Objective Metatheory of Dependent Type Theories

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References can be found in our draft:

"Gluing Models of Type Theory Along Flat Functors"

[S., Angiuli]

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What is Type Theory?

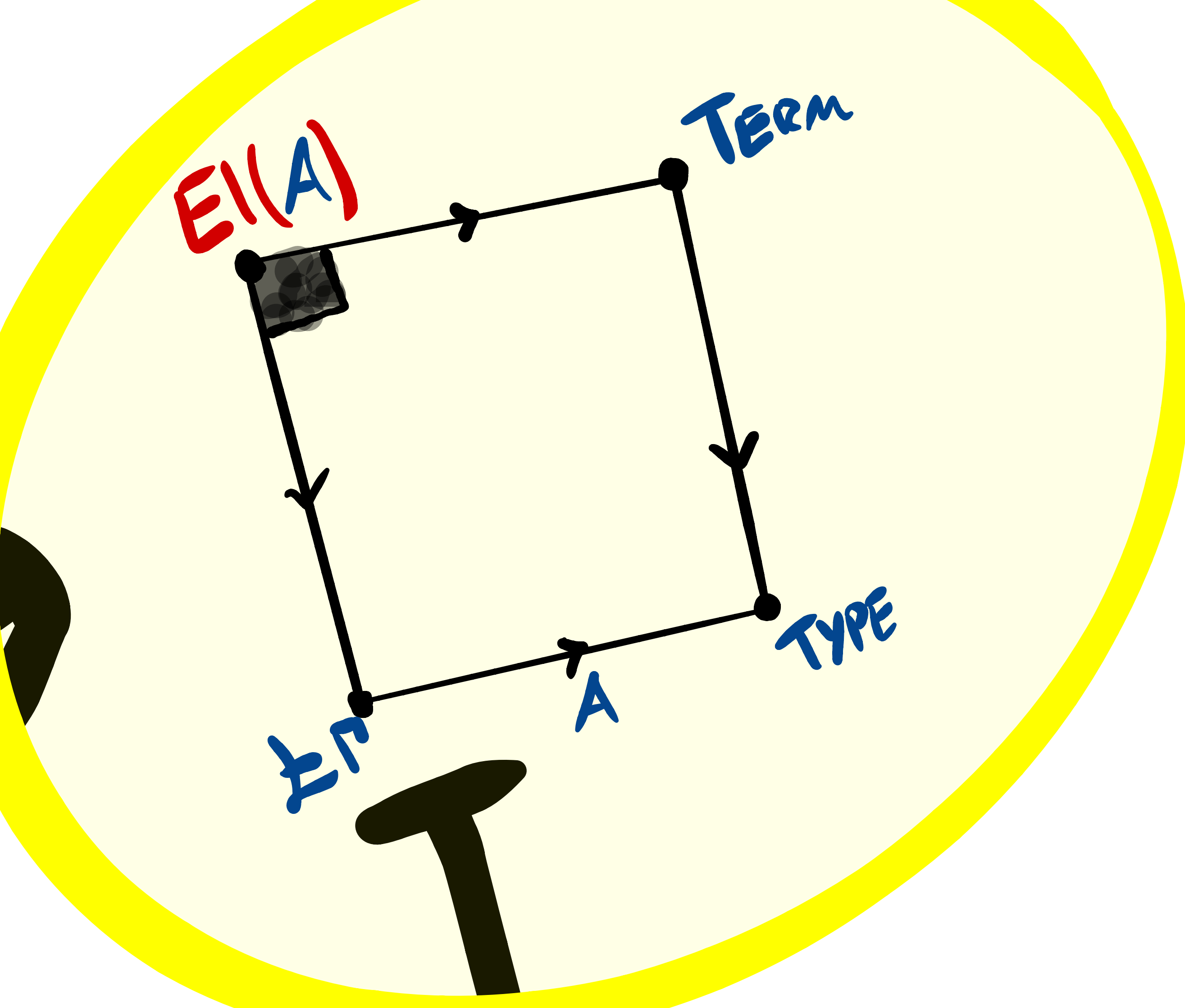
$$\begin{tikzcd} \mathrm{El}(A) \arrow[d, mapsto] \\ \Gamma \end{tikzcd}$$

$$
\begin{tikzcd}
\gamma^{*}\mathrm{El}(A) \arrow[r, dashed] \arrow[d, -{Triangle[open]}] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{El}(A) \arrow[d, -{Triangle[open]}] \\
\Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
$$

$$
\begin{tikzcd}
\gamma^* El(A) \arrow[r, dashed] \arrow[d, maps to] \arrow[dr, phantom, "\lrcorner", very near start] & El(A) \arrow[d, maps to] \\
\Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
$$


A
TYPE


A
TYPE


El(A)
Term
Type
⊢Γ
A

$$
\begin{tikzcd}
\mathrm{El}(A \circ \gamma) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{El}(A) \arrow[d] \\
\Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
$$

Type Theory is:

1) Relative p.o.v. (change of base built in)
2) (Weak) classifying objects

Who is a Type Theorist?

Who is a Type Theorist?

* Study initial model

Who is a Type Theorist?

* Study non-type-theoretic aspects of the initial model

Who is a Type Theorist?

* Study non-type-theoretic aspects of the initial model

⟪ Admissibility, Metatheory ⟫

Initial Model $\xrightarrow{\text{preserves } \langle\!\langle \cdot \vdash \mathcal{J} \rangle\!\rangle}$ Any Model

$$\underbrace{\mathcal{J} ::= (\Gamma \vdash A \ \mathsf{type}) \mid (\Gamma \vdash M : A) \mid (\Gamma \vdash A = B \ \mathsf{type}) \mid (\Gamma \vdash M = N : A)}_{\text{“type theoretic”}}$$

NOT PRESERVING:

$$\langle\!\langle \mathcal{J} \vdash K \rangle\!\rangle$$

Example: $(\Gamma \vdash (A \to B) = (A' \to B') \ \mathsf{type}) \vdash (\Gamma \vdash A = A' \ \mathsf{type})$

holds in initial model, not elsewhere!

