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Higher sites and their higher categorical logic

HoTTEST-Series

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Overview

A motivation

An ∞-topos is an ∞-category with an internal univalent type theory which is inherently proof-relevant. Predicates are generally non-monic type families, represented by a fibration of the form

$$\begin{array}{ccc} \sum_{b:B} E(x) & \text{e.g.} & \sum_{b_1,b_2:B\times B} b_1 =_B b_2 \\ \downarrow E & & \downarrow \mathrm{Id}_B \\ B & & B \times B \end{array}$$

according to the "Propositions as types"-paradigm. Particularly, due to univalence, a general fiber $b_1 =_B b_2$ of Id_B can be of virtually any homotopy type.

A motivation

In particular, this holds for the terminal ∞-topos: The ∞-category $\mathcal{S}$ of spaces. That means, the “external” logic of an ∞-topos is proof-relevant (and univalent) as well.

Guiding Principle: If the duality of logic and topology ought to be mathematically universal, and the logic in this context is proof relevant, then so should be its topology.

Thus, given a small ∞-category $\mathcal{C}$, want a canonical $\mathcal{C}$-indexed logical structure $\mathcal{O}_{\mathcal{C}}$ such that ∞-toposes embedded in $\hat{\mathcal{C}} := [\mathcal{C}^{op}, \mathcal{S}]$ correspond exactly to the topological ideals/logical quotients of $\mathcal{O}_{\mathcal{C}}$.

In order to capture **all** such ideals we have to allow to take quotients at **all** suitable multiplicative substructures.

A motivation

- In ordinary topos theory such suitable multiplicative substructures are presented by Grothendieck topologies: That is, certain $\mathcal{C}$-indexed collections of **sieves**

$$S\colon (\mathcal{C}_{/C})^{op} \to \{0,1\}$$
$$(f\colon D \to C) \mapsto \begin{cases} 1, & \text{if } f \in S, \\ 0, & \text{otherwise.} \end{cases}$$

- In higher topos theory such suitable multiplicative substructures should consist of general $\mathcal{C}$-indexed collections of **proof relevant** predicates

$$S\colon (\mathcal{C}_{/C})^{op} \to \mathcal{S}$$
$$(f\colon D \to C) \mapsto S(f)$$

where the spaces $S(f)$ can be of any homotopy type.

A motivation

Grothendieck topologies on a small category $\mathcal{C}$ are generated by an associated notion of covers over objects $C \in \mathcal{C}$: That is, collections of objects $X_i \to C$.
A J-sieve S thus generated consists of the maps $f \in \mathcal{C}_{/C}$ which **(merely) exhibit** a factorization through one of the X_i.
Two such generalized elements $f, g \colon D \to C$ of S coincide if and only if they coincide as generalized elements of the representable yC.
The ∞-categorical context allows to consider proof-relevant covers: Diagrams $X \colon I \to \mathcal{C}_{/C}$ which generate a cover $\mathrm{colim}\, yX \to yC$ whose generalized elements consist of **maps $f \in \mathcal{C}_{/C}$ together with a specified lift into a component X_i**.
Two such generalized elements f, g are equal if their lifts to the formal colimit $\mathrm{colim}\, yX \to yC$ coincide.

Ordinary Topos Theory

The proof-irrelevant logical structure sheaf

Let $\mathcal{C}$ be a small category. Consider the composition

$$\Omega_{\mathcal{C}} : \mathcal{C}^{op} \xrightarrow{\mathcal{C}_{/(\cdot)}} \mathrm{Cat}^{op} \xrightarrow{\Omega^{(\cdot)}} \mathrm{Frm}$$
$$C \mapsto \mathrm{Sv}(\mathcal{C}_{/C})$$

for Ω the subobject classifier in Set and Frm the category of frames and frame homomorphisms. This defines the "**proof-irrelevant logical structure sheaf on $\mathcal{C}$**".

Whenever $\mathcal{C}$ has finite products, $\Omega_{\mathcal{C}}$ is a first order hyperdoctrine on $\mathcal{C}$ with equality. We will generally think of $\mathcal{O}_{\mathcal{C}}$ as a canonical logical equipment of $\mathcal{C}$.

Definition

Let $\mathcal{C}$ be a small category. A sheaf $\mathcal{E}$ of $\Omega_{\mathcal{C}}$-ideals is a regular subfunctor

$$\mathcal{E} \subseteq \Omega_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Loc}$$

with $\mathcal{C}$-indexed reflector.

$\rightsquigarrow$ A sheaf $\mathcal{E}$ of $\Omega_{\mathcal{C}}$-ideals is literally a functor of exponential ideals of $\Omega_{\mathcal{C}}$, such that the associated “nuclei” $j_C \colon \Omega_{\mathcal{C}}(C) \twoheadrightarrow \mathcal{E}(C)$ for $C \in \mathcal{C}$ assemble to a natural transformation over $\mathcal{C}$ as well.

$\rightsquigarrow$ The **Grothendieck topology** J associated to $\mathcal{E}$ is exactly the collection of predicates $J(C) \subseteq \Omega_{\mathcal{C}}(C)$ nullified in $\mathcal{E}(C)$.

