Linear Homotopy Type Theory

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Intended Models

Space-parameterised families of Spectra

Or more generally:

$\mathcal{X}$-parameterised families of $\mathcal{C}$

where

- $\mathcal{X}$ is an $\infty$-topos,
- $\mathcal{C}$ is a symmetric monoidal closed $\infty$-category with a zero object.

(A $\mathcal{C}$ for which $\mathcal{X}$-parameterised families form an $\infty$-topos is called an ‘$\infty$-locus’, Hoyois 2019)

Every object has a nonlinear aspect and a linear aspect.
Intended Models

확정 모델

- ▲: (R., Finster, and Licata 2021)
- ▲: 'Fibrewise' tensor product
- $\otimes$: Unit of $\otimes$
- $\to$: Right adjoint to $\otimes$
The *homology and cohomology of* \(X\) *with coefficients in* \(E\) *can be defined by*

\[
E_n(X) \equiv \pi^s_n(\Sigma^\infty(X) \otimes E)
\]

\[
E^n(X) \equiv \pi^s_n(\Sigma^\infty(X) \rightarrow E)
\]

*where*

\[
\pi^s_n(E) \equiv \mathbb{H}(\mathbb{S} \rightarrow E)
\]

\[
\Sigma^\infty(X) \equiv X \wedge \mathbb{S}
\]
New Type Formers

We want the output of the type formers to be *ordinary types*.

Cannot use an indexed type theory (Vákár 2014; Krishnaswami, Pradic, and Benton 2015; Isaev 2021), or quantitative type theory (McBride 2016; Atkey 2018; Moon, Eades III, and Orchard 2021; Fu, Kishida, and Selinger 2020)
The Symmetry Proof We Want

Proposition
sym : \( A \otimes B \simeq B \otimes A \)

Proof.
To define \( \text{sym} : A \otimes B \rightarrow B \otimes A \), suppose we have \( p : A \otimes B \). Then \( \otimes \)-induction allows us to assume \( p \equiv x \otimes y \), and we have \( y \otimes x \).

\[
\text{sym} : \equiv \lambda p. \text{let } x \otimes y = p \text{ in } y \otimes x
\]

Then to prove \( \prod_{(p : A \otimes B)} \text{sym}(\text{sym}(p)) = p \), use \( \otimes \)-induction again: the goal reduces to \( x \otimes y = x \otimes y \) for which we have reflexivity.

\[
\text{inv} : \equiv \lambda p. \text{let } x \otimes y = p \text{ in refl}_{x \otimes y}
\]
We need to prevent terms like $\lambda x.x \otimes x : A \to A \otimes A$, so variable use needs to be restricted somehow.

- Every variable $x$ has a *colour* $c$.
- The relationships between colours are collected in a *palette*.

Palettes $\Phi$ are constructed by

\[
1 \quad \Phi_1 \otimes \Phi_2 \quad \Phi_1, \Phi_2 \quad c \quad c \prec \Phi
\]

Typical palettes:

\[
p \prec r \otimes b \quad w \prec (p \prec r \otimes b) \otimes y \quad p \prec (r \otimes b, r' \otimes b')
\]

(Similar to ‘bunched’ type theory P. W. O’Hearn and Pym 1999; P. O’Hearn 2003)
Using Colourful Variables

Building a term, we need to keep track of the current ‘top colour’. Suppose the palette is \( p \prec r \otimes b \), and we have variables

\[ x^r : A, \quad y^b : B, \quad z^p : C. \]

- The top colour here is \( p \).
- The only variable that can be used currently is \( z : C \). (Using \( x \) here would correspond to a projection from one side of a tensor.)
- Ordinary type formers bind variables with the current top colour:

\[
\sum_{(x:A)} B(x) \quad \Pi_{(x:A)} B(x) \quad (\lambda x.b) \\
\text{ind}_+(z.C, x.c_1, y.c_2, p) \quad \text{ind}_-(x.x'.p.C, x.c, p)
\]

- The rules for \( \otimes \) will grant us access to the other variables.
Rules for $\otimes$, Take 1

Let $p$ be the top colour.