Emergent Structure of the

Initial Model

injectivity of Π/Σ

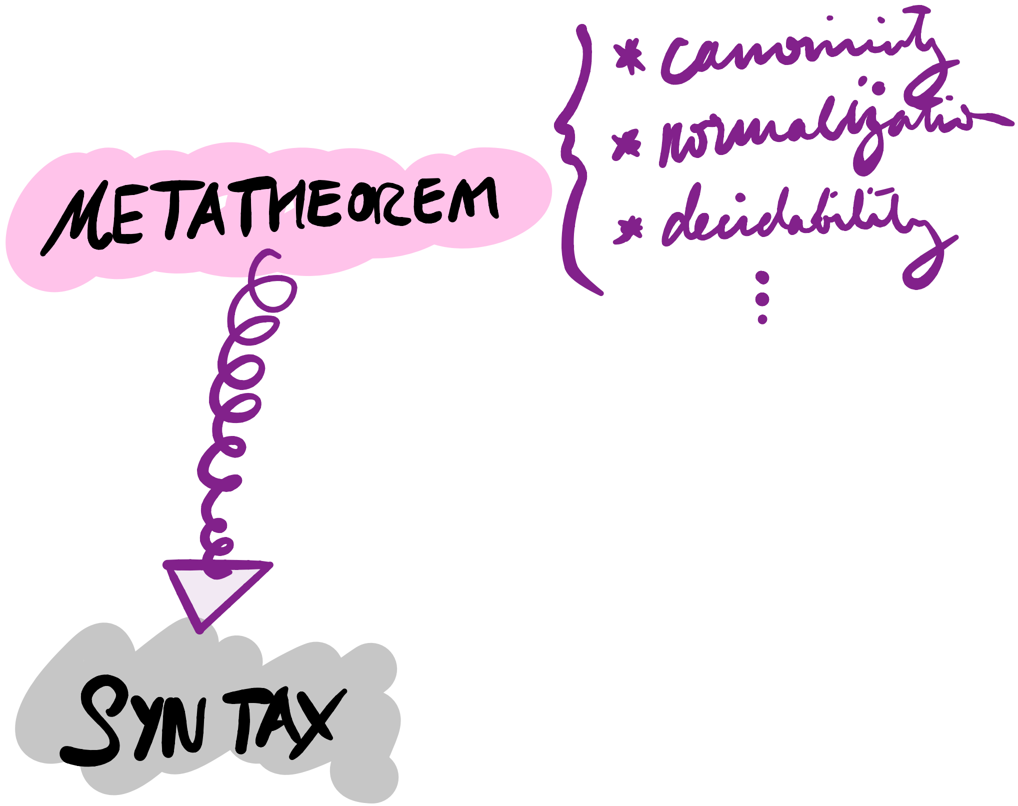
normalization

$\Longrightarrow$ Proof Assistants!

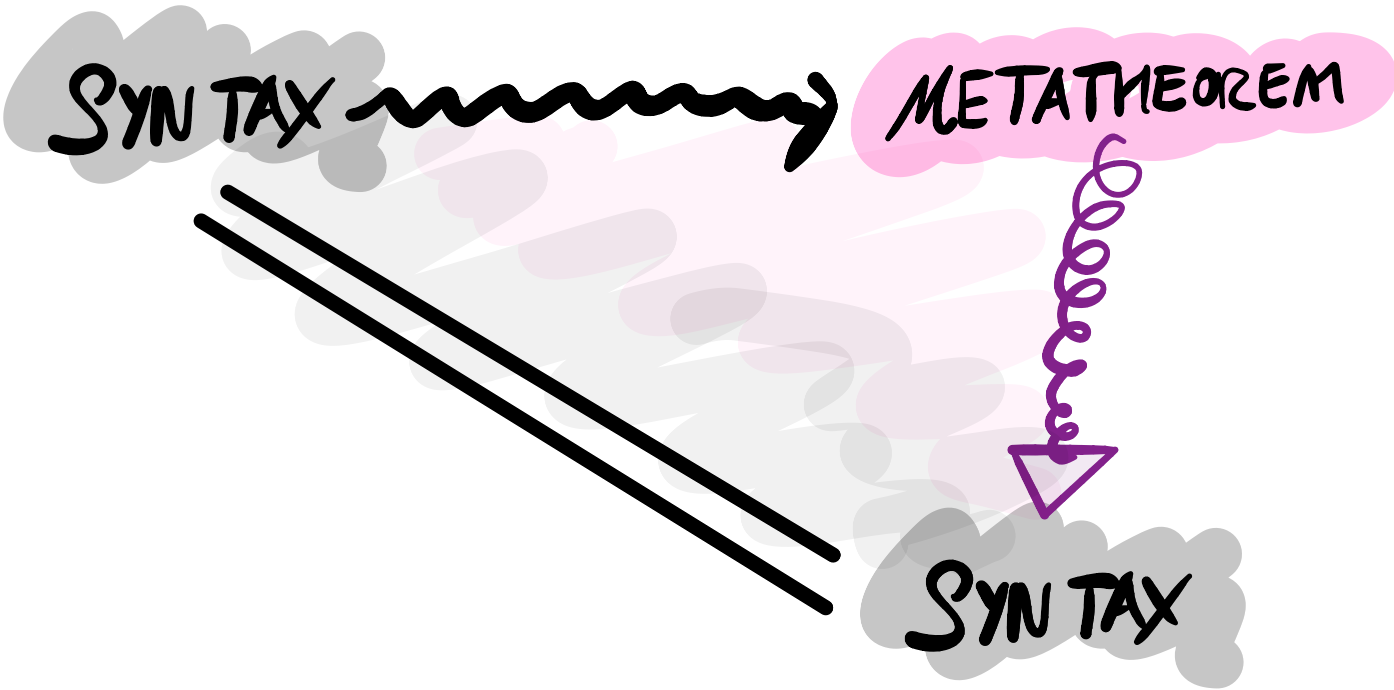
Emergent Structure of the

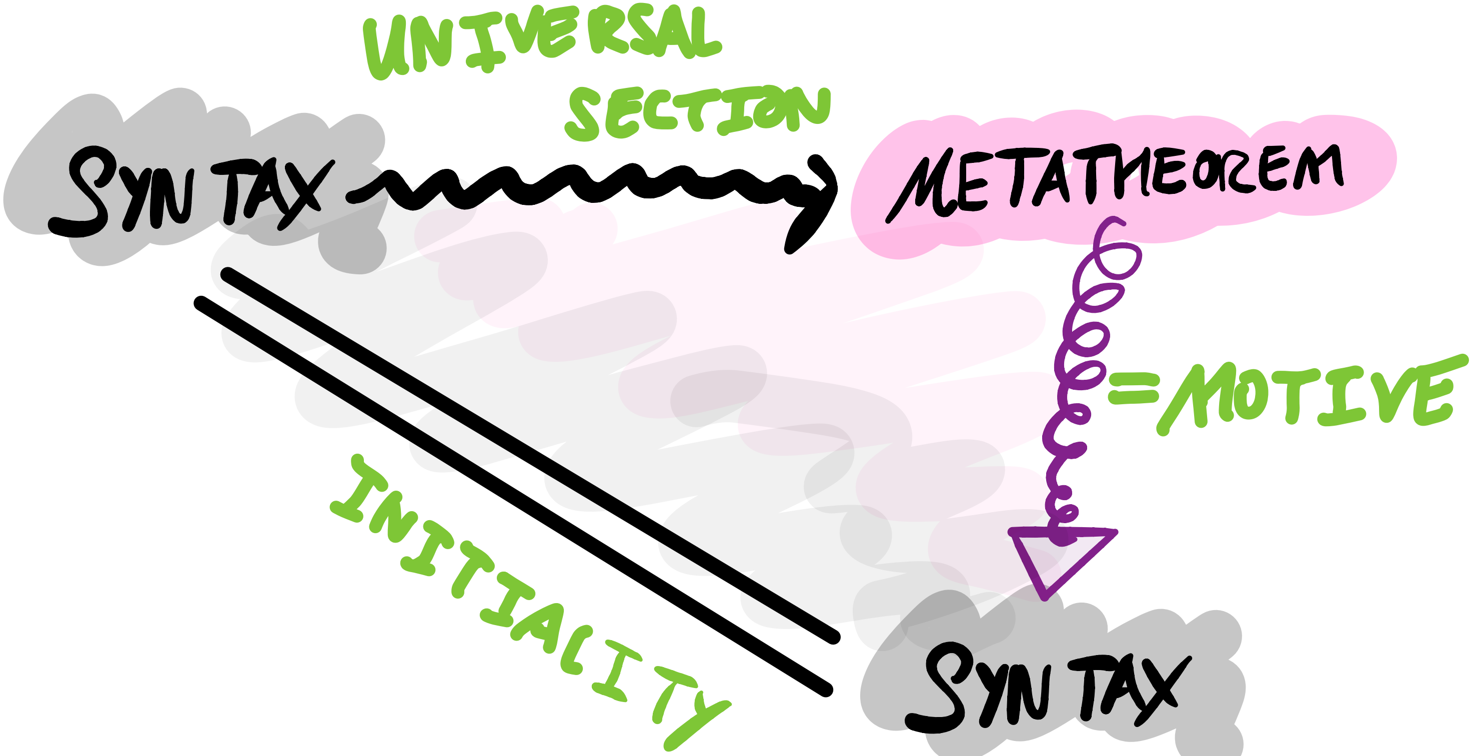
Initial Model

Artin Gluing!