Theorem (e.g. Mac Lane and Moerdijk's Book)

Let $\mathcal{C}$ be a small category. Then the following stand in 1-1 correspondence to one another.

1. *Equivalence classes of reflective left exact localizations of $\hat{\mathcal{C}}$.*
2. *A closure operator on $\hat{\mathcal{C}}$, that is, an indexed left exact factorization system of monomorphisms in $\hat{\mathcal{C}}$.*
3. *Sheaves of $\Omega_{\mathcal{C}}$-ideals.*
4. *Grothendieck topologies on $\mathcal{C}$.*

□

Remark

A closure operator on $\hat{\mathcal{C}}$ is an elementary subfibration

$$\begin{array}{ccc} \sum_{X \in \hat{\mathcal{C}}} \mathcal{E}(X) & \hookrightarrow & \sum_{X \in \hat{\mathcal{C}}} \mathrm{Sub}(X) \\ & \searrow & \downarrow \\ & & \hat{\mathcal{C}} \end{array}$$

with a cartesian fibered reflector $\overline{(\cdot)}\colon \sum_{X \in \hat{\mathcal{C}}} \mathrm{Sub}(X) \to \mathcal{E}$ which preserves meets fiberwise. Its pullback along the Yoneda embedding is exactly its associated sheaf of $\Omega_{\mathcal{C}}$-ideals represented as an elementary fibration.

Proposition

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a sheaf of $\Omega_{\mathcal{C}}$-ideals and J the associated Grothendieck topology on $\mathcal{C}$. Then the diagram $\mathcal{E} : \mathcal{C}^{op} \to \mathrm{Frm}$ is a J-stack of frames: For every J-cover S, the sequence

$$\mathcal{E}_C \longrightarrow \prod_{f \in S} \mathcal{E}_{\mathrm{dom} f} \rightrightarrows \prod_{f \in S, \mathrm{dom} f = \mathrm{cod} g} \mathcal{E}_{\mathrm{dom} g}$$

is an equalizer diagram of frames. □

Remark

Given a site $(\mathcal{C}, J)$, a presheaf X is a J-sheaf if and only if for all covering sieves $S \hookrightarrow yC$ the induced map $X(C) \to \{S, X\}$ of weighted limits is an isomorphism.

Higher Topos Theory

Global notions

Definition (Lurie, Rezk)

An ∞-category $\mathcal{B}$ is an **∞-topos** if it is equivalent to an accessible left exact localization of the ∞-category $\hat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves over a small ∞-category $\mathcal{C}$.

Given a small ∞-category $\mathcal{C}$, any accessible localization $\hat{\mathcal{C}} \to \mathcal{B}$ is reflective and hence may be presented by the ∞-subcategory of $\mathcal{E}$-local objects in $\hat{\mathcal{C}}$.

Equivalent definitions have been given via accordingly defined Giraud axioms as well as via Rezk's notion of descent.

Definition (Joyal)

Given an ∞-category $\mathcal{B}$, a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in $\mathcal{B}$ is a *factorisation system* whenever

1. $\mathcal{L} \perp \mathcal{R}$,
2. every map in $\mathcal{B}$ has an $(\mathcal{L}, \mathcal{R})$-factorisation,
3. each of the two classes $\mathcal{L}$ and $\mathcal{R}$ is closed under retracts.

A factorization system $(\mathcal{L}, \mathcal{R})$ is **of small generation** if there is a set $S \subseteq \mathcal{B}^{\Delta^1}$ such that $\mathcal{L} = {}^{\perp}(S^{\perp})$.

Definition (RSS, ABFJ)

Let $\mathcal{B}$ be an ∞-category with pullbacks. A factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$ is

1. a **modality** on $\mathcal{B}$ if the left class $\mathcal{L}$ is pullback stable.
2. a **left exact modality** if the full ∞-subcategory $\mathcal{L} \subseteq \mathcal{B}^{\Delta^1}$ is closed under finite limits.

Proposition

Let $\mathcal{B}$ be a presentable ∞-category with universal colimits. Then the following structures stand in bijective correspondence to one another.

1. *Equivalence classes of accessible left exact localizations of $\mathcal{B}$.*
2. *Left exact modalities of small generation on $\mathcal{B}$.*

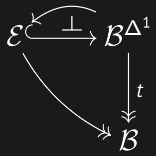
Proof.

Essentially ABFJ. $\square$

Proposition

Let $\mathcal{B}$ be an ∞-category with pullbacks. There are the following 1-1 correspondences.

1. ***Factorisation systems*** *on $\mathcal{B}$, and equivalence classes of fibered reflective localizations*

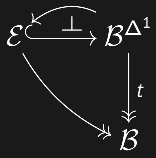


such that $\mathcal{E} \subseteq \mathcal{B}^{\Delta^1}$ is subfibration which contains all identities and is closed under composition.

Proposition

Let $\mathcal{B}$ be an ∞-category with pullbacks. There are the following 1-1 correspondences.

2. ***Modalities*** *on $\mathcal{B}$, and equivalence classes of fibered reflective localizations*

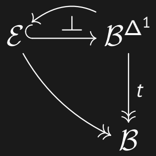


such that $\mathcal{E} \subseteq \mathcal{B}^{\Delta^1}$ is subfibration which contains all identities and is closed under composition, and ***the fibered reflector is a cartesian functor****.*

Proposition

Let $\mathcal{B}$ be an ∞-category with pullbacks. There are the following 1-1 correspondences.

