Double negation stable h-propositions in cubical sets

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In extensional type theory, we only have one notion of equality to worry about. We interpret it in a locally cartesian closed category as follows.

- Contexts B are interpreted as objects of the category.
- Types in context B are interpreted as maps $E \to B$.
- Terms are interpreted as sections $B \to E$ (we will also refer to sections as *points*).
- The lcc believes two terms are equal exactly when they are interpreted as the same map in the lcc.
- Hence if a type is an h-proposition it has at most one section.
- Propositional truncation "strictly identifies points."

$$\begin{tikzcd} E \arrow[d] \\ B \arrow[u, bend right] \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[dr] & & i \arrow[dl] \\ & B & \end{tikzcd}$$

In homotopical models of intensional type theory, such as cubical sets, things are not as simple.

The category of cubical sets, $\mathbf{Set}^{\square^{\mathrm{op}}}$ is a locally cartesian closed category, so is a model of extensional type theory, but also has a notion of *homotopy* which is important for modelling HoTT.

$$\begin{tikzcd} 1 \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \mathbb{I} \end{tikzcd}$$

Definition

We say two maps $s, t : B \to E$ are *homotopic* if there is a map h in the commutative diagram below:

$$\begin{tikzcd}
B \arrow[dr, "B\times\delta_0"] \arrow[drr, bend left, "s"] & & \\
& B \times \mathbb{I} \arrow[r, "h"] & E \\
B \arrow[ur, "B\times\delta_1"'] \arrow[urr, bend right, "t"'] & &
\end{tikzcd}$$

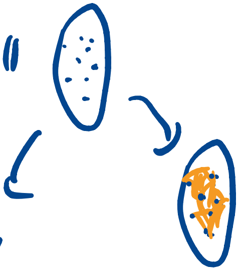
$$\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}$$

- Types in context B are interpreted as Kan fibrations $E \to B$ (a map together with "Kan fibration structure").
- Terms are interpreted as sections $B \to E$, as before.
- Definitionally equal terms are interpreted as equal sections.
- Two terms $s, t : B \to E$ are propositionally equal if they are homotopic.
- Hence h-propositions can have multiple sections, as long as any two are homotopic.

Typically propositional truncation does *not* strictly identify points:

- In cubical sets the map $|-|: E \to \|E\|$ is *always* a monomorphism.
- By Kraus' paradox there are examples of types E such that $|-|: E \to \|E\|$ is interpreted as a monomorphism in *every* model of HoTT (e.g. any E with decidable equality).

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[dr] & {[E]} \arrow[d] \\ & B \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, tail, "(-1)"] \arrow[dr] & \|E\| \arrow[d] \\ & B \end{tikzcd}$$



We can better understand h-propositions using some arguments due to Uemura, using the string of adjunctions between sets and cubical sets:

$$
\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left, "\Delta"] \arrow[rr, bend right, "\nabla"'] & & \mathbf{Set}^{\square^{\mathrm{op}}} \arrow[ll, "\Gamma" description]
\end{tikzcd}
$$

$1 \mapsto E$

- Δ sends each a set A to the constant cubical set (only points, no paths or homotopies).
- For a cubical set E, ΓE is the set of global sections (points of the space).
- ∇ maps into h-propositions.

Theorem

Let $p : E \to B$ be a Kan fibration in cubical sets. If $\|E\|_B \to B$ has a section, then so does the map $\Gamma E \to \Gamma B$.

Pf for $B = 1$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \nabla \Gamma E \\ \|E\| \arrow[ur, dotted, bend right] & \end{tikzcd}$$

$$\Gamma \|E\| \longrightarrow \Gamma E$$

Corollary

Let $p : X \twoheadrightarrow Y$ be a surjection in **Set**. *If $\|\Delta X\|_{B}$* ΔY *$\to \Delta Y$ is equivalent to a monomorphism, then p has a section. Hence if every h-proposition is equivalent to a monomorphism we deduce the axiom of choice.* in Set

$$
\begin{tikzcd}
X \arrow[d, two heads, "p"'] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta X \arrow[r, tail] \arrow[d] & \|\Delta X\| \arrow[r, bend left] \arrow[dl] & M \arrow[l, bend left] \arrow[dll, tail, "\simeq"] \\
\Delta Y & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \simeq \Gamma\Delta X \arrow[d] \\
Y \simeq \Gamma\Delta Y \arrow[u, dashed, bend right]
\end{tikzcd}
$$

Corollary

Suppose that cubical sets have a (homotopy) subobject classifier, and that power sets exist. Then we can deduce the law of excluded middle.