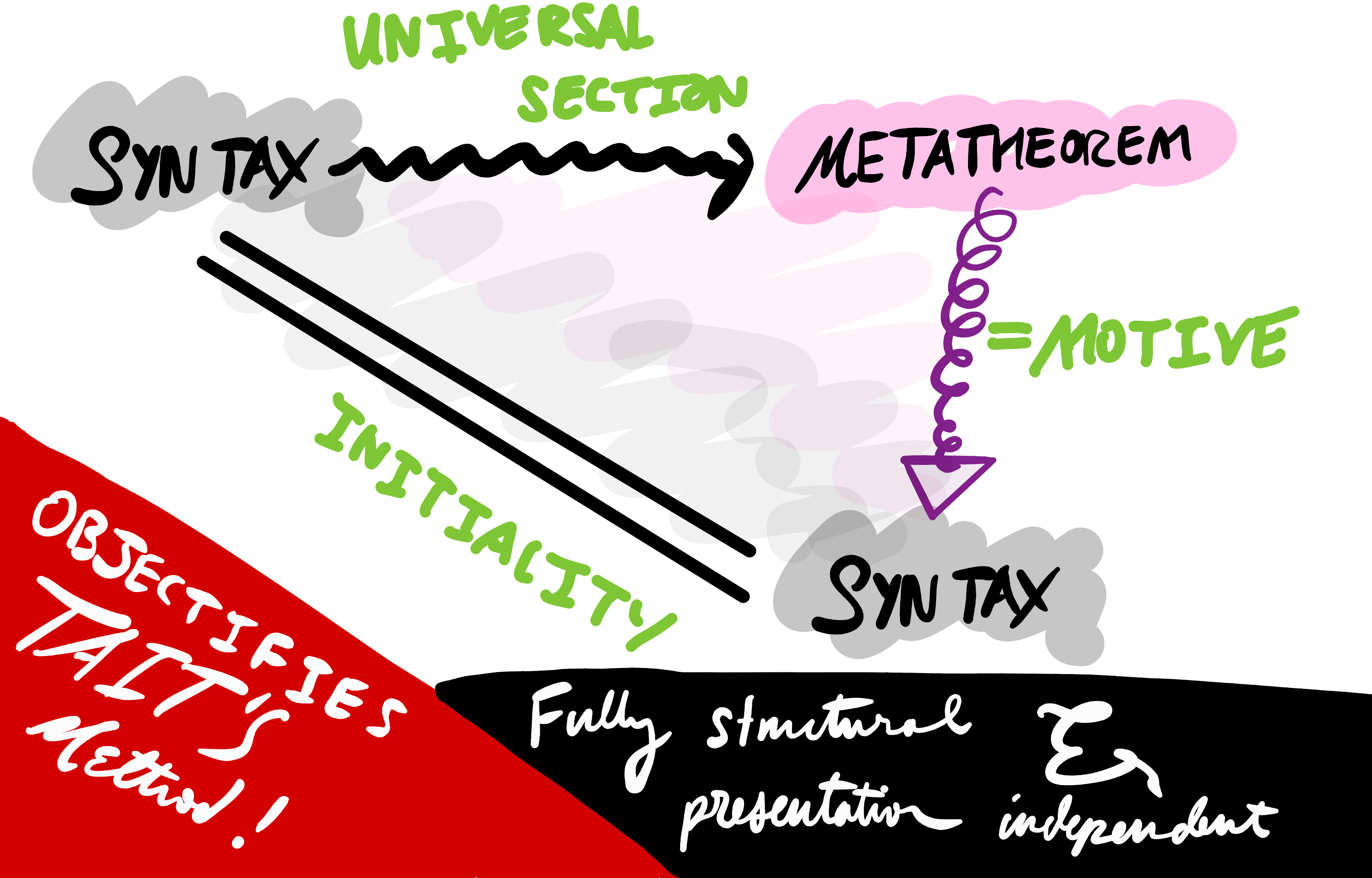

METATHEOREM
* canonicity
* normalization
* decidability
⋮
SYNTAX


METATHEOREM
SYNTAX


SYNTAX
METATHEOREM
SYNTAX

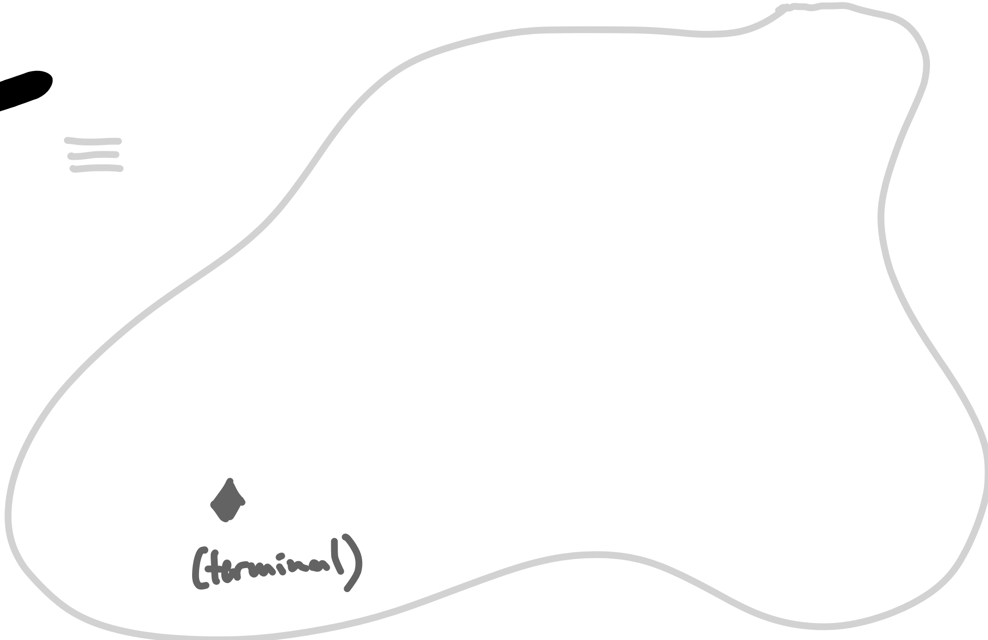


SYNTAX
UNIVERSAL SECTION
METATHEOREM
= MOTIVE
INITIALITY
SYNTAX


UNIVERSAL SECTION
SYNTAX
METATHEOREM
=MOTIVE
INITIALITY
SYNTAX
OBJECTIFIES TAIT'S Method!
Fully structural presentation & independent