3. ***Left exact modalities*** *on $\mathcal{D}$, and equivalence classes of fibered reflective localizations*

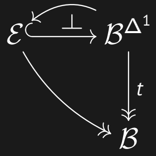


with (fiberwise) ***left exact*** *cartesian reflector, such that $\mathcal{E} \subseteq \mathcal{B}^{\Delta^1}$ is closed under composition.*

Proposition

Let $\mathcal{B}$ be a presentable ∞-category with universal colimits. There are the following 1-1 correspondences.

4. ***Modalities of small generation*** *on $\mathcal{B}$, and equivalence classes of fibered reflective localizations*



such that $\mathcal{E} \subseteq \mathcal{B}^{\Delta^1}$ is a ***fiberwise accessible subfibration*** *which contains all identities and is closed under composition, and the fibered reflector is a cartesian functor.*

□

Remark

A reflective localization as in the Proposition exhibits $\mathcal{E}$ as a cocartesian fibration as well. Given a modality $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$, its associated right adjoint

$$\mathcal{B}^{\Delta^1} \to \sum_{B \in \mathcal{B}} \mathcal{R}(B)$$

is essentially the operation which makes $\sum_{B \in \mathcal{B}} \mathcal{R}(B) \twoheadrightarrow \mathcal{B}$ into a full cartesian Lawvere ∞-category (Jacobs). The second part hence constitutes a 1-1 correspondence between modalities on $\mathcal{B}$ and "full cartesian Lawvere ∞-categories with strong sums" over $\mathcal{B}$.

Lemma

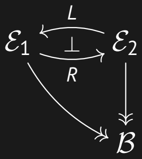
Let $\iota \colon \mathcal{E} \hookrightarrow \mathcal{B}^{\Delta^1}$ be a fibered reflective localization. Then $\mathcal{E} \subseteq \mathcal{B}^{\Delta^1}$ is closed under composition if and only if for every $f \colon A \to B$ in $\mathcal{B}$ the cocartesian action $\Sigma_f \colon \mathcal{E}(A) \to \mathcal{E}(B)$ is conservative. □

Higher Topos Theory

Towards higher Lawvere-Tierney topologies

Fact

A fibered adjoint pair between cartesian fibrations



over an ∞-category $\mathcal{B}$ is a homotopy-coherent adjunction in the ∞-cosmos $\mathrm{Cart}(\mathcal{B})$ of cartesian fibrations over $\mathcal{B}$ if and only if the underlying maps L and R are cartesian functors.

Beck Monadicity (Riehl-Verity): Adjunctions in $\mathrm{Cart}(\mathcal{B})$ correspond to homotopy-coherent monads in $\mathrm{Cart}(\mathcal{B})$.

Proposition

Let $\mathcal{B}$ be presentable. Then there is a 1-1 correspondence between the following structures.

1. *Fibered accessible reflective localizations $\mathcal{E} \hookrightarrow \mathcal{B}^{\Delta^1}$ with left exact reflector such that $\mathcal{E}$ is closed under composition.*
2. *Fibered accessible left exact idempotent (homotopy-coherent) monads T on $\mathcal{B}^{\Delta^1}$ such that $T(g \circ f) \simeq T(g) \circ T(\eta_g \circ T(f))$ for every pair of composable arrows in $\mathcal{B}$, where η_g is the unit of T applied to g.*

□

Definition

A *modal operator T* on an ∞-category $\mathcal{B}$ with pullbacks is a fibered left exact idempotent monad on $\mathcal{B}^{\Delta^1}$.

To internalize such operators, we make use of the following result.

Proposition

Let $\mathcal{B}$ be a presentable ∞-category. The externalization functor

$$\mathrm{Ext}\colon \mathrm{IntCat}(\mathcal{B}) \to \mathrm{Cart}(\mathcal{B})$$

is a cosmological embedding. □

Furthermore, whenever the base $\mathcal{B}$ is presentable, the target fibration $\mathcal{B}^{\Delta^1} \twoheadrightarrow \mathcal{B}$ can be filtrated by the subfibrations $(\mathcal{B}^{\Delta^1})_\kappa \twoheadrightarrow \mathcal{B}$ of κ-small maps for any cofinal sequence of large enough regular cardinals κ.

Proposition

Let $\mathcal{B}$ be an ∞-topos. For every regular cardinal κ large enough, the object classifier π_κ for κ-small maps gives rise to an internal ∞-category $N(\pi_\kappa)$ such that $\mathrm{Ext}(N(\pi_\kappa)) \simeq (\mathcal{B}^{\Delta^1})_\kappa$ in $\mathrm{Cart}(\mathcal{B})$. □

Theorem (Work in progress)

Let $\mathcal{B}$ be an ∞-topos. Then there is a 1-1 correspondence between the following structures.

