Filter Products and Elementary Models of Homotopy Type Theory

Nima Rasekh

École Polytechnique Fédérale de Lausanne

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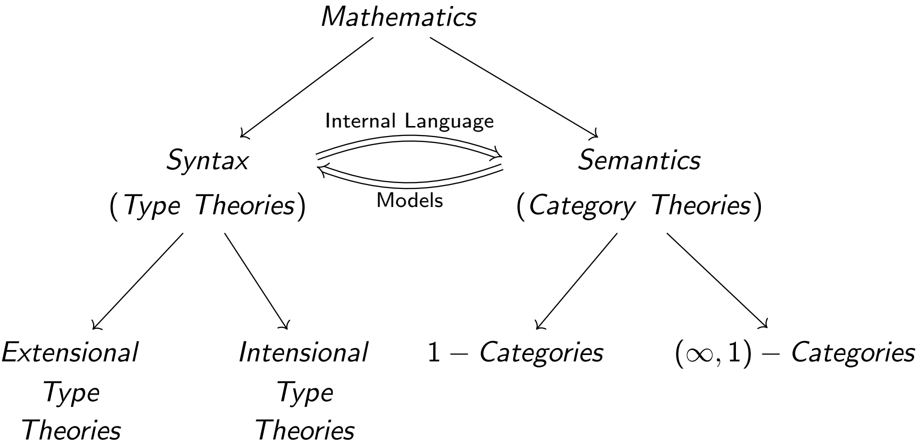
Syntax vs. Semantics



Mathematics

Syntax
(*Type Theories*)

Internal Language

Models

Semantics
(*Category Theories*)

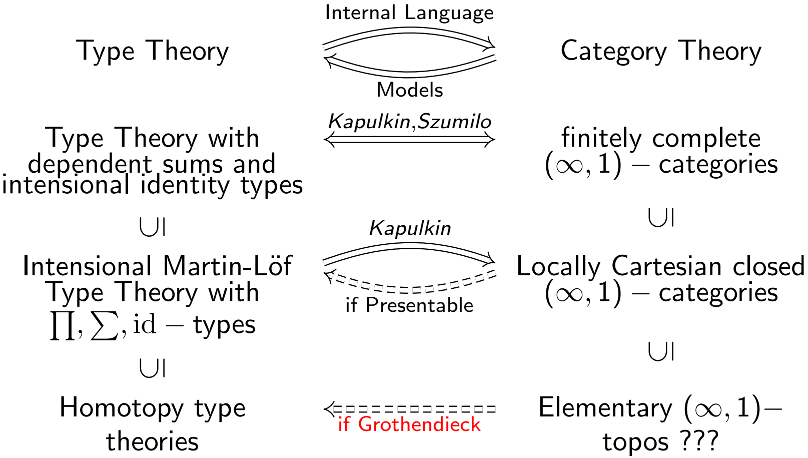
Extensional Type Theories

Intensional Type Theories

$1-$ *Categories*

$(\infty, 1)-$ *Categories*

Intensional Type Theories vs. $(\infty, 1)$-Categories



Type Theory | Internal Language | Models | Category Theory

Type Theory with dependent sums and intensional identity types | *Kapulkin,Szumilo* | finitely complete $(\infty, 1)-$categories

