Synthetic Spectra via a Monadic and Comonadic Modality

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Pointed Types

Recall:

Definition

- A pointed type is a pair of A : Type and $a : A$.
- A pointed function $(A, a) \to_\star (B, b)$ is a function $f : A \to B$ and path $p : f(a) = b$.

Carrying these paths p through constructions can be tedious.

We might prefer to talk about functions that preserve the point *strictly*. But we cannot arrange this in ordinary type theory.

Spectra

Definition

- A *prespectrum* E is a sequence of pointed types $E : \mathbb{N} \to \mathsf{Type}_\star$ together with pointed maps $\alpha_n : E_n \to_\star \Omega E_{n+1}$.
- A *spectrum* is a prespectrum such that the α_n are pointed equivalences.

Examples

- Each abelian group G yields a spectrum with $E_n :\equiv K(G, n)$, the ‘Eilenberg-MacLane spaces’.
- The *zero spectrum* with $E_n :\equiv 1$.
- The *sphere prespectrum* has $E_n :\equiv S^n$, with α_n the transpose of $\Sigma S^n \to_\star S^{n+1}$

Working With Spectra

Definition

A map of spectra $f : E \to F$ is a sequence of pointed maps $f_n : E_n \to_\star F_n$ that commute with the structure maps of E and F.

Not many operations on spectra have been defined in type theory!

Do It Synthetically

Can we find a model where functions automatically respect the point?

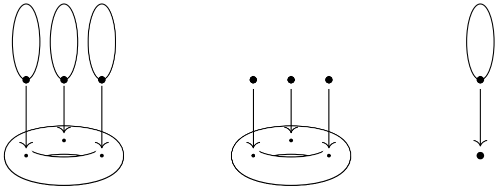
Pointed spaces or spectra don't form a good model of type theory.

Space indexed families of pointed spaces/spectra do!

Parameterised Pointed Spaces

Definition

A *parameterised pointed space* is a space-indexed family of pointed spaces.



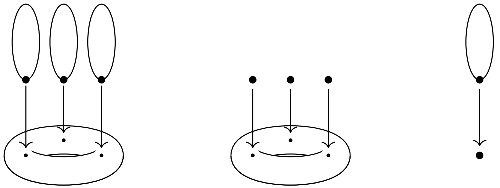
Theorem

The ∞-category of parameterised pointed spaces, $P\mathcal{S}_{\star}$, is an ∞-topos.

Parameterised Spectra

Definition

A *parameterised spectrum* is a space-indexed family of spectra.



Theorem (Joyal 2008, jww. Biedermann)

The ∞-category of parameterised spectra, $P\mathrm{Spec}$, is an ∞-topos.

Types As

HoTT

Types as ∞-groupoids.

In This Talk

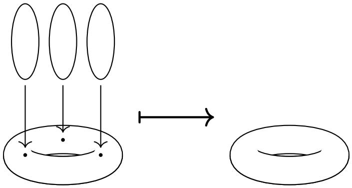
Types as ∞-groupoids indexing a family of pointed things.

Spatial Type Theory (Shulman 2018)

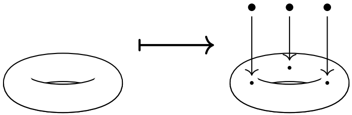
Types as ∞-groupoids equipped with additional topological structure.

Underlying Space

For every parameterised family, there is an operation that forgets the family.

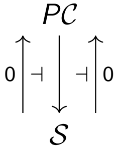


And given a space, we can equip it with the trivial family.



Underlying Space

As a diagram of categories:



Let $\natural$ be the round-trip on $P\mathcal{C}$, this is an idempotent monad and comonad that is adjoint to itself.

Goal:
We want an extension of HoTT with a type former $\natural$ that captures this situation.

Review: Spatial Type Theory

The $\natural$ Modality

Axioms

A Synthetic Smash Product

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Spatial Type Theory

The $\flat/\sharp$ fragment of cohesive type theory (Shulman 2018).

The intended models are 'local toposes':

$$\begin{array}{c} \mathcal{E} \\ \mathrm{Disc}\uparrow\ \dashv\ \downarrow\Gamma\ \dashv\ \uparrow\mathrm{CoDisc} \\ \mathcal{S} \end{array}$$

with the outer functors fully faithful.

- $\flat :\equiv \mathrm{Disc} \circ \Gamma$ is a lex idempotent comonad,
- $\sharp :\equiv \mathrm{CoDisc} \circ \Gamma$ is an idempotent monad,
- with $\flat \dashv \sharp$.

We want $\flat$ and $\sharp$ as unary type formers in our theory.