1. *Equivalence classes of accessible left exact localizations of $\mathcal{B}$.*
2. *Equivalence classes of sequences of eventually pairwise compatible modal operators T_λ on $N(\pi_\lambda) \in \mathrm{IntCat}(\mathcal{B})$ which each satisfy the composition formula and are accessible in a suitable sense.* □

Remark

The composition formula essentially seems to say that the underlying endofunctor $T\colon N(\pi_\lambda) \to N(\pi_\lambda)$ is determined by the unit of the monad and its restriction $T^{\simeq}\colon N(\pi_\lambda)^{\simeq} \to N(\pi_\lambda)^{\simeq}$ to the core.

Higher topos theory

The logical structure sheaf

Let $\mathcal{C}$ be a small ∞-category, and consider the composition

$$\mathcal{C}^{op} \xrightarrow{y^{op}} (\hat{\mathcal{C}})^{op} \xrightarrow{\hat{\mathcal{C}}_{/-}} \mathrm{CAT}_\infty.$$

Here, CAT_∞ is the ∞-category of *large* ∞-categories. Each such value

$$\hat{\mathcal{C}}_{/yC} \simeq \widehat{\mathcal{C}_{/C}} \simeq \mathrm{RFib}(\mathcal{C}_{/C})$$

is an ∞-topos, and the induced transition maps $f^*\colon \hat{\mathcal{C}}_{/yD} \to \hat{\mathcal{C}}_{/yC}$ for $f\colon C \to D$ in $\mathcal{C}$ are part of the étale geometric morphisms (Σ_f, f^*, Π_f). We will denote the induced composition by

$$\mathcal{O}_{\mathcal{C}}\colon \mathcal{C}^{op} \to (\mathrm{LTop}, \mathrm{Et})$$

and refer to $\mathcal{O}_{\mathcal{C}}$ as the **proof relevant logical structure sheaf**.

Definition

Let RTop be the ∞-category of ∞-toposes and geometric morphisms, and LTop be its opposite. Thus, the arrows in LTop are the left exact cocontinuous functors.

A **geometric embedding** is a geometric morphism between ∞-toposes with fully faithful right adjoint.

A geometric morphism $f_*\colon \mathcal{F} \to \mathcal{E}$ between ∞-toposes is **étale** if its left adjoint f^* is equivalent to one of the form

$$(\_ \times E)\colon \mathcal{E} \to \mathcal{E}_{/E}$$

for some object $E \in \mathcal{E}$.

Let $(\mathrm{RTop}, \mathrm{Et})$ be the ∞-category of ∞-toposes and étale geometric morphisms between them, and let $(\mathrm{LTop}, \mathrm{Et})$ again be the opposite.

Proposition (Recognition Criterion, Lurie)

A geometric morphism $f_* \colon \mathcal{F} \to \mathcal{E}$ is étale if and only if the following three conditions hold.

1. *The left adjoint f^* admits a further left adjoint $f_! \colon \mathcal{F} \to \mathcal{E}$.*
2. *The left adjoint $f_!$ is conservative.*
3. *The pair $(f_!, f^*)$ satisfies the projection formula, i.e. for every $X \to Y$ in $\mathcal{E}$, every object $Z \in \mathcal{F}$ and every morphism $f_! Z \to Y$, the induced square*

$$\begin{array}{ccc} f_!(f^*X \times_{f^*Y} Z) & \longrightarrow & f_!Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is cartesian in $\mathcal{E}$.

Definition

Let $\mathcal{C}$ be a small ∞-category. A sheaf $\mathcal{E}$ of $\mathcal{O}_{\mathcal{C}}$-ideals is a fiberwise accessible full subfibration

$$\begin{array}{ccc} \mathcal{E} & \overset{\iota}{\hookrightarrow} & \mathcal{O}_{\mathcal{C}} \\ & \searrow\!\!\!\searrow & \downarrow\!\!\!\downarrow \\ & & \mathcal{C} \end{array}$$

with a fibered cartesian left exact left adjoint such that the associated indexed ∞-topos $\mathcal{E}\colon \mathcal{C}^{op} \to \mathrm{LTop}$ factors through $(\mathrm{LTop}, \mathrm{Et})$ as well.

$\rightsquigarrow$ For $f\colon C \to D$ in $\mathcal{C}$ we obtain homotopy-commutative squares

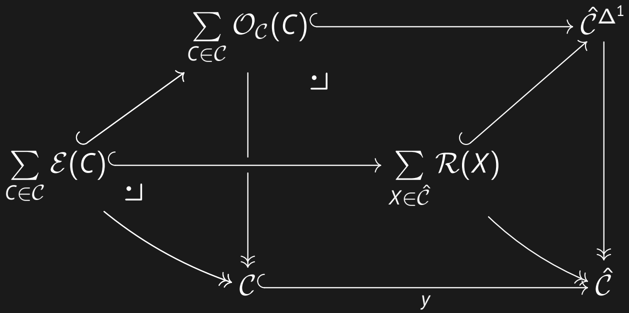
$$\begin{array}{ccc} \mathcal{E}_D & \underset{\bar{(\cdot)}_D}{\overset{\iota_D}{\rightleftarrows}} & \hat{\mathcal{C}}_{/yD} \\ {\scriptstyle (\mathcal{E}_f)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{E}_C & \underset{\bar{(\cdot)}_C}{\overset{\iota_C}{\rightleftarrows}} & \hat{\mathcal{C}}_{/yC} \end{array}$$

Theorem

Let $\mathcal{C}$ be a small ∞-category. Then the following structures stand in 1-1 correspondence to one another.