- **Formation:** For closed* $A : \mathcal{U}$ and $B : \mathcal{U}$, there is a type $A \otimes B : \mathcal{U}$.

- **Introduction:** In palette* $p \prec r \otimes b$, for any terms $a : A$ with colour $r$ and $b : B$ with colour $b$, there is a term

  $$a_{r \otimes b} b : A \otimes B$$

- **Elimination:** Any term $p : A \otimes B$ may be assumed to be of the form $x_{r \otimes b} y$ for some variables $x^r : A, y^b : B$ with $p \prec r \otimes b$, in a term $c : C[x_{r \otimes b} y/z]$.

  $$\text{(let } x_{r \otimes b} y = p \text{ in } c) : C[p/z]$$

- **Computation:** If the term really is of the form $a_{r' \otimes b'} b$, then

  $$\text{(let } x_{r \otimes b} y = a_{r' \otimes b'} b \text{ in } c) \equiv c[r'/r \otimes b'/b | a/x, b/y]$$
Proposition

*There is a function* \( \text{sym} : A \otimes B \rightarrow B \otimes A \)

Proof.

Suppose have \( p : A \otimes B \). Then \( \otimes \)-induction on \( p \) gives \( x^r : A \) and \( y^b : B \), where \( p \approx r \otimes b \).

We need to form a purple term of \( B \otimes A \), so ‘split \( p \) into \( b \) and \( r \)’. Then we can form \( y^b \otimes r^x : B \otimes A \).

\[
\text{sym} \equiv \lambda p. \text{let } x \otimes b \ y = p \text{ in } y^b \otimes r^x
\]

But we don’t have \( p \approx b \otimes r \) literally, we need to build in the symmetric monoidal structure.
Palette Splits

Need a more general judgement for when the palette linearly splits into two monoidally combined pieces: $\Phi \vdash \overrightarrow{r} | \overrightarrow{b}$ split

**Symmetry:** In palette $p \prec r \otimes b$,

$$b \mid r \text{ split}$$

**Associativity:** In palette $w \prec (p \prec r \otimes b) \otimes y$,

$$r \mid (b \otimes y) \text{ split}$$

**Cartesian weakening:** In palette $p \prec (r \otimes b, r' \otimes b')$,

$$r' \mid b' \text{ split}$$
Rules for $\otimes$, Take 2

Let $p$ be the top colour.

- **Formation:** For closed* $A : \mathcal{U}$ and $B : \mathcal{U}$, there is a type $A \otimes B : \mathcal{U}$.

- **Introduction:** For any palette split $\vec{r} \mid \vec{b}$ and terms $a : A$ with colour $\vec{r}$ and $b : B$ with colour $\vec{b}$, there is a term

  $$a \otimes_{\vec{r}} b : A \otimes B$$

- **Elimination:** Any term $p : A \otimes B$ may be assumed to be of the form $x \otimes_{\vec{r}} y$ for some variables $x^{\vec{r}} : A$, $y^{\vec{b}} : B$ with $p \prec r \otimes b$ in a term $c : C[x \otimes_{\vec{r}} y]$. 

  $$(\text{let } x \otimes_{\vec{r}} y = p \text{ in } c) : C[p/z]$$

- **Computation:** If the term really is of the form $a \otimes_{\vec{r}'} b'$, then

  $$(\text{let } x \otimes_{\vec{b}} y = a \otimes_{\vec{r}'} b' \text{ in } c) \equiv c[r'/r \otimes b'/b \mid a/x, b/y]$$
Eg: Uniqueness principle for $\otimes$

**Proposition**

If $C : A \otimes B \rightarrow \mathcal{U}$ is a type family and $f : \prod_{p : A \otimes B} C(p)$, then for any $p : A \otimes B$ we have

$$(\text{let } x \otimes y = p \text{ in } f(x \otimes y)) = f(p)$$

**Proof.**

By $\otimes$-induction we may assume $p \equiv x' \otimes y'$. Our goal is now

$$(\text{let } x \otimes y = x' \otimes y' \text{ in } f(x \otimes y)) = f(x' \otimes y')$$

Which by computation reduces to $f(x' \otimes y') = f(x' \otimes y')$, for which we have reflexivity. □

(Cannot state this in indexed type or quantitative type theories)
Dependency in \[\otimes\]

From last time:

- Any assumption \(x : A\) can be used ‘marked’: \(\_x : A\).
- A \(\_x\) usage ignores the ‘linear aspect’ of \(x\).
- A term \(a\) is *dull* if all free variables in \(a\) are marked.