Menni: generic prop

Setoids don't satisfy prop. resizing

Definition

We say a monomorphism $m : A \to B$ in a locally cartesian closed category is *extensional* if the following holds in the internal language:

$$\prod_{b,b':B} (A_b \leftrightarrow A_{b'}) \to b = b'$$

mono ⇒ h-proposition

Theorem

Suppose that $p : E \to B$ is both a monomorphism and a Kan fibration, and that $\Gamma(p) : \Gamma(E) \to \Gamma(B)$ is a pullback of some extensional monomorphism $m : X \to Y$. Then p is a pullback of $\Delta(m) : \Delta(X) \to \Delta(Y)$.

Corollary

Suppose that $1 \to \Omega$ is the subobject classifier in **Set**. *Then every $p : E \to B$ which is both a monomorphism and Kan fibration is a pullback of $1 \to \Delta(\Omega)$.*

We see that the classifier for monic fibrations not only exists, but is preserved by Δ.

Although monomorphisms are well behaved semantically, there does not seem to be any way to talk about them internally in type theory. One way to fix this is to augment type theory with a type of *strict propositions* as developed by Gilbert, Cockx, Sozeau and Tabareau.

For this work we take an alternative approach of restricting to a subclass of h-propositions that is very easy to define internally in HoTT.

Definition

A type P is $\neg\neg$-*stable* if it is an h-proposition and $\neg\neg P \to P$.

We can think of double negation as "removing computational information from a proposition."

For example, writing RH for the Riemann hypothesis, $RH + \neg RH$ is an h-proposition (because RH can't be both true and false) but to prove it constructively, we need to either provide a proof of RH or provide a proof of $\neg RH$, i.e. the computational information that tells us whether RH is true or false. On the other hand $\neg\neg(RH + \neg RH)$ is just provable without needing to show which one holds.

With care, we can ensure that $RH + \neg RH$ is interpreted as a monomorphism in $\mathbf{Set}^{\square^{\mathrm{op}}}$.

Theorem

Suppose that $E \to B$ is a $\neg\neg$-stable h-proposition in $\mathbf{Set}^{\square^{\mathrm{op}}}$. Then it is equivalent to a monomorphism.

Proof.

Sketch: By assumption E is equivalent to $\neg\neg E$ (in context B), so it suffices to show $\neg\neg E$ is interpreted as a monomorphism $\neg\neg E \to B$. When we interpret exponentials in HoTT into cubical type theory, we just use the underlying locally cartesian closed structure of cubical sets. Also we interpret $\perp$ as the initial object in the usual sense. Hence it suffices to show $\neg\neg E \to B$ is an h-proposition in extensional type theory, which it is. □

$\neg A \rightarrowtail 1$

$A : \neg\tilde{C}$ in $\mathbf{Set}^{\square^{\mathrm{op}}}/B$

$\neg\neg\tilde{C} \rightarrowtail B$

We now have three different notions of proposition in cubical sets:

$$\text{h-propositions} \quad \supsetneq \quad \underbrace{\text{monos} \quad \supsetneq \quad \neg\neg\text{-stable monos}}_{\text{Strict Props}}$$

1. H-propositions are spaces where any two points are joined by a path, any two paths by a homotopy, etc
2. Monomorphisms have no homotopical structure but can carry computational information (we can't prove $P + \neg P$ without assuming excluded middle in **Set**)
3. $\neg\neg$-stable monomorphisms have no homotopical structure or computational information.

Either 2 or 3 can be used to implement strict propositions.

By the same arguments as earlier,

Theorem
Suppose that $\Omega_{\neg\neg}$ is a classifier for $\neg\neg$-stable monomorphisms in **Set**. *Then $\Delta(\Omega_{\neg\neg})$ is a classifier for $\neg\neg$-stable h-propositions in cubical sets.*

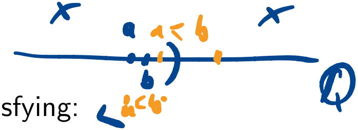
A classifier for $\neg\neg$-stable h-propositions suffices for many constructions in type theory, including:

- The Dedekind real numbers. $\mathbb{N} \to \mathbb{N}$
- To formulate Extended Church's Thesis (all partial functions are computable), and prove its consistency.
- To construct the modality of 0-truncated $\neg\neg$-sheafification.