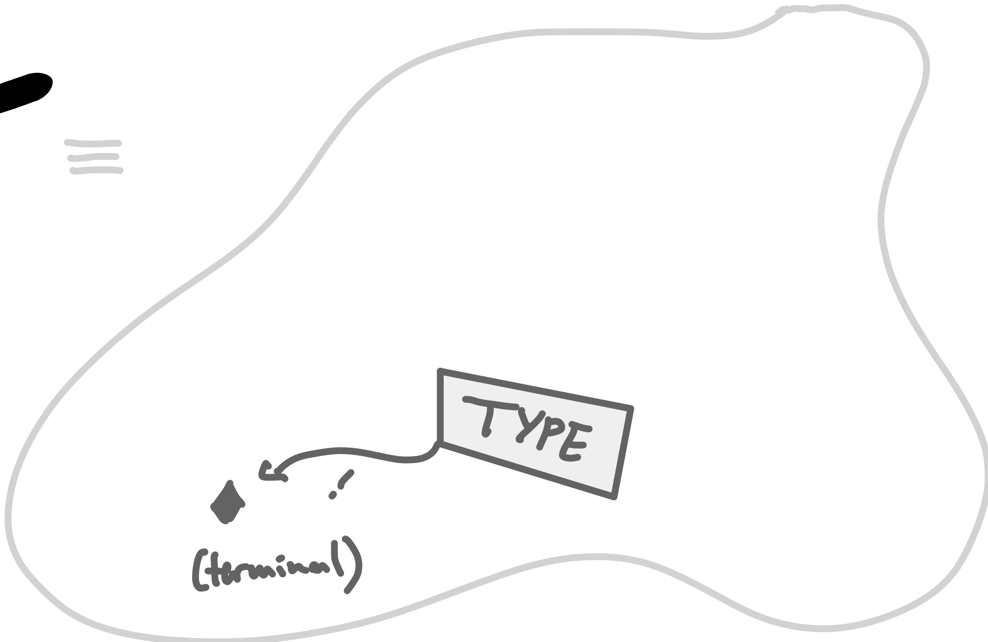
Uemura's General Framework

Π =

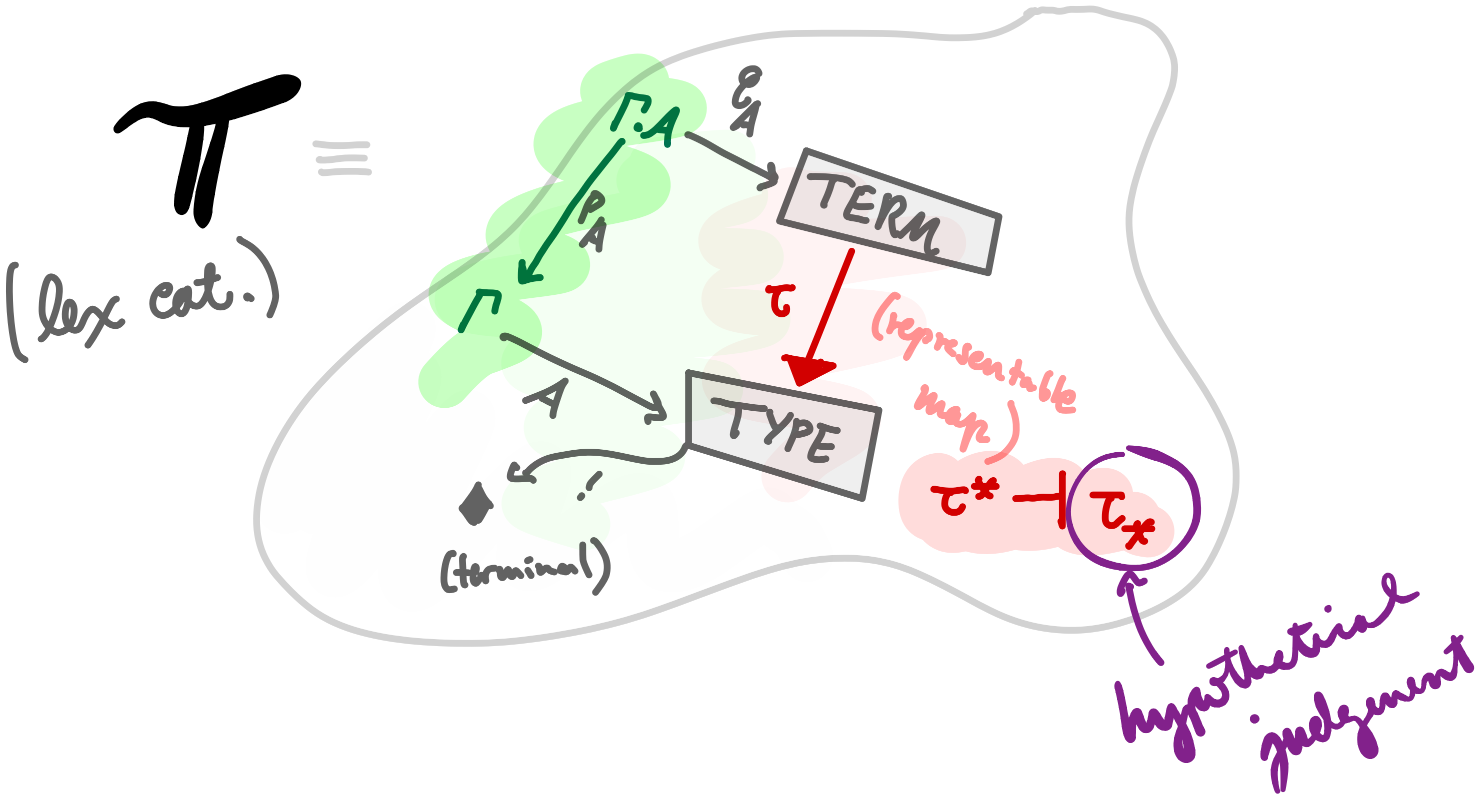


Uemura's General Framework

$\mathcal{T} \equiv$



Uemura's General Framework



$\mathcal{C}_\bullet$

$$\Pi \xrightarrow{\text{functorial semantics}} \hat{\mathcal{C}}$$

$$
\begin{tikzcd}
\Pi^{\to} \arrow[rr, bend left, "\hat{\mathcal{C}}_{\bullet}^{\to}"] & & \hat{\mathcal{C}}^{\to} \\
\mathrm{Rep}_{\Pi} \arrow[u, hook] \arrow[r] & \mathrm{Rep}_{\hat{\mathcal{C}}} \arrow[ur, hook] &
\end{tikzcd}
$$

Rep. Maps à la Grothendieck:

$$
\begin{tikzcd}
\text{よ}(\Gamma.A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Term} \arrow[d, "\tau"] \\
\text{よ}\Gamma \arrow[r, "A"'] & \mathrm{Type}
\end{tikzcd}
$$

Awodey's Natural Models

Let $\mathcal{C}$ be a model of $\mathbb{T}$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\mathrm{gl}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^{\rightarrow} \arrow[d, "\partial_1"] \\
\mathcal{C} \arrow[r] & \mathcal{E}
\end{tikzcd}$$

(global sections
cubical nerve
"renaming" nerve)

Theorem:

If F a flat functor and $\mathcal{E}$ a Grothendieck topos, then $\mathcal{G}$ carries the structure of a model of $\mathbb{T}$, and gl preserves it.

(Angiuli, S.)