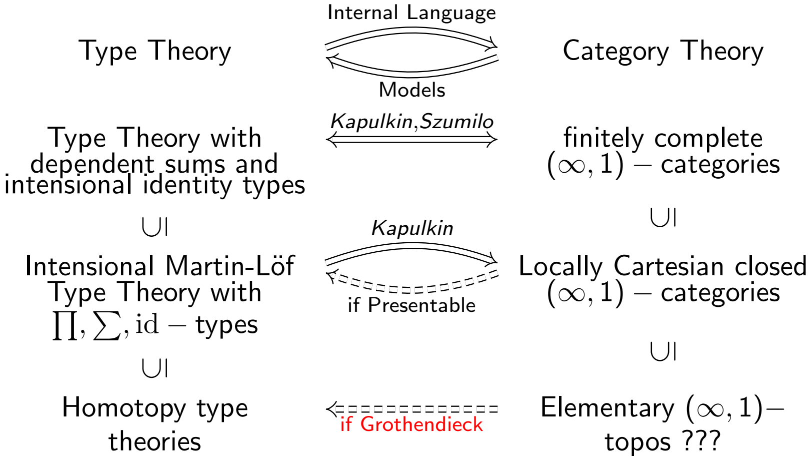
$\cup\!|$ | $\cup\!|$

Intensional Martin-Löf Type Theory with $\prod, \sum, \mathrm{id}-$types | *Kapulkin* | if Presentable | Locally Cartesian closed $(\infty, 1)-$categories

$\cup\!|$ | $\cup\!|$

Homotopy type theories | if Grothendieck | Elementary $(\infty, 1)-$topos ???

Intensional Type Theories vs. $(\infty, 1)$-Categories



Type Theory

Internal Language

Models

Category Theory

Type Theory with dependent sums and intensional identity types

Kapulkin,Szumilo

finitely complete $(\infty, 1)-$ categories

⋃|

⋃|

Intensional Martin-Löf Type Theory with $\prod, \sum, \mathrm{id} -$ types

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if Presentable

Locally Cartesian closed $(\infty, 1)-$ categories

⋃|

⋃|

Homotopy type theories

if Grothendieck

Elementary $(\infty, 1)-$ topos ???

Let's focus on the last line!

What's a model for HoTT: Vague Edition

A model for HoTT is an $(\infty, 1)$-category in which we can interpret various type theoretic constructions.
In some cases it is reasonably clear what that means:

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2. The type of natural numbers $\rightsquigarrow$ natural number object.

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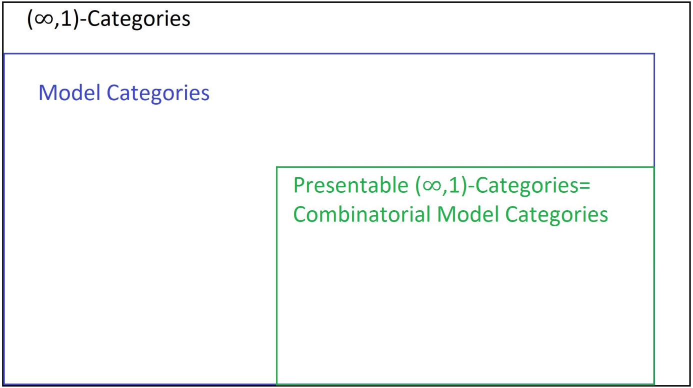
In some cases it's not really clear what that means:

3. Univalent universe?

Key problem: General $(\infty, 1)$-categories are very non-strict!

Model Categories

The fix is an appropriate use of model categories!



(∞,1)-Categories

Model Categories

Presentable (∞,1)-Categories=
Combinatorial Model Categories

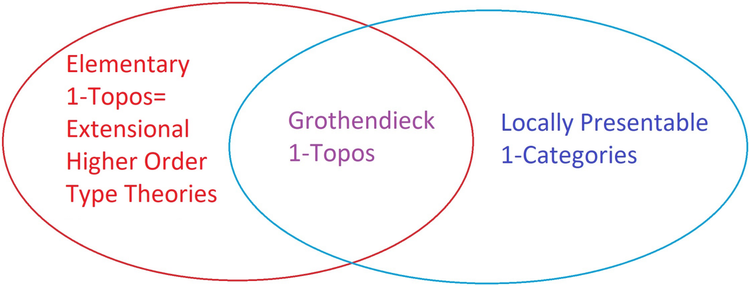
Model categories are strict 1-categories!