Spatial Type Theory

Following the pattern of adjoint logic, we put in a judgemental version of $\flat$ and have the type formers interact with it.

$$\Delta \mid \Gamma \vdash a : A \qquad \text{corresponds to} \qquad a : \flat\Delta \times \Gamma \to A$$

We need two variable rules:

$$\textsc{var}\quad \frac{}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A} \qquad \textsc{var-crisp}\quad \frac{}{\Delta, x :: A, \Delta' \mid \Gamma \vdash x : A}$$

The second rule comes from the counit $\flat A \to A$.

Figuring Out $\sharp$

The unary type former $\sharp$ is supposed to be right adjoint to $\flat$, so we make it adjoint to the judgemental context $\flat$.

What does $\flat$ do to contexts? Recall $\Delta \mid \Gamma$ means $\flat\Delta \times \Gamma$.

$$\flat(\flat\Delta \times \Gamma) \cong \flat\flat\Delta \times \flat\Gamma \cong \flat\Delta \times \flat\Gamma \cong \flat(\Delta \times \Gamma)$$

So applying $\flat$ to $\Delta \mid \Gamma$ gives $\Delta, \Gamma \mid \cdot$.

$$\textsc{$\sharp$-form}\ \frac{\Delta, \Gamma \mid \cdot \vdash A \ \mathsf{type}}{\Delta \mid \Gamma \vdash \sharp A \ \mathsf{type}} \qquad \textsc{$\sharp$-intro}\ \frac{\Delta, \Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^{\sharp} : \sharp A}$$

Figuring Out $\sharp$ Elim

First go:

$$\dfrac{\Delta \mid \Gamma \vdash s : \sharp A}{\Delta, \Gamma \mid \cdot \vdash s_\sharp : A}\ \sharp\text{-ELIM-V1?}$$

Going from the conclusion to the premise, demoting Γ only makes it more difficult to use:

$$\dfrac{\Delta \mid \cdot \vdash s : \sharp A}{\Delta \mid \cdot \vdash s_\sharp : A}\ \sharp\text{-ELIM-V2?}$$

Context in the conclusion should be fully general:

$$\dfrac{\Delta \mid \cdot \vdash s : \sharp A}{\Delta \mid \Gamma \vdash s_\sharp : A}\ \sharp\text{-ELIM}$$

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Almost Spatial Type Theory

Comparing the setting of spatial type theory with ours:

$$\begin{array}{c} \mathcal{E} \\ \mathrm{Disc}\Big\uparrow \dashv \Big\downarrow \Gamma \dashv \Big\uparrow \mathrm{CoDisc} \\ \mathcal{S} \end{array} \qquad\qquad \begin{array}{c} P\mathcal{C} \\ 0\Big\uparrow \dashv \Big\downarrow \dashv \Big\uparrow 0 \\ \mathcal{S} \end{array}$$

We could use Spatial Type Theory to study our setting on the right, if we impose that $\flat A \to A \to \sharp A$ is always an equivalence.

But transport across equivalence this would need to occur everywhere. We want a version that captures such a modality directly.

The Roundtrip

- The primary difficulty is that the structure maps include a non-trivial round trip $A \to \natural A \to A$.
- In Spatial Type Theory the counit was *silent*, not annotated in the term.

$$\frac{}{\Delta, x :: A, \Delta' \mid \Gamma \vdash x : A}$$

  At least one of the unit or counit has to be explicit.
- We chose to make the counit *explicit*, and the unit silent.

Variables

Our contexts again have two zones, where $\Delta \mid \Gamma$ morally means $\natural\Delta \times \Gamma$.

$$\frac{}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A}\ \textsc{var}$$

$$\frac{}{\Delta, \underline{x} :: A, \Delta' \mid \Gamma \vdash \underline{x} : A}\ \textsc{var-zero}$$

$$\frac{}{\Delta \mid \Gamma, x : A, \Gamma' \vdash \underline{x} : \underline{A}}\ \textsc{var-roundtrip}$$

- VAR-ZERO corresponds to a use of the counit,
- VAR-ROUNDTRIP corresponds to the unit followed by the counit.
- With this convention, whenever $x : A$ is used via $\natural A$, it is marked.

$\natural$ on Contexts

What does $\natural$ do to contexts? Like last time:

$$\natural(\natural\Delta \times \Gamma) \cong \natural\natural\Delta \times \natural\Gamma \cong \natural\Delta \times \natural\Gamma \cong \natural(\Delta \times \Gamma)$$

But we can't write $\Delta, \Gamma \mid \cdot$ exactly, because the counit is not silent! The types in Γ have to have all uses of other variables from Γ marked.