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d, "\mathrm{gl}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^{\to} \arrow[d, "\partial_1"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

$$\mathbb{T} \xrightarrow{\mathcal{G}_\bullet} \hat{\mathcal{G}}$$

$$\mathcal{C} \xrightarrow[\Gamma_{lat}]{} \mathcal{E}$$

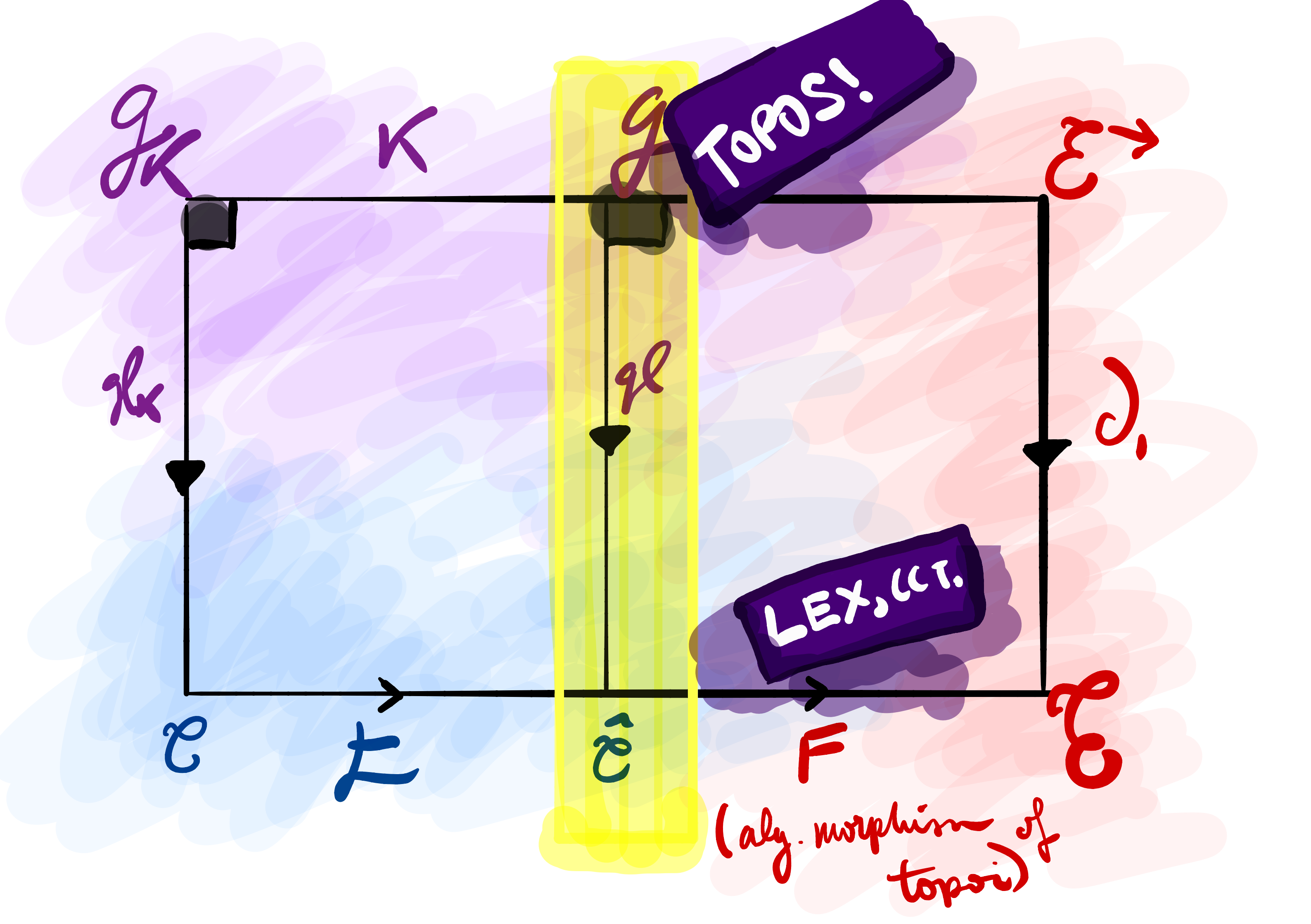
$$
\begin{tikzcd}
 & \hat{\mathcal{C}} \arrow[dr, dashed, "\hat{F}_{\mathrm{lex}}"] & \\
\mathcal{C} \arrow[ur, bend left, "\text{よ}"] \arrow[rr, "F_{\mathrm{lat}}"'] & & \mathcal{E}
\end{tikzcd}
$$

not lex

[Diaconescu]

$$
\begin{tikzcd}
\mathcal{G}_K \arrow[r, "K"] \arrow[d, "\mathcal{U}_K"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{G} \arrow[r] \arrow[d, "\mathcal{U}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^{\to} \arrow[d, "\partial_1"] \\
\mathcal{C} \arrow[r, "\mathcal{L}"'] & \hat{\mathcal{C}} \arrow[r, "F"'] & \mathcal{T}
\end{tikzcd}
$$

(alg. morphism of topoi)



$\mathcal{G}_K$
K
$\mathcal{G}$
TOPOS!
$\mathcal{E}^{\to}$
q_K
q
∂_1
LEX, CCT.
$\mathcal{C}$
$\mathcal{L}$
$\hat{\mathcal{C}}$
F
$\mathcal{E}$
(alg. morphism of topoi)

DENSE! [A.; S.]

$$
\begin{tikzcd}
\mathcal{G}_K \arrow[r, "K"] \arrow[d, "q_K"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{G} \arrow[r] \arrow[d, "q_\ell"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^{\to} \arrow[d, "\partial_1"] \\
\mathcal{C} \arrow[r, "E"'] & \hat{\mathcal{C}} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}
$$

TOPOS!

LEX, CCT.

Therefore, we have **fully faithful nerve**

$$\mathcal{G} \xrightarrow{N_K} \widehat{\mathcal{G}_K}$$

DENSE! [A., S.]

$$\begin{tikzcd}
\mathcal{G}_K \arrow[r, "K"] \arrow[d, "\pi_K"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G} \arrow[r] \arrow[d, "\mathrm{gl}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^{\to} \arrow[d, "\partial_1"] \\
\mathcal{C} \arrow[r, "\text{よ}"'] & \hat{\mathcal{C}} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}$$

TOPOS!

LEX, LCT.

THEREFORE:

fully faithful nerve

$$\Pi \longrightarrow \mathcal{G} \xrightarrow{N_K} \widehat{\mathcal{G}_K}$$

[Easy]