1. *Left exact modalities $(\mathcal{L}, \mathcal{R})$ of small generation on $\hat{\mathcal{C}}$.*
2. *Sheaves $\mathcal{E}$ of $\mathcal{O}_{\mathcal{C}}$-ideals.*

Proof.



Essentially, left Kan extension/restriction along y + descent. □

Higher topos theory

Higher Grothendieck topologies

Given a sheaf $\mathcal{E}$ of $\mathcal{O}_{\mathcal{C}}$-ideals, consider the classes of pointwise nullified "predicates":

$$J(C) = \bar{(\cdot)}_C^{-1}[\{\text{contractible objects}\}] \subseteq \hat{\mathcal{C}}_{/yC}.$$

That is, the full ∞-subcategory in $\hat{\mathcal{C}}_{/yC}$ generated by the preimage of the contractible objects in $\mathcal{E}(C)$. $\rightsquigarrow$ These yield a notion of "higher' or

'"proof-relevant" Grothendieck topologies when axiomatized abstractly.

Definition

Let $\mathcal{C}$ be a small ∞-category. A *proof-relevant topology* $J = \bigcup_{C \in \mathcal{C}} J(C)$ on $\mathcal{C}$ is a family of full ∞-subcategories $J(C) \subseteq \hat{\mathcal{C}}_{/yC}$ such that the following conditions hold.

1. $1_{yC} \in J(C)$ for all $C \in \mathcal{C}$ (Unitality).
2. $f^*[J(C)] \subseteq J(D)$ for all maps $f\colon D \to C$ in $\mathcal{C}$ (Stability).
3. Given a pair
$$B \xrightarrow{b} A \xrightarrow{a} C$$
such that $C \in \mathcal{C}$, $a \in J(C)$ and $d^*b \in J(D)$ for all $d\colon yD \to A$, then the composition ba is contained in $J(C)$ as well (Transitivity).
4. The full ∞-subcategory $J(C) \subseteq \hat{\mathcal{C}}_{/yC}$ is closed under pullbacks for all $C \in \mathcal{C}$ (Local left exactness).

Definition (Continuation)

5. Given a decomposition of a representable yC into a pushout $A \cup_C B$, and an extension of the span $S = (A \leftarrow C \to B)$ to a span $s = (f \leftarrow h \to g)$ in $(J_{\mathrm{loc}})_{/yC}$ with $t(s) = S$, then the pushout $f \cup_h g$ is contained in $J(C)$ (Local right exactness).

Furthermore, let $\mathcal{T}$ be the (superlarge) class of all proof-relevant topologies on $\mathcal{C}$. Let $G \subset \sum_{C \in \mathcal{C}} \mathcal{O}_{\mathcal{C}}(C)$ be a set. Then

$$J(G) := \bigcap_{J \in \mathcal{T}, G \subset J} J$$

is the smallest proof-relevant topology on $\mathcal{C}$ which contains G.

6. A proof-relevant topology J on $\mathcal{C}$ is *of small generation* if there is a small set $G \subset \sum_{C \in \mathcal{C}} \mathcal{O}_{\mathcal{C}}(C)$ such that $J = J(G)$.

We will refer to a tuple $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small ∞-category and J is a proof-relevant topology of small generation on $\mathcal{C}$ as an $(\infty, 1)$-site.

Theorem

Let $\mathcal{C}$ be a small ∞-category. Then there is a 1-1 correspondence between the following structures.

1. *Sheaves $\mathcal{E}$ of $\mathcal{O}_{\mathcal{C}}$-ideals.*
2. *Proof-relevant topologies J of small generation on $\mathcal{C}$.* □

Remark (Relation to "proof-irrelevant" Grothendieck topologies)

- *Every proof-relevant topology J (of small generation) induces a Grothendieck topology $J_{-1} := J \cap \{\mathrm{Monos}\}$ on the same ∞-category $\mathcal{C}$. The respective localizations $\hat{\mathcal{C}} \to \mathrm{Sh}_{J_{-1}}(\mathcal{C}) \to \mathrm{Sh}_{J}(\mathcal{C})$ give a factorization of the composite into a topological followed by a cotopological localization.*

Remark

- *A Grothendieck topology J_{-1} on a small ∞-category $\mathcal{C}$ is not quite a proof-relevant topology itself because monomorphisms are not closed under pushouts. But each such J_{-1} generates a proof-relevant topology J on $\mathcal{C}$ such that $J_{-1} = J \cap \{\mathrm{Monos}\}$. This J yields the same left exact (topological) localization on $\hat{\mathcal{C}}$, and hence the same notion of sheaf.*
- *The definition of a proof-relevant topology however does yield the definition of a Grothendieck topology "in the proof-irrelevant context" $\sum_{C \in \mathcal{C}} \mathrm{Sub}(yC) \subseteq \sum_{C \in \mathcal{C}} \hat{\mathcal{C}}_{/yC}$.*

Lastly, to justify the "sheaf"-denotation, we have the following.

