Why cubical sets are different to simplicial sets

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January 20, 2022

In the classical homotopy theory of simplicial sets, there are various ways to define an important class of maps, known as *Kan fibrations*.

Definition (Kan)

A map $f : X \to Y$ is a *Kan fibration* if it has the right lifting property against a class of maps called *Horn inclusions*. That is for every square below, we can find a diagonal map making two commutative triangles:

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta_n & \longrightarrow & Y \end{array}$$

Theorem (Joyal)

f is a Kan fibration if and only if it has the right lifting property against the pushout product of any cofibration $m : A \to B$ and endpoint inclusion of an interval $\delta_i : 1 \to \mathbb{I}$:

$$\begin{array}{ccc} A \times \mathbb{I} \cup B \times 1 & \longrightarrow & X \\ {\scriptstyle m \hat{\times} \delta_i} \big\downarrow & \nearrow & \big\downarrow {\scriptstyle f} \\ \Delta_n & \longrightarrow & Y \end{array}$$

In classical homotopy theory these definitions give the same class of maps and are often treated as the same. However, they are different in two ways:

1. Constructively: There is no way to prove the classes of maps are the same without assuming the law of excluded middle.
2. Algebraically: If we consider not just the class of maps with property, but use the same definitions to define structure (i.e. choices of diagonal map not just mere existence), then they are different, even in classical logic.

Theorem (Voevodsky)

Homotopy type theory can be modelled in simplicial sets, using the law of excluded middle and the axiom of choice.

Types in Voevodsky's model are implemented as maps with extra structure (including Kan fibration structure).

Constructively, the situation is as follows.

Theorem (Cohen, Coquand, Huber and Mörtberg)

Homotopy type theory can be modelled in cubical sets, with pushout product Kan fibration structure.

Theorem (Henry, Gambino, Sattler, Szumiło)

Simplicial sets have a model structure and univalent universe, using the "horn inclusion" definition of Kan fibration structure.

It is natural to ask why different definitions of Kan fibration were used in each case:

1. Why couldn't CCHM use the horn-inclusion definition of Kan fibration?
2. If they needed to use a different definition of Kan fibration, why did they also need to use cubical sets? Couldn't they use the pushout-product definition with simplicial sets?

CCHM defined Kan fibrations the way that they did in order to ensure they are closed under dependent products. Constructively this is not possible with the “horn-inclusion” definition of Kan fibration (Bezem-Coquand-Palmgren).

We can illustrate this with a related but simpler fact.

We say an object X is *fibrant* if the unique map $X \to 1$ is a fibration.

Note that $f : X \to 1$ has the right lifting property against horn inclusions precisely if each map below has a section:

$$\hom(\Delta_n, X) \longrightarrow \hom(\Lambda^i_n, X)$$

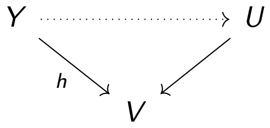
Say that X is *strongly fibrant* if each map below has a section.

$$X^{\Delta_n} \longrightarrow X^{\Lambda^i_n}$$

Theorem

If X is strongly fibrant, then so is X^Y for any Y.

In both cubical sets and simplicial sets there is a "well behaved" universe $\dot{U} \to U$: we have "forgetful" map to the (extensional) Hofmann-Streicher universe $U \to V$, such that $\dot{U}$ is the pullback of $\dot{V}$, and if $X \to Y$ is a pullback of $\dot{V} \to V$ along $h : Y \to V$, then Kan filling operations on f correspond precisely to maps $Y \to U$ making a commutative triangle, as below



In particular $\dot{U} \to U$ itself has a canonical Kan filling operation corresponding to the identity map.

Theorem (Licata-Orton-Pitts-Spitters, Awodey)

We can define a "well behaved" universe for Kan fibrations in any topos with a universe for extensional type theory, as long as the interval $\mathbb{I}$ *appearing in the definition of Kan filling operation is tiny, i.e.* $(-)^{\mathbb{I}}$ *has a right adjoint.*

They observe that the interval object in simplical sets is not tiny and so their proof does not apply there.