Recall that we can define the Dedekind real numbers as the collection of all *left cuts*:

Definition

A *Dedekind left cut* is a set $L \subseteq \mathbb{Q}$ satisfying:



1. (Boundedness) There exist rational numbers $a \in L$ and $b \notin L$.
2. (Openness) For all $a \in L$ there merely exists $b \in L$ such that $b > a$.
3. (Locatedness) For all $a < b \in \mathbb{Q}$ either $a \in L$ or $b \notin L$.

Left cuts do not need to be $\neg\neg$-stable, so in order to construct the Dedekind reals, we switch to the dual notion of *cocut*:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, mapsto] \arrow[dr, "C"'] & \Omega \\ & \Omega_{\neg\neg} \arrow[u, dotted] \end{tikzcd}$$

Definition

A cocut is a set $C \subseteq \mathbb{Q}$ satisfying:

1. (Boundedness) There exist rational numbers $a \notin C$ and $b \in C$.
2. (Closedness) For all $a \in \mathbb{Q}$, if $b \in C$ for all $b > a$, then $a \in C$.
3. (Locatedness) For all $a < b \in \mathbb{Q}$ either $a \notin C$ or $b \in C$.

Theorem

Suppose that there is a classifier for all $\neg\neg$-stable h-propositions. Then the type of Dedekind real numbers exists.

Proof. - Every cocut is $\neg\neg$-stable

- Using $\Omega_{\neg\neg}$ we can construct the type of all cocuts.
- Every left cut is of the form $L = \{a \in \mathbb{Q} \mid \exists b \in \mathbb{Q} \setminus C \; a < b\}$ for a unique cocut C.

$\square$

Side remark: in fact we don't need the full power of $\neg\neg$-resizing in our metatheory: We can show that $\mathbf{Set}^{\square^{op}}$ has Dedekind real numbers only using the fact that we are working in a locally cartesian closed category with natural number object.

Theorem $\quad$ MLTT + UA + $\mathbb{R}_D$ ≡ MLTT + UA

$\mathbf{MLTT}^- + \mathbf{UA} + \mathbb{R}_D$ has the same consistency strength as $\mathbf{MLTT}^- + \mathbf{UA}$, where $\mathbf{MLTT}^-$ is Martin-Löf type theory without W-types, $\mathbf{UA}$ is the univalence axiom and $\mathbb{R}_D$ asserts the existence of the Dedekind reals.

Open Problem

What is the consistency strength of $\mathbf{MLTT}^- + \mathbf{UA} + \mathbb{R}_{HIT}$, where $\mathbb{R}_{HIT}$ is the existence of the HIT reals as defined in Chapter 11 of the HoTT book?

Theorem (Swan, Uemura)

There is a reflective subuniverse of cubical assemblies that satisfies Church's thesis. $\mathbb{N} \to \mathbb{N}$

In many cases we don't want to just consider total computable, but also partial computable functions. Because of this, it's common in constructive mathematics to consider a stronger version of Church's thesis that applies also to partial functions. We will use $\Omega_{\neg\neg}$ to first define what we mean by partial function.

Definition

For a type A, we define ∂A, the type of $\neg\neg$-stable partial elements to consist of pairs p, α, where $p : \Omega_{\neg\neg}$ and $f : p \to A$. For $\alpha = (p, f) : \partial A$, we write $\alpha\downarrow$ to mean p is true, and write $\alpha \downarrow= a$ to mean p is true and $f* = a$.

In particular, note that we can think of maps $A \to \partial B$ as *partial functions* $A \rightharpoondown B$. This allows us to formulate the stronger version of Church's thesis.

Definition

The axiom of *extended Church's thesis* (**ECT**) states that for every map $f : \mathbb{N} \to \partial\mathbb{N}$ there is a natural number $e : \mathbb{N}$ such that whenever $f(n)\downarrow$, we have $\varphi_e(n) = f(n)$.