Proposition

Given an $(\infty, 1)$-site $(\mathcal{C}, J)$, its associated sheaf of $\mathcal{O}_{\mathcal{C}}$-ideals

$$\mathcal{E} \colon \mathcal{C}^{op} \to (\mathrm{LTop}, \mathrm{Et})$$

is a J-stack in the following sense. Every J-cover $s \colon S \to yC$ of presheaves induces an equivalence

$$\mathcal{E}_C \simeq \{yC, \mathcal{E}\} \xrightarrow{s^*} \{S, \mathcal{E}\}$$

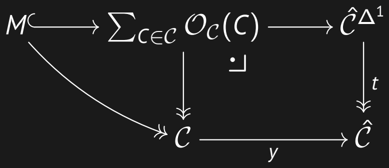
of weighted limits. □

$\rightsquigarrow$ Higher categorical version of the “local character” of an underlying space-valued (and hence proof-relevant) sheaf semantics.

Bases of topologies via modulators

Definition (Anel, Leena-Subramaniam)

A *pre-modulator* M on $\mathcal{C}$ is a collection of sets of objects $M(C) \subset \hat{\mathcal{C}}_{/yC}$ such each $M(C)$ contains the identity 1_{yC}.
A *modulator* M on $\mathcal{C}$ is a full subfibration



such that each fiber $M(C) \subset \hat{\mathcal{C}}_{/yC}$ is a small set and contains the identity 1_{yC}.
A *lex modulator* M on $\mathcal{C}$ is a modulator whose fibers $M(C) \subset \mathcal{O}_{\mathcal{C}}(C)$ are co-filtered.

Bases of topologies via modulators

Id-modulators

Definition

A modulator M on $\mathcal{C}$ is an *Id-modulator* if for every $m \in M(C)$ and every pair of sections s_1, s_2 to m, the equalizer $\mathrm{Eq}(s_1, s_2)_{yC} \to yC$ in $\hat{\mathcal{C}}_{/yC}$ is again contained in $M(C)$.

Lemma

Let $\mathcal{C}$ be a small ∞-category.

1. *Every modulator on $\mathcal{C}$ which is fiberwise closed under finite limits (and hence lex) is an Id-modulator.*
2. *Every Id-modulator on $\mathcal{C}$ is a Δ-modulator.*

□

Corollary

The factorization system on $\hat{\mathcal{C}}$ generated by an Id-modulator M on $\mathcal{C}$ is a left exact modality. The transfinitely iterated plus-construction associated to such M computes the corresponding sheafification.

Proof.

Immediate from the Lemma and [AS]. □

Remark

- *Every proof-relevant topology of small generation on a small ∞-category $\mathcal{C}$ is generated by a lex modulator which is fiberwise closed under finite limits.*
- *Every Grothendieck topology J_{-1} is trivially an Id-modulator: Given $m \in J_{-1}$, there is at most one section to m up to homotopy, and so $\mathrm{Eq}(s_1, s_2)_{/yC} \to yC$ is an equivalence for any two such sections s_1, s_2.*

  $\rightsquigarrow$ *Massive overkill. It would be enough for this equalizer to be covering.*

Examples

The extensive topology

Given a small ∞-category $\mathcal{C}$ with finite coproducts, consider the pre-modulator $M_\sqcup$ defined at an object $C \in \mathcal{C}$ as

$$M_\sqcup(C) := \{\coprod_{i \in I} yC_i \to y(\coprod_{i \in I} C_i) \mid \text{I is a finite set, } \{C_i \mid i \in I\} \subseteq \mathcal{C}\}.$$

Lemma

Let $\mathcal{C}$ be a small ∞-category with finite coproducts. Then a presheaf $X \in \hat{\mathcal{C}}$ is $M_\sqcup$-local if and only if $X \colon \mathcal{C}^{op} \to \mathcal{S}$ preserves finite products. Whenever $\mathcal{C}$ is extensive, the localization $\mathrm{Sh}_\sqcup(\mathcal{C})$ consists exactly of the sheaves for the extensive Grothendieck topology. □

Corollary

Let $\mathcal{C}$ be a small extensive ∞-category. Then the localization $\mathrm{Sh}_\sqcup(\mathcal{C})$ of $\hat{\mathcal{C}}$ at $M_\sqcup$ is topological (and left exact). □

Proposition

Let $\mathcal{C}$ be a small extensive ∞-category. Then the ∞-topos $\mathrm{Sh}_{\sqcup}(\mathcal{C})$ is hypercomplete.

Proof.

The geometric inclusion $\iota\colon \mathrm{Sh}_{\sqcup}(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}}$ preserves sifted colimits. In particular, it preserves effective epimorphisms. □

Corollary

Let $\mathcal{C}$ be a small lextensive ∞-category, i.e. $\mathcal{C}$ is extensive and left exact. Then the ∞-topos $\mathrm{Sh}_{\sqcup}(\mathcal{C})$ has enough points. These are up to equivalence exactly the left exact and finite coproduct preserving functors of type $\mathcal{C} \to \mathcal{S}$.

Proof.

The first statement follows immediately from the Proposition together with the Lurie-Deligne Completeness Theorem. The second statement is a standard argument via left Kan extension along $y\colon \mathcal{C} \to \hat{\mathcal{C}}$. □

Examples

The regular topology

Definition

An ∞-category $\mathcal{C}$ is *regular* if it is finitely complete, the Čech nerve of every morphism $f \in \mathcal{C}$ is effective, and effective epimorphisms in $\mathcal{C}$ are pullback stable.