Q. Is there an alternative construction of a universe for Kan fibrations defined by pushout product that works for simplicial sets?
A: No! (As long as we want the universe to be "well behaved")

Let $\mathbb{C}$ be a category with pullbacks. Write $\mathsf{Cart}(\mathbb{C}^{\rightarrow})$ for the wide subcategory of $\mathbb{C}^{\rightarrow}$ whose morphisms are pullback squares.

Definition (Shulman)

A *notion of fibred structure on* $\mathbb{C}$ is a discrete fibration, $\chi : \mathbb{D} \to \mathsf{Cart}(\mathbb{C}^{\rightarrow})$.

Suppose we are given a notion of fibred structure $\chi : \mathbb{D} \to \mathrm{Cart}(\mathbb{C}^{\to})$. For each $f \in \mathbb{E}$, we can define a presheaf $\bar{\chi}_f$ on $\mathbb{C}/\operatorname{cod}(f)$ as follows. Given a map $\sigma : I \to \operatorname{cod}(f)$, we take $\bar{\chi}_f(\sigma)$ to be the set of objects of $\chi^{-1}(\sigma^*(f))$.

Definition (Shulman)

We say χ is *locally representable* if it satisfies any of the equivalent conditions below.

1. For every $f \in \mathbb{C}^{\to}$ the presheaf $\bar{\chi}_f$ is representable.
2. χ has a right adjoint as a functor $\mathbb{D} \to \mathrm{Cart}(\mathbb{C}^{\to})$ in **Cat**.
3. χ is comonadic as a functor $\mathbb{D} \to \mathrm{Cart}(\mathbb{C}^{\to})$ in **Cat**.

Theorem (S)

None of the following definitions of Kan fibration on $\mathbf{Set}^{\Delta^{op}}$ *listed below are locally representable as notions of fibred structure.*

1. *Right lifting property against pushout product of monomorphism and interval endpoint inclusion*[1]
2. *Right lifting property against pushout product of boundary inclusion and interval endpoint inclusion*[2]
3. *Monoidal lifting property against horn inclusions*

[1] Definition of Kan fibration in cubical sets, B_3 in Gabriel and Zisman

[2] B_2 in Gabriel and Zisman

We illustrate the proof with the following simpler version of the same idea. Note that we have an interval object $\partial_0, \partial_1 : 1 \rightrightarrows \Delta_1$ in simplicial sets.

Definition (Barthel-Riehl)

Let $f : X \to Y$ be a map in simplicial sets. A *Hurewicz fibration structure* on f is a section of the map

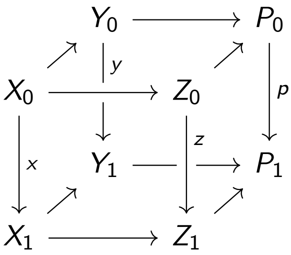
$$X^{\Delta_1} \to X \times_Y Y^{\Delta_1}$$

Theorem

Hurewicz fibration structures are not locally representable as notions of fibred structure in simplicial sets.

Lemma

Suppose we are given locally representable notion of fibred structure $\mathbb{F} \to \mathrm{Cart}(\mathbb{C}^{\to})$ *and a commutative cube as below, where the top and bottom faces are pushouts and the remaining side faces are pullbacks.*



If we are given structures on x, y *and* z *that are preserved by the pullback squares. Then there is a unique structure on* p *such that the pullback squares preserve fibration structures.*

Proof.

$\mathbb{F} \to \mathrm{Cart}(\mathbb{C}^{\to})$ is comonadic and so creates colimits. □

The second key observation is that the inclusion Δ_1 has a linear order in the internal language of $\mathbf{Set}^{\Delta^{\mathrm{op}}}$. In fact this is a key property of $\mathbf{Set}^{\Delta^{\mathrm{op}}}$.