The Gluing Model of Π

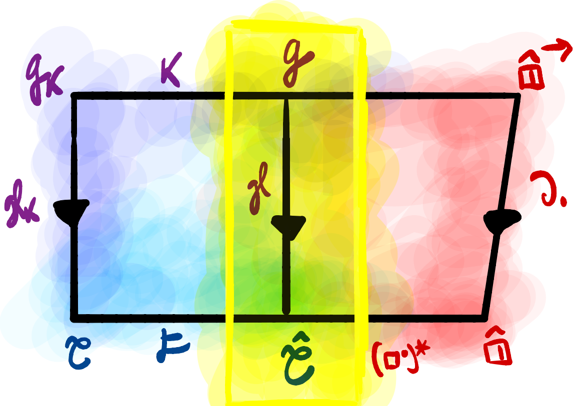
$\square$-ical Canonicity

1) $\mathbb{I} : \mathcal{C}$

2) $\square \xrightarrow[\text{f.p.}]{\square^{\circ}} \mathcal{C}$

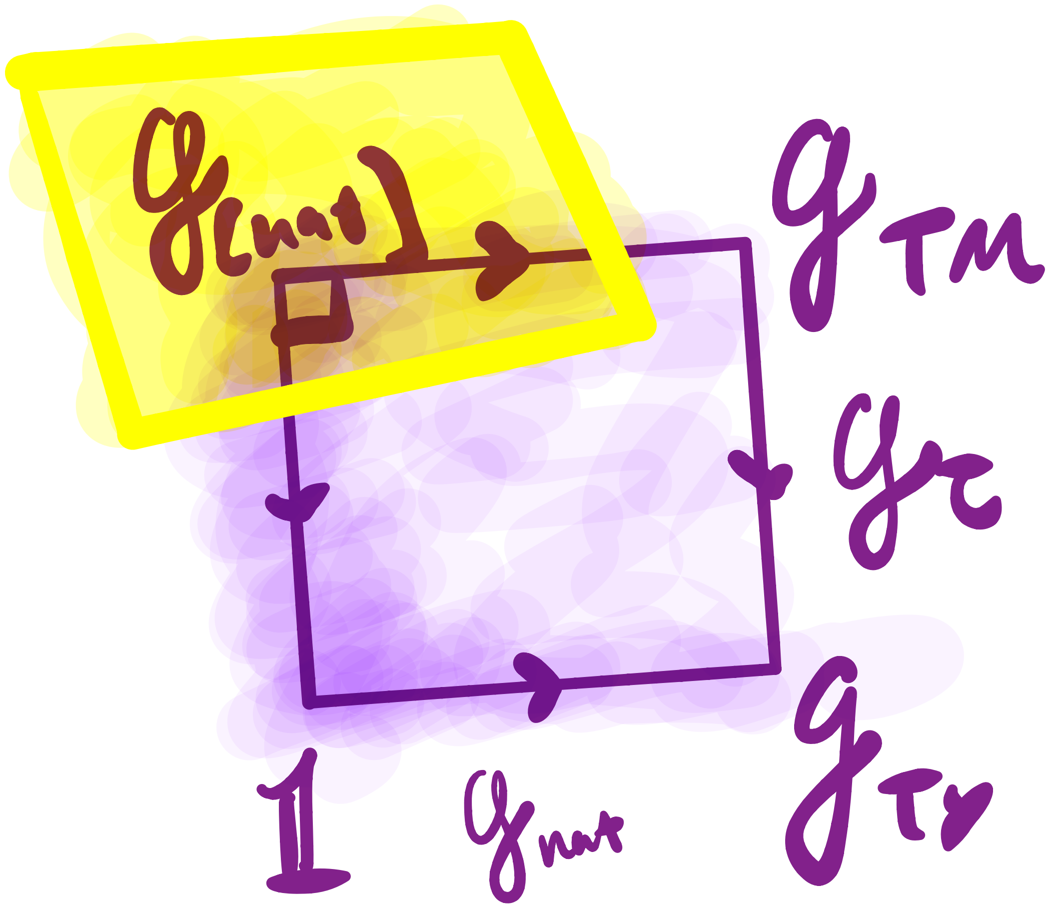
[Awodey + Fiore]

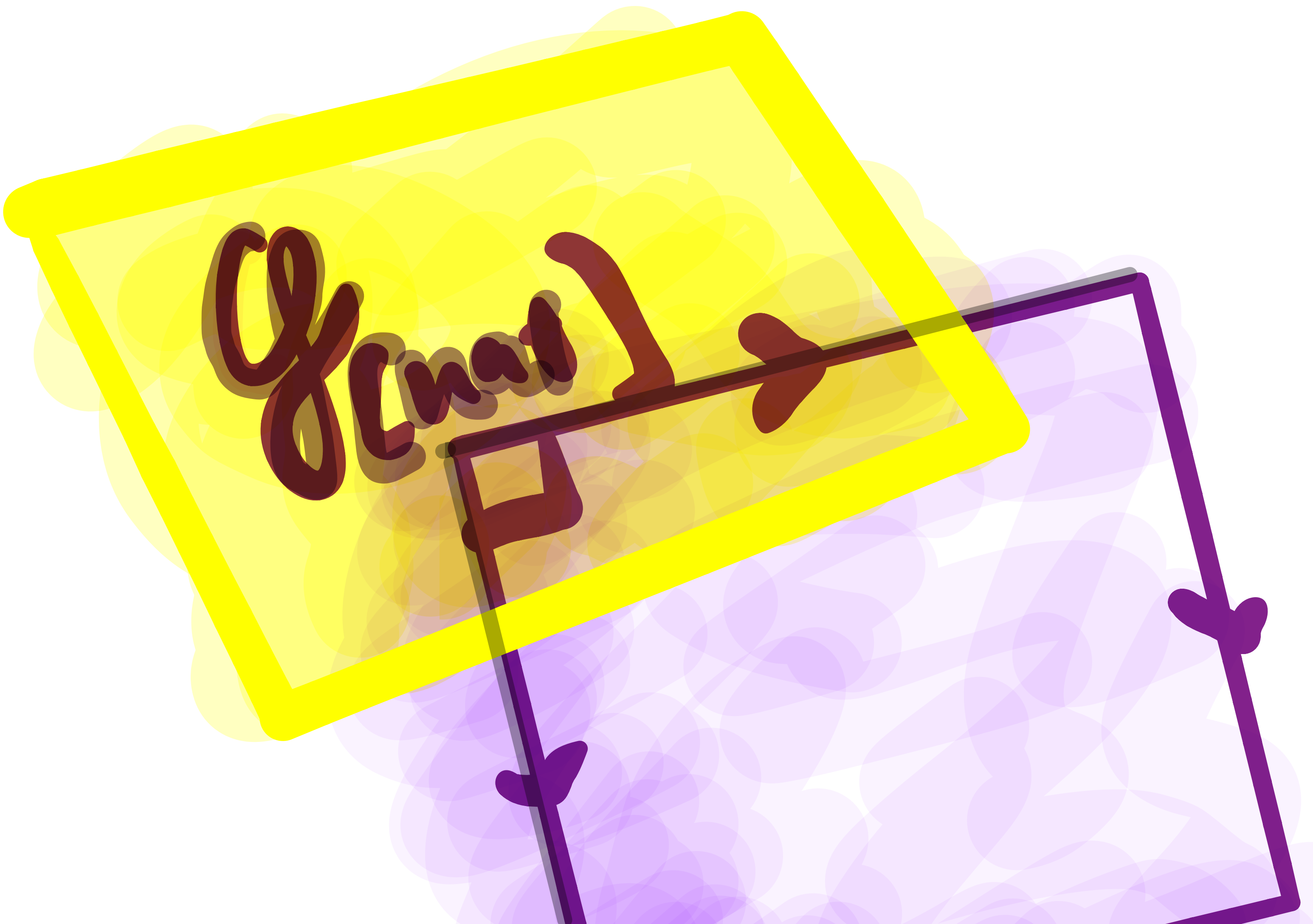
3) $\mathcal{C} \xrightarrow{N_{\square}} \hat{\square}$

4) 

[A., Gratzer, S.]

$$
\begin{tikzcd}
C_{[nat]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{TM} \arrow[d, "C_{\tau}"] \\
1 \arrow[r, "C_{nat}"'] & C_{TY}
\end{tikzcd}
$$

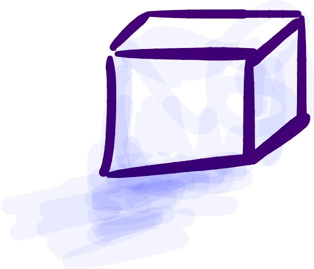

g(nat)
gTM
gc
1
gnat
gTV



$$\begin{tikzcd} \mathcal{G}_{[nat]} \arrow[d, mapsto] \\ \mathcal{C}_{[nat]} \end{tikzcd} \qquad \text{(is)} \qquad \begin{tikzcd} \mathbb{N} \arrow[d, "\mathcal{G}_{[nat]}"] \\ (\square^{\bullet})^{*}(\mathcal{C}_{[nat]}) \end{tikzcd}$$

(universal comparison map)

Corollary:

-ical Type Theory has Canonicity

∎

Next: cubical normalization (β/η)

WHY: Essential for usability (automatically discharge boundary conditions, cf. Cubical Agda & redtt)