Let's write $\Delta, 0\Gamma \mid \cdot$ for this.

E.g.: $\underline{x} :: A \mid y : B, z : C(y)$ becomes $\underline{x} :: A, \underline{y} :: B, \underline{z} :: C(\underline{y}) \mid \cdot$.

Marking Terms

Precomposition with the structural rules can be extended to terms:

$$\textsc{counit}\ \frac{\Delta \mid \Gamma \vdash a : A}{\Delta, 0\Gamma \mid \cdot \vdash \underline{a} : \underline{A}} \qquad \textsc{unit}\ \frac{\Delta, 0\Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a : A}$$

When using $x : A$ via the round-trip, also have to round-trip the type:

$$\frac{\textsc{var-roundtrip}}{\Delta \mid \Gamma, x : A, \Gamma' \vdash \underline{x} : \underline{A}}$$

Figuring Out $\natural$

$$\frac{\Delta, 0\Gamma \mid \cdot \vdash A \text{ type}}{\Delta \mid \Gamma \vdash \natural A \text{ type}}\ \natural\text{-FORM}$$

$$\frac{\Delta, 0\Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^{\natural} : \natural A}\ \natural\text{-INTRO}$$

$$\frac{\Delta \mid \Gamma \vdash a : \natural A}{\Delta, 0\Gamma \mid \cdot \vdash a_{\natural} : A}\ \natural\text{-ELIM-V1?}$$

Here we don't have to drop Γ as we did with $\sharp$, instead we can precompose the result with the unit:

$$\frac{\Delta \mid \Gamma \vdash a : \natural A}{\Delta \mid \Gamma \vdash a_{\natural} : A}\ \natural\text{-ELIM}$$

Rules for $\natural$

$\natural$-FORM

$$\frac{\Delta, 0\Gamma \mid \cdot \vdash A \text{ type}}{\Delta \mid \Gamma \vdash \natural A \text{ type}}$$

$\natural$-INTRO

$$\frac{\Delta, 0\Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^{\natural} : \natural A}$$

$\natural$-ELIM

$$\frac{\Delta \mid \Gamma \vdash v : \natural A}{\Delta \mid \Gamma \vdash v_{\natural} : A}$$

$\natural$-BETA

$$\frac{\Delta, 0\Gamma \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^{\natural}{}_{\natural} \equiv a : A}$$

$\natural$-ETA

$$\frac{\Delta \mid \Gamma \vdash v : \natural A}{\Delta \mid \Gamma \vdash v \equiv \underline{v}_{\natural}{}^{\natural} : \natural A}$$

Properties of $\natural$

- $\natural$ is a lex monadic modality in the sense of the HoTT book, like $\sharp$
- $\natural$ is also comonadic, like $\flat$
- $\natural$ is self-adjoint: $\natural(\natural A \to B) \simeq \natural(A \to \natural B)$

Definition

- A type X is a *space* if $(\lambda x.\underline{x}^{\natural}) : X \to \natural \underline{X}$ is an equivalence.
- A type E is a *spectrum* if $\natural \underline{E}$ is contractible.

(To be more model agnostic you might call these 'modal' and 'reduced')

Using $\natural$

Proposition

For any A, the type $\natural\underline{A}$ is a space.

Proof.

We have to show that $(\lambda v.\underline{v}^\natural) : \natural\underline{A} \to \natural\natural\underline{A}$ is an equivalence. For an inverse, use the counit $(\lambda z.z_\natural) : \natural\natural\underline{A} \to \natural\underline{A}$.

In one direction:

$$\underline{z_\natural}^\natural \equiv \underline{z}_\natural{}^\natural \equiv z$$

and in the other:

$$\underline{v^\natural}_\natural \equiv \underline{v} \equiv \underline{\underline{v}}_\natural{}^\natural \equiv \underline{v}_\natural{}^\natural \equiv v.$$

□

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Stability

Our spectra don't behave much like actual spectra yet.

Axiom S

For any 'dull' spectra $\underline{E}$ and $\underline{F}$, the wedge inclusion $\iota_{\underline{E},\underline{F}} : \underline{E} \vee \underline{F} \to \underline{E} \times \underline{F}$ is an equivalence.

(The 'spectra' don't form a stable category in every slice, only in slices over spaces!)

Proposition

A dull square of spectra is a pushout square iff it is a pullback square.

Proposition

Dull spectra and dull maps between them are ∞-connected.

Normalisation

Fix a distinguished spectrum $\mathbb{S}$: Type.