Given a small regular ∞-category, consider

$$M_{\mathrm{Eff}}(C) = \{f\colon yD \to yC | f \in \mathcal{C} \text{ is an effective epimorphism}\}.$$

The modulator M_{Eff} is not an Id-modulator. In fact, under further mild assumptions on $\mathcal{C}$, the left exact localization of $\hat{\mathcal{C}}$ at M_{Eff} consists exactly of the constant sheaves on $\mathcal{C}$.

$\rightsquigarrow$ What is the largest Id-modulator contained in M_{Eff}?

Definition

A map f in a regular ∞-category $\mathcal{C}$ is called ∞-connected if all its higher diagonals (including the 0-th) are effective epimorphisms. Let Eff_∞ be the class of ∞-connected maps in $\mathcal{C}$.

Lemma

Let $\mathcal{C}$ be small regular. Then the collection $y[\mathrm{Eff}_\infty]$ is a lex modulator. □

Proposition

In general, for $\mathcal{C}$ small regular, the localization $\hat{\mathcal{C}} \to \mathrm{Sh}_{y[\mathrm{Eff}_\infty]}(\mathcal{C})$ is not topological, and not sub-canonical.

Proof.

One can show that all representables are sheaves with respect to the topological part of the localization, but only the representables of hypercomplete objects in $\mathcal{C}$ are sheaves for $y[\mathrm{Eff}_\infty]$. □

Remark

There is a diagram of localizations of the following form.

$$
\begin{array}{ccccc}
 & & & & \mathrm{Sh}_{\mathrm{Reg}}\mathcal{C} \\
 & & & \nearrow & \\
\hat{\mathcal{C}} & \longrightarrow & \mathrm{Sh}_{y[\mathrm{Eff}_\infty]_{-1}}(\mathcal{C}) & & \\
 & & & \searrow & \\
 & & & & \mathrm{Sh}_{y[\mathrm{Eff}_\infty]}\mathcal{C}
\end{array}
$$

Generally, neither of the two leaves are contained in one another
Yet, the points of $\mathrm{Sh}_{\mathrm{Reg}}(\mathcal{C})$ *are exactly the left exact functors* $\mathcal{C} \to \mathcal{S}$ *which preserves effective epimorphisms. The points of* $\mathrm{Sh}_{\mathrm{Eff}_\infty}(\mathcal{C})$ *are exactly the left exact functors* $\mathcal{C} \to \mathcal{S}$ *which preserves* ∞*-connected maps. It follows that* $\mathrm{pt}(\mathrm{Sh}_{\mathrm{Reg}}(\mathcal{C})) \subseteq \mathrm{Sh}_{\mathrm{Eff}_\infty}(\mathcal{C})$.

Examples

The colimit topology of covering diagrams

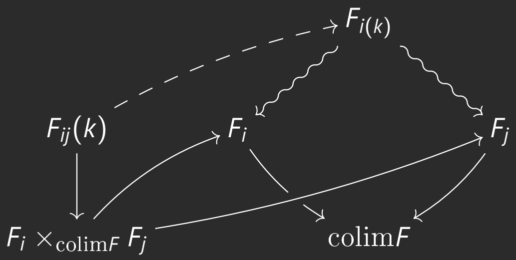
Recall that on a coherent category $\mathcal{C}$, the coherent covering sieves are generated by (finite) collections X_i of objects over a given object C whose coproduct is epimorphic over C.

$$\begin{array}{ccc} & & \coprod_{i \in I} X_i \\ & \overset{\exists ?}{\nearrow} & \downarrow\!\!\!\downarrow \\ D & \longrightarrow & C \end{array}$$

Given a small simplicial set I such that a given ∞-category $\mathcal{C}$ admits I-shaped colimits, we may instead consider covering maps of the form

$$\eta_F\colon \mathrm{colim} yF \to y(\mathrm{colim} F)$$

for functors $F\colon I \to \mathcal{C}$.
To obtain an Id-modulator M which contains such an η_F, we need that each pullback $F_i \times_{\mathrm{colim} F} F_j$ can be expressed as the colimit of a suitable diagram $F_{ij}\colon I_{ij} \to \mathcal{C}_{/(F_i \times_{\mathrm{colim} F} F_j)}$ such that, universally,



Remark

Given a κ-coherent ∞-category, the most straight-forward way to obtain such a modulator M is the classical one:
Consider sets $X \in \mathrm{Set}_\kappa$ and the Grothendieck construction of the simplicial set

$$X^{|(\cdot)|}\colon \Delta^{op} \to \mathrm{Set}_\kappa.$$

Let M be given by the functors $\left(F\colon \sum_{[n]\in\Delta^{op}} X^{|[n]|}\right) \to \mathcal{C}$ such that

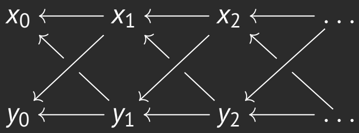
$$F([n], \overrightarrow{i}) \simeq F([0], i_0) \times_{\mathrm{colim}F} \cdots \times_{\mathrm{colim}F} F([0], i_n).$$

Then the lifts to zig-zags exist globally since the spans associated to the pullback $F([n], \overrightarrow{i}) \times_{\mathrm{colim}F} F([m], \overrightarrow{j})$ are contained in the image of F.
Indeed, the set of maps $\mathrm{colim} yF \to y(\mathrm{colim}F)$ for tuples (X, F) as specified above for a coherent ∞-category $\mathcal{C}$ yields an Id-modulator, which generates the κ-coherent Grothendieck topology on $\mathcal{C}$.