Axiomatizations of Model Categories

1. **Arendt, Kapulkin:** Introduce *logical model categories* and prove they model Σ, Π, id-types.
2. **Shulman:** Introduces *type-theoretic model categories* and prove their Π-types satisfy function extensionality.
3. **Shulman, Lumsdaine:** Introduce *good model categories*, which model certain higher inductive types (coproduct type, circle, ...), and *excellent model categories*, which model further higher inductive types (W-types, truncations, localizations, ...).
4. **Shulman:** Introduces *type-theoretic model toposes*, which is a model topos and is a special case of all the previous examples, but also models strict **univalent universes**.

Why Grothendieck Toposes?

Motivated by 1-Categories



Elementary 1-Topos= Extensional Higher Order Type Theories

Grothendieck 1-Topos

Locally Presentable 1-Categories

Grothendieck Toposes and Model Toposes

Definition

A *Grothendieck* 1-*topos* is a category $\mathcal{G}$ that fits into an adjunction

$$\begin{tikzcd}\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \arrow[r, shift left, "a"] \arrow[r, phantom, "\perp"] & \mathcal{G} \arrow[l, shift left]\end{tikzcd}$$

where $\mathcal{C}$ **small**, a is **left-exact**.

? if this fails $\mathcal{G}$ is locally presentable

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Definition

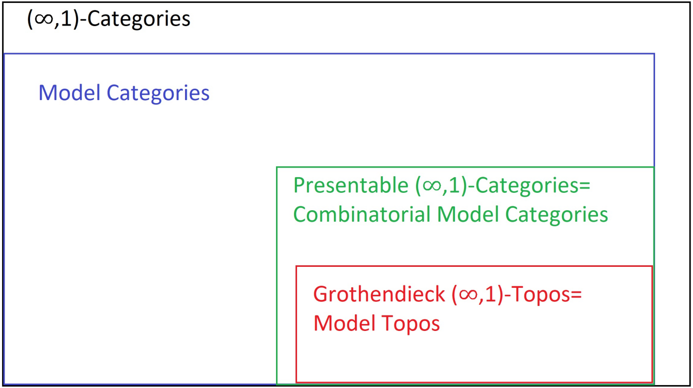
A *Grothendieck model topos* is a simplicial model category $\mathcal{X}$ that fits into a Quillen adjunction:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{s}\mathcal{S}\mathrm{et}^{Kan})^{proj} \arrow[r, shift left, "a"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift left, hook] \end{tikzcd}$$

where $\mathcal{C}$ **small**, a is **left-exact**.

Model Topos vs. Grothendieck $(\infty, 1)$-Topos

We generalize the previous diagram:



(∞,1)-Categories

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Presentable (∞,1)-Categories=
Combinatorial Model Categories

Grothendieck (∞,1)-Topos=
Model Topos

What is a Type-Theoretic Model Topos $\mathcal{E}$?

Definition (Shulman)

1. Grothendieck 1-topos.
2. right proper, simplicial, combinatorial model structure with cofibrations monos.
3. simplicially locally Cartesian closed
4. locally representable, relatively acyclic notion of fibred structure that covers all fibrations

Note: It is in fact a **Grothendieck model topos**.

Univalent Universes in TTMT

Fix large enough κ.

$$\begin{aligned} \mathbb{F}\mathrm{ib}^{\kappa} : \mathcal{E}^{op} &\longrightarrow \mathcal{G}\mathrm{rpd} \\ X &\longmapsto ((\mathbb{F}\mathrm{ib}_{/X})^{\kappa})^{\simeq} \end{aligned}$$

underlying groupoid

The last condition implies the existence of a cofibrant object $\mathcal{U}$ and an *acyclic fibration*

$$\mathcal{E}(-,\mathcal{U}) \twoheadrightarrow \mathbb{F}\mathrm{ib}^{\kappa}$$

Moreover, $\mathcal{U}$ is fibrant and univalent.