We can use this to build an adjunction

$$\mathrm{Space}_\star \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Spec}$$

$$\Sigma^\infty X :\equiv X \wedge \mathbb{S}$$
$$\Omega^\infty \underline{E} :\equiv \natural(\mathbb{S} \to_\star \underline{E})$$

Definition

The homotopy groups of a spectrum $\underline{E}$ are

$$\pi_n^s(\underline{E}) :\equiv \pi_n(\Omega^\infty \underline{E})$$

Normalisation

In fact, this factors into a sequence of adjunctions:

$$\mathrm{Space}_\star \underset{0th}{\overset{\mathrm{susp}}{\rightleftarrows}} \mathrm{SeqPreSpec} \underset{\iota}{\overset{\mathrm{spec}}{\rightleftarrows}} \mathrm{SeqSpec} \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Spec}$$

where $\mathrm{SeqPreSpec}$ and $\mathrm{SeqSpec}$ are the types of sequential prespectra and spectra described earlier.

$$\begin{aligned} LJ &:\equiv \mathrm{colim}(\Sigma^\infty J_0 \to \Omega\Sigma^\infty J_1 \to \Omega^2\Sigma^\infty J_2 \to \dots) \\ (R\underline{E})_n &:\equiv \Omega^\infty\Sigma^n \underline{E} \end{aligned}$$

(The details of the $\mathrm{SeqPreSpec} \to \mathrm{SeqSpec}$ adjunction have not yet been done in type theory)

Normalisation

Axiom N

The $L \dashv R$ adjunction between $\mathrm{SeqSpec}$ and Spec is a (dull) adjoint equivalence: $\mathrm{Mor}(J, R\underline{E}) \simeq \natural(LJ \to_\star \underline{E})$

Proposition

$\pi_n^s(\mathbb{S}) \simeq \mathrm{colim}_k\, \pi_{n+k}(S^k)$

Proof.

$$\begin{aligned}\pi_n^s(\mathbb{S}) &\equiv \pi_n(\Omega^\infty \mathbb{S}) \simeq \pi_n(\Omega^\infty(S^0 \wedge \mathbb{S})) \simeq \pi_n(\Omega^\infty \Sigma^\infty S^0) \\ &\simeq \pi_n(\operatorname*{colim}_k \Omega^k \Sigma^k S^0) \simeq \operatorname*{colim}_k \pi_n(\Omega^k \Sigma^k S^0) \\ &\simeq \operatorname*{colim}_k \pi_{n+k}(\Sigma^k S^0) \simeq \operatorname*{colim}_k \pi_{n+k}(S^k)\end{aligned}$$

□

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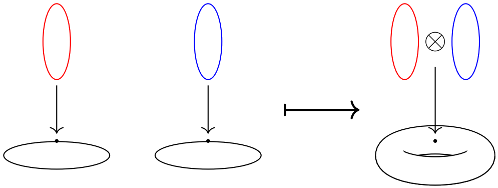
Coming Soon

For two types A and B there should be a type $A \otimes B$ corresponding to the 'external smash product'.



This is a symmetric monoidal product with no additional structural rules. (i.e., no weakening or contraction)

Bunched Contexts

We can take a cue from 'bunched logics', where there are two ways of combining contexts, an ordinary cartesian one and a linear one.

$$\frac{\Gamma_1 \ \mathsf{ctx} \qquad \Gamma_2 \ \mathsf{ctx}}{\Gamma_1, \Gamma_2 \ \mathsf{ctx}} \qquad\qquad \frac{\Gamma_1 \ \mathsf{ctx} \qquad \Gamma_2 \ \mathsf{ctx}}{(\Gamma_1)(\Gamma_2) \ \mathsf{ctx}}$$

For the comma *only*, we have weakening and contraction as normal.

Smash and Dependency

- When does a 'dependent external smash' $(x : A) \otimes B(x)$ make sense?
- When $B(x)$ only depends on the base space of $x : A$, so when we have $(x : A) \otimes B(\underline{x})$.
- Having the modality first is critical for dependent smash to work!

Thank You!

- Described a human-usable type theory for a $\natural$ modality with the correct properties.
- Gave an axiom making synthetic spectra form a stable category, and another for ‘normalisation’ of $\mathbb{S}$.
- Hinted at how the smash type former will work.

Questions?

References I

Joyal, André (2008). *Notes on logoi*. URL: http://www.math.uchicago.edu/~may/IMA/JOYAL/Joyal.pdf.

Shulman, Michael (2018). "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Math. Structures Comput. Sci.* 28.6, pp. 856–941. ISSN: 0960-1295. DOI: 10.1017/S0960129517000147. URL: https://doi.org/10.1017/S0960129517000147.