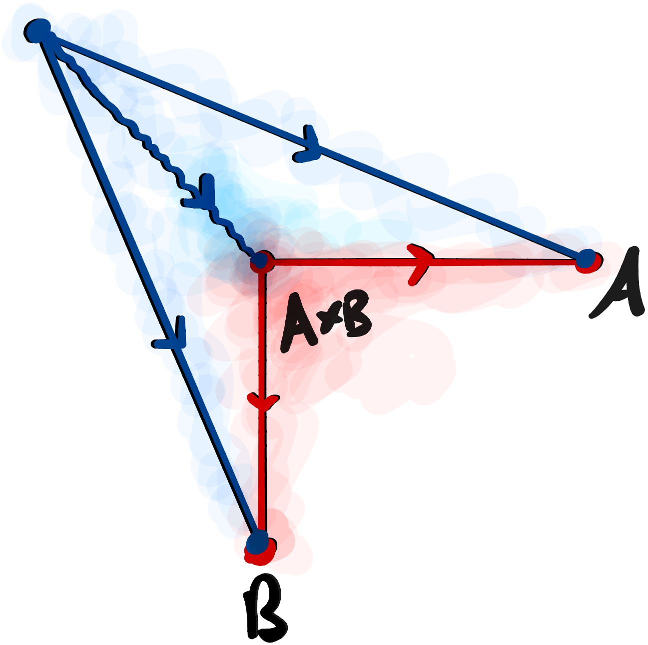
HOW: Glue along nerve induced by "formal renaming algebra":

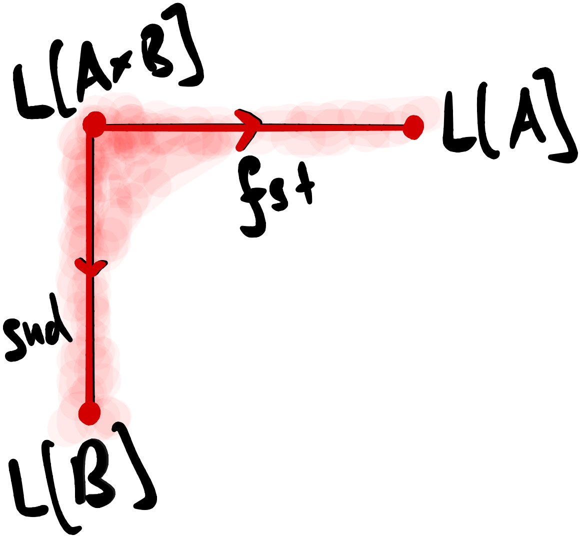
(renaming algebra)

(Formal Renamings) $\rightsquigarrow$ $\mathcal{R} \xrightarrow{\varepsilon} \mathcal{C}$ (Substitutions)

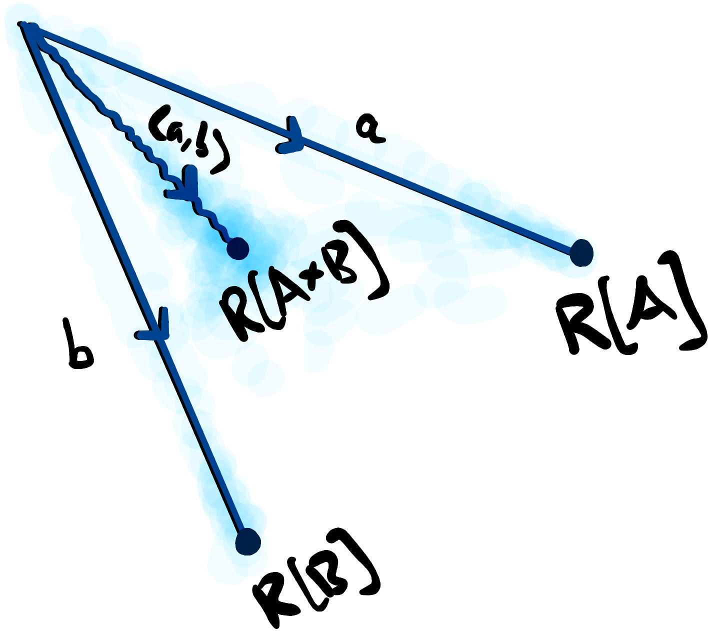
[Fiore 2002]

$$
\begin{tikzcd}
\bullet \arrow[dr, squiggly] \arrow[drr] \arrow[ddr] & & \\
& A\times B \arrow[r] \arrow[d] & A \\
& B &
\end{tikzcd}
$$

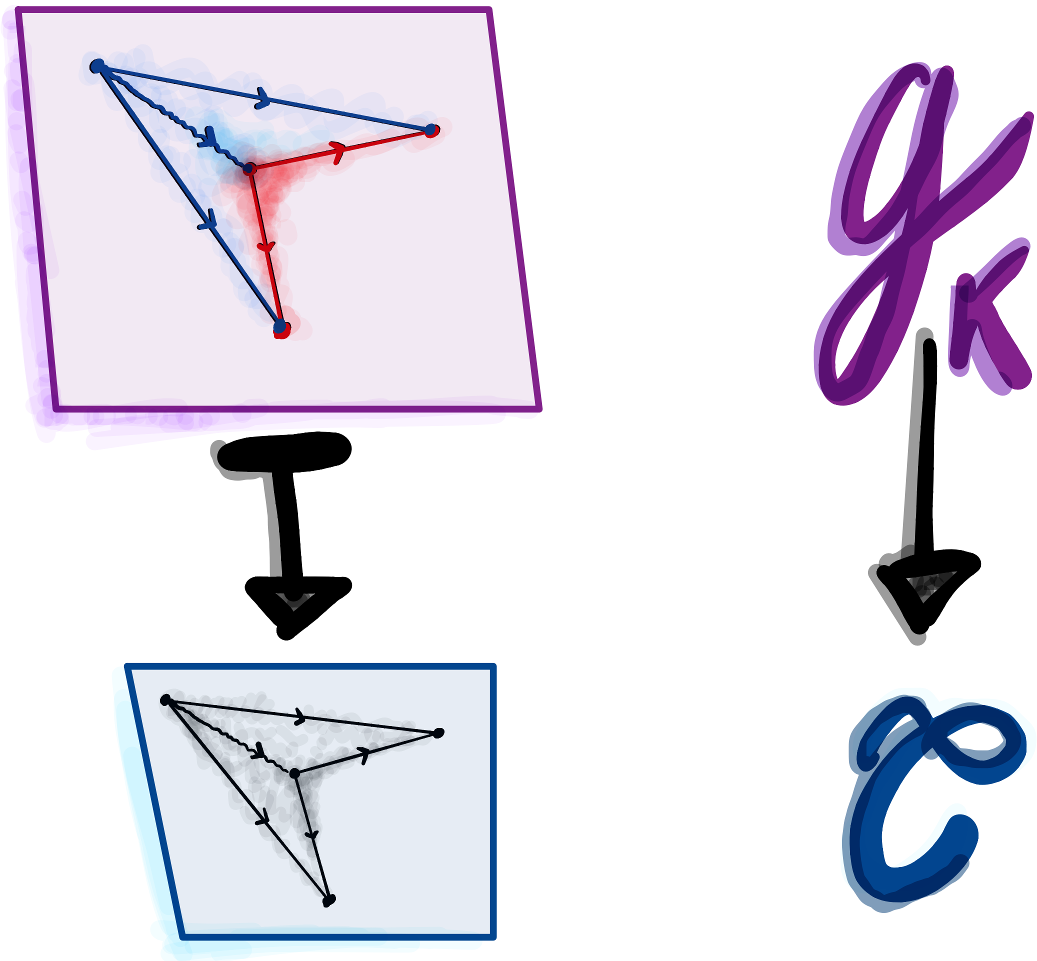

A
A×B
B



"left normal forms"



"right normal forms"