Model Topos = Type-Theoretic Model Topos

Type-theoretic model toposes in fact recover all model toposes.

$$\begin{tikzcd}\mathrm{Fun}(\mathcal{C}, s\mathcal{S}\mathrm{et}^{Kan})^{proj} \arrow[r, shift left, "a"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift left, hook']\end{tikzcd}$$

Three steps:

1. The **Kan model structure** is a TTMT (also observed by Kapulkin-Lumsdaine).
2. **Injective model structure** on diagrams into TTMT is a TTMT.
3. **Left-exact Bousfield localizations** of a TTMT is a TTMT.

What about the non-presentable case?

We expect **non-presentable models** for homotopy type theories. However, arbitrary non-presentable $(\infty, 1)$-categories don't come from model categories.

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We expect **non-presentable models** for homotopy type theories. However, arbitrary non-presentable $(\infty, 1)$-categories don't come from model categories.

1. *General Case*: Embed every category in its presheaf category.
2. *Specific Case*: Use specific constructions.
    - **Realizability topos**
    - **Filter Product**

Elementary 1-Toposes

In order to move from the presentable to the non-presentable world we need to generalize our toposes.

Definition

An elementary 1-topos is a locally Cartesian closed category with subobject classifier.

Proposition

A locally presentable category is an elementary 1*-topos if and only if it is a Grothendieck* 1*-topos.*

Slogan: The "correct" generalization of Grothendieck 1-toposes.

Filters

Let's start with filters:

Definition

Let I be a set. A filter $\Phi \subseteq P(I)$ is a subset of the **power set** satisfying:

1. **Non-Empty:** $I \in \Phi$.
2. **Intersection Closed:** $J_1, J_2 \in \Phi \to J_1 \cap J_2 \in \Phi$.
3. **Upwards Closed:** $J_1 \in \Phi, J_1 \subseteq J_2 \to J_2 \in \Phi$.

We can think of elements in Φ as "large" subsets of I.

$I \in \Phi \Leftrightarrow I^c$ "small"

Filter Products

Definition

For a category $\mathcal{C}$ define the **filter-product** $\prod_\Phi \mathcal{C}$ as:

- Obj: $(c_i)_{i \in I}$, $c_i \in Obj(\mathcal{C})$.
- Mor:

$(J, f_i : c_i \to c_i')_{i \in J}$

$$\mathrm{Hom}_{\prod_\Phi \mathcal{C}}((c_i)_{i\in I}, (c_i')_{i \in I}) = \left(\coprod_{J \in \Phi} \prod_{i \in J} \mathrm{Hom}_{\mathcal{C}}(c_i, c_i') \right) / \sim$$

$$(f_i)_{i \in J_1} \sim (g_i)_{i \in J_2} \Leftrightarrow \exists J_3 \subseteq J_1 \cap J_2, J_3 \in \Phi, \{i \in J_3 : f_i = g_i\} \in \Phi$$

So, morphisms that agree on a "large" index set are identified.

Filter Products in Elementary Topos Theory

Filter products are relevant in topos theory.

Theorem (Adelman-Johnstone 1982)

Let $\mathcal{E}$ be an elementary 1*-topos, I a set and Φ a filter. Then $\prod_\Phi \mathcal{E}$ is also an elementary* 1*-topos.*

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Example

Let $\mathcal{E} = \mathcal{S}\text{et}$, $I = \mathbb{N}$ and Φ the **Fréchet filter** (of cofinite sets).
Then $\prod_\Phi \mathcal{S}\text{et}$ is a non-Grothendieck elementary topos.

finite = J "small" if $J^c \in \Phi$

Generalize to Model Categories

We want to generalize the example from categories to model categories!