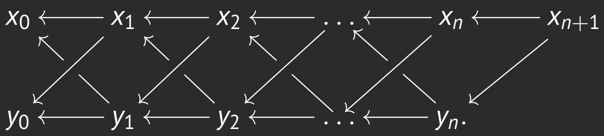
Generally, for a diagram $F\colon I \to \mathcal{C}$ we need to make the condition that the natural map

$$\operatorname*{colim}_{i \leftarrow k \rightarrow j} F_k \to F_i \times_{\mathrm{colim} F} F_j$$

is an equivalence "stable under diagonals". Therefore, we have to require this condition for iterated spans and iterated pullbacks of components over $\mathrm{colim} F$ as well.
Intuitively, we want the tuple (I, F) to be structured well enough so that all higher homotopical data of $\mathrm{colim} F$ can be computed in the components $F(i)$, $i \in I$.
We therefore make the following definitions.
Let S^∞ be the poset generated by the diagram



Let S^n be the truncation of S^∞ at stage n, and D^{n+1} be $\Delta^0 * S^n$, that is, the poset given by



Given a map $p\colon S^n \to I$ into an ∞-category I, let

$$\begin{array}{ccc} \mathrm{Fun}(D^{n+1}, I)_{|S^n = p} & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \{p\}} \\ \mathrm{Fun}(D^{n+1}, I) & \xrightarrow[(\iota_n)^*]{} & \mathrm{Fun}(S^n, I) \end{array}$$

Definition

Let $\mathcal{C}$ be an ∞-category with pullbacks and κ-small colimits and I be a κ-small ∞-category with pullbacks. Let $F\colon I \to \mathcal{C}$ be a functor which preserves pullbacks. Say that F is *covering* if for all $n \geq 0$, and all maps $p\colon S^n \to I$, the natural map

$$\operatorname{colim}\left(\operatorname{Fun}(D^{n+1}, I)_{|S^n = p} \xrightarrow{\mathrm{ev}_{x_{n+1}}} I \xrightarrow{F} \mathcal{C}\right)$$

$$\downarrow$$

$$F(p(x_n)) \times_{\left(F(p(x_{n-1})) \times_{\vdots_{\operatorname{colim} F}} F(p(y_{n-1}))\right)} F(p(y_n))$$

is an equivalence.

Theorem

Let κ be a regular cardinal. Let $\mathcal{C}$ be a small ∞-category with pullbacks and universal κ-small colimits. Let

$$M_\kappa(C) := \{\mathrm{colim} yF \to y\mathrm{colimF} | F\colon I \to \mathcal{C} \text{ is a covering functor,} \\ I \text{ is } \kappa\text{-small with pullbacks.}\}$$

Then M is an Id-modulator on $\mathcal{C}$. □

Remark

- *By construction, the sheaves for M_κ are exactly the presheaves X on $\mathcal{C}$ which take colimits of covering functors to limits. It hence is sub-canonical.*
- *The slice $(\mathcal{C}_{/\_})^{\simeq}$ is a sheaf for M_κ whenever $\mathcal{C}$ has descent (for κ-small covering diagrams).*

Remark

- *Suppose that $\mathcal{C}$ has disjoint coproducts, so it is extensive. Sets are ∞-categories with pullbacks, and one easily sees that every set-indexed functor is covering. It follows that the generating κ-extensive covers are contained in M_κ, and so every M_κ-sheaf is a κ-extensive sheaf.*
- *More generally, every κ-coherent cover is an M_κ-cover (when $\kappa = \omega$, need M_{κ^+}). Thus, we obtain a factorization of localizations of the form*

$$\hat{\mathcal{C}} \to \mathrm{Sh}_{\kappa\text{-}coh}(\mathcal{C}) \to \mathrm{Sh}_{M_\kappa}(\mathcal{C}).$$

- *The points of $\mathrm{Sh}_{M_\kappa}(\mathcal{C})$ correspond to left exact functors $\mathcal{C} \to \mathcal{S}$ which preserve colimits of covering functors.*

$\rightsquigarrow$ *Towards models of intensional type theories with (higher) inductive types and their classifying ∞-toposes?*

Thanks for your time!

[1] M. Anel, A. Joyal, E. Finster, and G. Biedermann, *Higher sheaves and left-exact localizations of ∞-topoi*, `https://arxiv.org/abs/2101.02791`.

[2] M. Anel and C. Leena Subramaniam, *Small object arguments, plus-construction and left-exact localizations*, `https://arxiv.org/abs/2004.00731`, Last update 12 April 2020.

[3] E. Rijke, M. Shulman, and B. Spitters, *Modalities in homotopy type theory*, Logical Methods in Computer Science **16** (2020), no. 1, 2:1–2:79.