}(I, \mathrm{Fun}(A, Q))$

This is a localization.

Now take $I = sSet^+/A^\# \Rightarrow N(sSet^+/A^\#) \to \mathrm{Fun}(A, Q)$

Get an induced functor

$$\mathcal{L}\left(sSet^{+}/_{A^{\#}}\right) \longrightarrow Fun(A, Q)$$

↗

∞-Cat localization

This preserves colimits and is an equiv. on homotopy cats.

Thank You!