We need to generalize the definition appropriately:

Type-Theoretic Elementary Model Topos

Type-Theoretic Elementary Model Topos

	TT Grothendieck MT	**TT Elementary MT**
(1)	~~Grothendieck~~ topos	
(2)	right proper, simplicial, ~~combinatorial,~~ cofibrations are monos	
(3)	simplicially lcc	
(4)	notion of fibred structure $\mathbb{F}$ locally representable relatively acyclic ? $\lvert\mathbb{F}\rvert = \mathbb{F}\text{ib}$	

Type-Theoretic Elementary Model Topos

	TT Grothendieck MT	**TT Elementary MT**
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(3)	simplicially lcc	simplicially lcc
(4)	notion of fibred structure $\mathbb{F}$ locally representable relatively acyclic $\lvert\mathbb{F}\rvert = \mathbb{F}\mathrm{ib}$	$\mathbb{F}\mathrm{ib}$ has a fibrant-cofibrant univalent universe

What does it mean: "has a universe"?

$$\mathbb{F}\mathrm{ib}^{\kappa} : \mathcal{E}^{op} \longrightarrow \mathrm{Grp}$$
$$X \longmapsto \left(\mathbb{F}\mathrm{ib}_{/X}\right)^{\simeq}$$

$\mathcal{E}$ has a universe, if there is a filtration

$$\mathcal{E}^{\kappa_1} \subseteq \mathcal{E}^{\kappa_2} \subseteq \ldots$$

of $\mathcal{E}$ such that for all $\mathcal{E}^{\kappa}$ in the filtration there exists a fibrant-cofibrant, univalent universe $\mathcal{U}$, meaning a acyclic ~~trivial~~ fibration:

$$\mathcal{E}(-, \mathcal{U}) \twoheadrightarrow \mathbb{F}\mathrm{ib}^{\kappa}.$$

Is this the best we can do?

I don't know!

1. It **generalizes** type-theoretic Grothendieck model topos.
2. It still **includes** many relevant examples:
    - Logical model categories
    - Type-theoretic model categories
    - good model categories
    - but **not** excellent model categories

   In particular, it interprets **Martin-Löf Type theory** with Σ**-types**, Π**-types with function extensionality**, **identity types**, the **natural numbers type**, the **sphere types** S^n,**universe types**, ...
3. It has **non-trivial examples**.

Filter-Product Model Structure

Theorem (R)

Let $\mathcal{M}$ be a model structure (with **finite** *(co)limits), I a set and Φ a filter on I. Then there is a model structure on $\prod_\Phi \mathcal{M}$ given by*

$$(f_i)_{i \in J} \in \mathfrak{F} \Leftrightarrow \{i \in J : f_i \in \mathfrak{F}\} \in \Phi$$

where $\mathfrak{F}$ is one of the classes of fibrations/cofibration/weak equivalences.

The proof is routine checking.

Remark

Fun fact: We really need all three conditions of a filter!

Transfer of Properties

The following properties will transfer from $\mathcal{M}$ to $\prod_{\Phi} \mathcal{M}$:

- finite (co)limits
- Cartesian closure
- left/right proper
- simplicial
- compatibility with Cartesian closure
- cofibrations monomorphism

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The following will **not** transfer:

- infinite (co)limits
- local presentability
- cofibrantly generated

Comparison with $(\infty, 1)$-Version

We also have following comparison theorem.

Theorem (R)

Let $\mathcal{M}$ be a simplicial model category, I a set and Φ a filter. Then we have an equivalence of $(\infty, 1)$-categories

$$N(\prod_{\Phi} \mathcal{M}) \simeq \prod_{\Phi} N(\mathcal{M})$$

Constructing Elementary Models of HoTT

Theorem (R)

Let $\mathcal{E}$ be a type-theoretic elementary model topos, I a set and Φ a filter. Then $\prod_{\Phi} \mathcal{E}$ is a type-theoretic elementary model topos.

This directly generalizes the result by Adelman-Johnstone.

Example I

We are now finally in a position to put the theory in practice and give examples.

Example

Let

- $\mathcal{E} = \mathrm{s}\mathcal{S}\mathrm{et}$ with $\mathcal{K}\mathrm{an}$ model structure
- $I = \mathbb{N}$
- $\Phi =$ Fréchet filter (cofinite subsets) $I \in \Phi \iff I^c$ finite

Then, by the previous theorem, $\prod_\Phi \mathcal{K}\mathrm{an}$ is a type-theoretic elementary model topos.