Then we can allow the following dependency in \(\otimes\):

- If \(A : \mathcal{U}\) and \(B : \mathcal{U}\) are *dull* types then \(A \otimes B : \mathcal{U}\).
- If \(A : \mathcal{U}\) is a dull type and \(B\) is a dull type assuming \(x : A\), then \(\bigotimes (x : A) B : \mathcal{U}\).
Eg. Associativity

Like dependent associativity of $\times$,

\[
\text{assoc} : \left( \sum_{x:A} \sum_{y:B(x)} C(x)(y) \right) \\
\simeq \left( \sum_{v:\sum_{x:A} B(x)} C(\text{pr}_1 v)(\text{pr}_2 v) \right)
\]

There is dependent associativity of $\otimes$:

\[
\text{assoc} : \left( \bigotimes_{x:A} \bigotimes_{y:B(x)} C(x)(y) \right) \\
\simeq \left( \bigotimes_{v:\sum_{x:A} B(x)} \text{let } x \otimes y = v \text{ in } C(x)(y) \right)
\]
\[\Gamma \times A \vdash B\]
\[\Gamma \vdash A \rightarrow B\]
\[\Gamma \otimes A \vdash B\]
\[\Gamma \vdash A \rightarrow B\]
$$\Gamma \times (x : A) \vdash b : B$$
$$\Gamma \vdash \lambda x. b : \prod_{(x : A)} B$$

$$\Gamma \otimes (y : A) \vdash b : B$$
$$\Gamma \vdash \partial y. b : \prod_{(y : A)} B$$
Hom

\[\frac{r \mid \Gamma, x^r : A \vdash b : B}{r \mid \Gamma \vdash \lambda x. b : \prod_{(x : A)} B}\]

\[\frac{p \prec r \otimes b \mid \Gamma, y^b : A \vdash b : B}{r \mid \Gamma \vdash \partial y. b : \prod_{(y^b : A)} B}\]
Hom Extensionality

**Axiom Homext**
For any $f, g : \prod_{x:A} B(x)$, the function

$$(f = g) \rightarrow \prod_{x:A} f(x) = g(x)$$

is an equivalence.

**Theorem**
*Univalence implies hom extensionality.*
Bigger Picture
Applications

- Formalising some arguments in synthetic homotopy theory: (Schreiber 2017, Section 5.5)
- Acting as a specification language for quantum circuits: (Fu, Kishida, Ross, et al. 2020; Fu, Kishida, and Selinger 2020)
Modal Type Theories

- **Specialised modal extensions of MLTT:** (Shulman 2018; Birkedal et al. 2020; Gratzer, Sterling, and Birkedal 2019; Zwanziger 2019; Bizjak et al. 2016)

- **MTT Framework:** Adjoint Modalities, Dependent Types, No Substructural Types (Gratzer, Kavvos, et al. 2020; Gratzer, Cavallo, et al. 2021)

- **Fibrational Framework:** Any Modalities, Non-dependent Types, Substructural Types (Licata and Shulman 2016; Licata, Shulman, and R. 2017)

Linear HoTT does not currently fit into either framework!

Lars Birkedal et al. (2020). “Modal dependent type theory and dependent right adjoints”. In: Mathematical Structures in Computer Science 30.2. DOI: 10.1017/S0960129519000197.


References II


Benjamin Moon, Harley Eades III, and Dominic Orchard (2021). “Graded Modal Dependent Type Theory”. In: Programming Languages and Systems. DOI: 10.1007/978-3-030-72019-3_17.