$\mathcal{G}_K$
$\mathcal{C}$

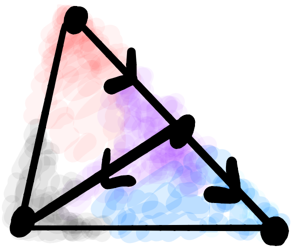
Reify

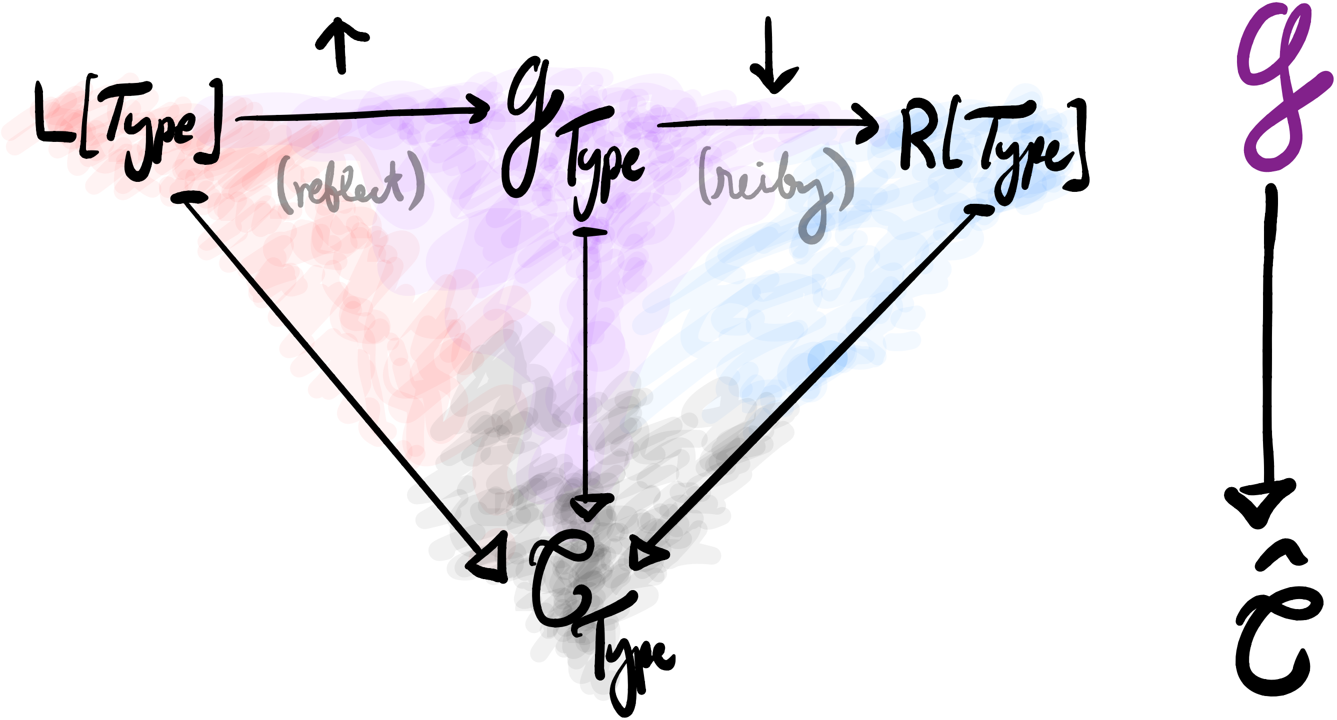
$\tilde{X} \to R[X]$

Reflect

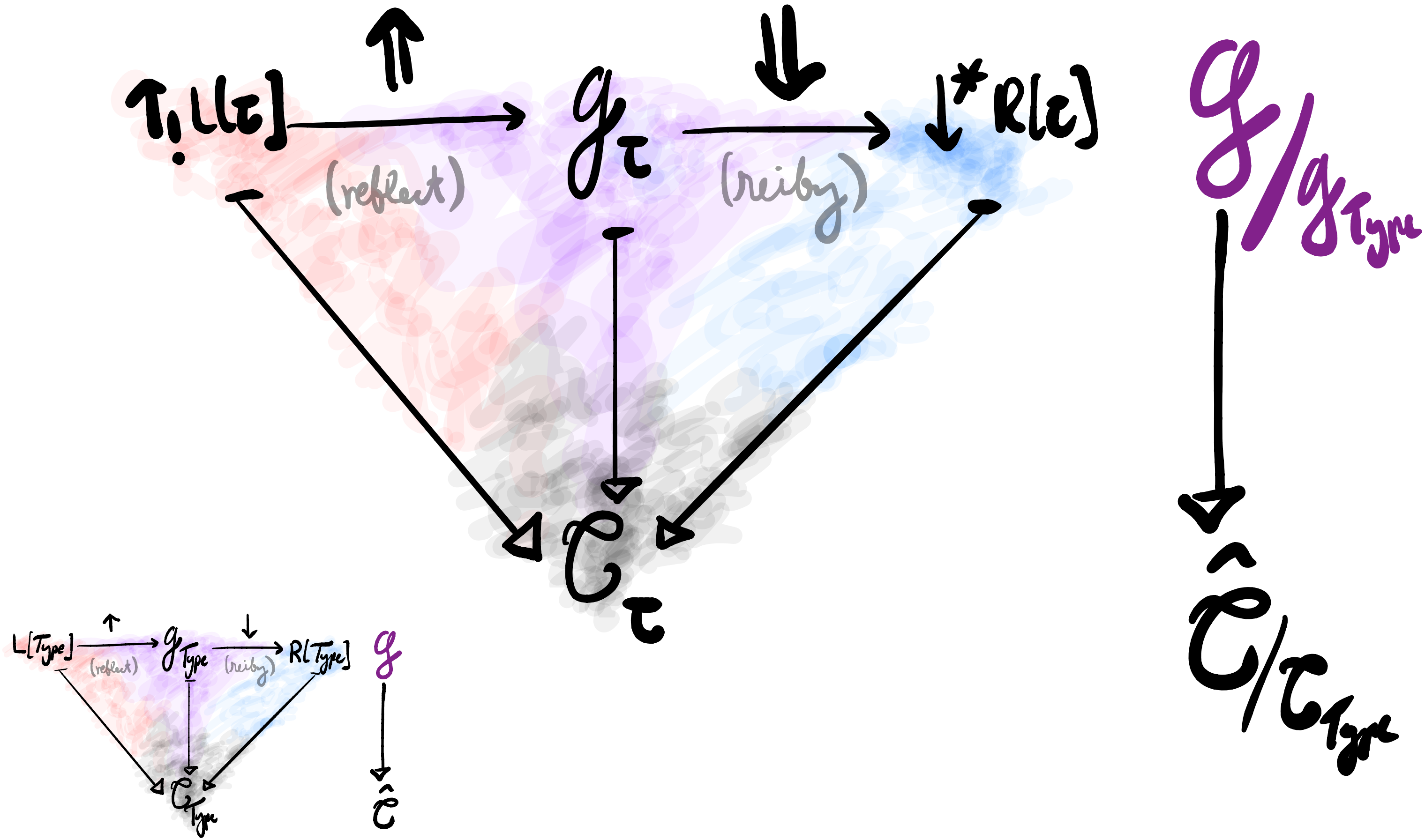
$L[X] \to \tilde{X}$

Yoga





L[Type]
↑
(reflect)
$\mathcal{G}_{Type}$
↓
(reify)
R[Type]
$\mathcal{C}_{Type}$
$\mathcal{G}$
$\hat{\mathcal{C}}$


⇑
⇓
(reflect)
(reify)
L[Type]
R[Type]
(reflect)
(reify)