Claim: It's not Grothendieck!

Example II

1. The underlying category $\prod_{\Phi} s\mathcal{S}et$ is not locally presentable. So, in particular, $\prod_{\Phi} \mathcal{K}an$ is not combinatorial.
2. It does not even have infinite colimits.
3. The natural number object is non-standard.

$\coprod_{\mathbb{N}} 1$ NNO

Kan × Kan

$(n,m) \in \mathbb{N} \times \mathbb{N}$

$n \times \underline{(1,0)} + m \times (0,1)$

Example III

Obj $(K_1, K_2, K_3, \dots)$

Mor $(f_1, f_2, f_3, \dots)$

$$(f_n) \sim (g_n) \iff \exists N \, \forall n \geq N \, (f_n = g_n)$$

Observation NNO: $(\mathbb{N}, \mathbb{N}, \mathbb{N}, \dots)$ non-standard by Elephant D5.1.7

1

$1d$

$$1 \xrightarrow{s^n} \mathbb{N}$$

n-th spot

$$\begin{tikzcd}
\phi = (\emptyset, \dots, *, \emptyset, \dots) \arrow[r] \arrow[d] & (*, *, *, \dots) \arrow[d, "{(1,2,3,\dots)}"', "\Delta"] \\
(*, *, *, \dots) \arrow[r, "{(n,n,n,\dots)}"] & (\mathbb{N}, \mathbb{N}, \mathbb{N}, \dots)
\end{tikzcd}$$

Where do we go from here?

- These results don't generalize to other elementary models!
- Could be an application of HoTT to non-classical algebraic topology
  - HoTT indexes homotopy groups by the internal natural number object.
  - HoTT proofs of algebraic topological results still hold
  - ...

Proof I

Need to check four conditions:

1. ✓ $\prod_{\Phi} \mathcal{E}$ is an elementary 1-topos.
2. ✓ $\prod_{\Phi} \mathcal{E}$ has a right proper, simplicial model structure where cofibrations are monos
3. ✓ $\prod_{\Phi} \mathcal{E}$ is simplicially locally Cartesian closed.
4. $\mathbb{F}\mathrm{ib}_{\Phi}$ has a fibrant-cofibrant univalent universe

The first three follow from previous analysis.

Proof II

Claim: If $\mathcal{U}$ is a universe in $\mathcal{E}$, then $(\mathcal{U})_{\underline{I}}$ is also a universe in $\prod_{\underline{\Phi}} \mathcal{E}$

$$\begin{tikzcd}
A_i \arrow[d, "cofib"] & \prod_{\underline{\Phi}} \mathcal{E}(-,(A_i)_{\underline{I}}) \arrow[r] \arrow[d] & \prod_{\underline{\Phi}} \mathcal{E}(-,(\mathcal{U})_{\underline{I}}) \arrow[d] \\
B_i & \prod_{\underline{\Phi}} \mathcal{E}(-,(B_i)_{\underline{I}}) \arrow[r] \arrow[ur, dashed] & \mathbb{F}\mathrm{ib}_{\underline{\Phi}}
\end{tikzcd}$$

$$\mathbb{F}\mathrm{ib}_{\underline{\Phi}}(X_i) = \mathbb{F}\mathrm{ib}_{/(X_i)}$$

Proof III

$$\begin{tikzcd}
\prod_I \mathcal{E}(-,(A)_i)/_\sim \arrow[r] \arrow[d] & \prod_I \mathcal{E}(-,\mathcal{U})/_\sim \arrow[d] \arrow[ddr, bend left] & \\
\prod_I \mathcal{E}(-,(B)_i)/_\sim \arrow[r] \arrow[ur, dashed] \arrow[drr, bend right] & \mathbb{F}\mathrm{ib}_\Phi \arrow[dr] & \\
 & & \prod \mathbb{F}\mathrm{ib}/_\sim
\end{tikzcd}$$

Claim:
∃ equivalence
of groupoids

The End!

Thank you!

Questions?