Theorem (Joyal)

$\mathbf{Set}^{\Delta^{\mathrm{op}}}$ *is the classifying topos for linear orders with endpoints, with universal model* $\Delta_2 \rightarrowtail \Delta_1 \times \Delta_1$ *(with inclusion specified by the two degeneracy maps* $\Delta_2 \to \Delta_1$*).*

Hence we can write $\Delta_1 \times \Delta_1$ as the union of two subobjects $T_0 := \{(x, y) \mid x \leq y\}$ and $T_1 := \{(x, y) \mid x \geq y\}$.

This is already enough to see $(-)^{\Delta_1}$ does not preserve the pushout witnessing the union $T_0 \cup T_1$:

Given a path $p : \Delta_1 \to \Delta_1 \times \Delta_1$, we write $p \subseteq T_n$ as notation for $\forall i . p(i) \in T_n$ for $n = 0, 1$.

Working in the internal language, we define a family of paths $p_j : \Delta_1 \to \Delta_1 \times \Delta_1$ by $p_j(i) := (j, i)$.

If $p_j \subseteq T_0$, then in particular $p_j(1) \in T_0$, so $j \leq 0$, and we deduce $j = 0$. Similarly if $p_j \subseteq T_1$ then $j = 1$.

We then have,

$$\begin{aligned}\{j \in \Delta_1 \mid p_j \subseteq T_0 \vee p_j \subseteq T_1\} &= \{j \in \Delta_1 \mid j = 0 \vee j = 1\} \\ &\subsetneq \Delta_1\end{aligned}$$

Key idea: We define a map $f : X \to \Delta_1 \times \Delta_1$ with two different Hurewicz fibration structures that agree on paths belonging to $T_0^{\Delta_1} \cup T_1^{\Delta_1}$, but disagree for other paths.

We define X "fibrewise" in the internal language as follows. Each fibre $X_{i,j}$ will be a subobject of Ω.

$$X_{i,j} := \{\varphi \in \Omega \mid i \geq j \Rightarrow \varphi\}$$

We see that for $(i, j) \in T_1$ we have $X_{i,j} = \{\top\}$ and that $X_{0,1} = \Omega$. In between X is "wedge shaped."

We define two Hurewicz fibration structures $\alpha, \beta : X \times_Y Y^{\Delta_1} \rightrightarrows X^{\Delta_1}$ as follows.

$$\alpha(x,p)(i) := (p(i) \in T_1) \vee (i = 0 \wedge x) \vee (\pi_1(p(0)) = 0 \wedge \pi_1(p(i)) = 1)$$
$$\beta(x,p)(i) := (p(i) \in T_1) \vee (i = 0 \wedge x) \vee (p(0) = (0,0) \wedge \pi_1(p(i)) = 1)$$

If $p \subseteq T_1$, then $X_{p(i)}$ is contractible for all i, so $\alpha(x,p) = \beta(x,p)$.
If $p \subseteq T_0$, then $\pi_1(p(i)) = 0$ if and only if $p(i) = (0,0)$, so again $\alpha(x,p) = \beta(x,p)$.
For $j \in \Delta_1$, define p_j by $p_j(i) := (j,i)$. We have $\alpha(\top, p_j)(1) = \top$ and $\beta(\top, p_j) = [\![j = 0 \vee j = 1]\!]$. □

1. CCHM use the “pushout product” definition of Kan fibration to ensure fibrations are closed under dependent product.
2. This definition of Kan fibration is locally representable in cubical sets, but not in simplicial sets.
3. In order to get a well behaved definition of universe in simplicial sets, we need to use the “horn inclusion” definition of Kan fibration.

This will appear in a paper, alongside some more general theory and other results, *Locally representable algebraic weak factorisation systems*