Note that this doesn't talk about which partial functions are *equal* to a computable function, but rather the partial functions that are *extended* by some computable function.

Theorem

There is a reflective subuniverse of cubical assemblies that satisfies Extended Church's thesis. Hence **ECT** *is consistent HoTT.*

Key ideas: Largely this follows the same outline as the earlier proof for Church's thesis, namely:

1. We take the largest reflective subuniverse that forces **ECT** to be true.
2. We use the fact that **ECT** holds in assemblies to ensure the subuniverse is non trivial.

In order for this to work, it is important that we can closely relate the statement of **ECT** in assemblies with the statement of **ECT** in cubical assemblies. This uses the fact that $\Delta : \mathbf{Asm} \to \mathbf{Asm}^{\square^{op}}$ preserves all the type formers used to state **ECT** except truncation. In particular the proof requires that Δ preserves the classifier for $\neg\neg$-stable h-propositions.

Definition

A *modal operator* is a map $\bigcirc : \mathcal{U} \to \mathcal{U}$ together with a family of maps $\eta_A : A \to \bigcirc A$ that we call the *unit* maps.
We say a type A is $\bigcirc$-*modal* or $\bigcirc$-*stable* if η_A is an equivalence.

Definition (Rijke, Shulman, Spitters)

A *(uniquely eliminating) modality* is a modal operator $\bigcirc, \eta$ such that for every A, the map

$$- \circ \eta_A : \prod_{z:\bigcirc A} \bigcirc(P(z)) \to \prod_{x:A} \bigcirc(P(\eta_A(x)))$$

is an equivalence for every $P : \bigcirc A \to \mathcal{U}$.

$$\begin{tikzcd}
A \arrow[r, bend left] \arrow[d, "\eta_A"'] & \bigcirc A \arrow[dl, bend left] \\
\bigcirc A \arrow[ur, dotted] &
\end{tikzcd}$$

Earlier we stated that double negation "erases computational information." However, in fact it does two things: it erases information *and* it truncates: $\neg\neg A$ is always an h-proposition for any type A. We can think of $\neg\neg$-sheafification as a generalisation of double negation to arbitrary (untruncated) types.

Usually we would define it as follows:

Definition

An h-proposition P is $\neg\neg$-*dense* if $\neg\neg P$ is true.
$\neg\neg$-*sheafification* is the smallest modality ∇ such that every $\neg\neg$-dense h-proposition is ∇-connected. That is,
$\neg\neg P \rightarrow \nabla P \simeq 1$.

Rijke, Shulman and Spitters showed ∇ can be constructed as a HIT, *as long as we have propositional resizing*.

We can define 0-truncated $\neg\neg$-sheafification only using $\Omega_{\neg\neg}$ by a standard construction in (1-)topos theory. This is enough to generalise double negation from h-propositions to h-sets.

Definition

For a type A, we define $\nabla_0 A$ to consist of $\neg\neg$-stable subsets of A, $c : A \to \Omega_{\neg\neg}$ with the properties:

1. c is not empty: $\neg\neg \sum_{a:A} c(a)$
2. c has at most one element up to double negation: $\prod_{a,b:A} c(a) \to c(b) \to \neg\neg a = b$

Theorem

$\nabla_0 : \mathcal{U} \to \mathcal{U}$ reflects onto types that are both 0-truncated and $\neg\neg$-sheaves.

We think of $\nabla_0 A$ as the result of stripping computational information from a set without adding paths between its points. We can use it to formulate objects in type theory that we can *explicitly define* but not *compute*.

To make this precise, first note that we can define elements of $\nabla_0 A$ by cases like in classical logic:

Lemma

Let P an h-proposition and A a $\neg\neg$-sheaf. For $\alpha, \beta : A$, there is a unique $\gamma : A$ such that $P \to \gamma = \alpha$ and $\neg P \to \gamma = \beta$, i.e.

$$\gamma = \begin{cases} \alpha & P \\ \beta & \textit{otherwise} \end{cases}$$

$(X, \Vdash)$ $\quad$ $X \in$ Set $\quad$ $\Vdash \subseteq \mathbb{N} \times X$

$(2, \Vdash_2)$

$n \Vdash 0$ iff $n = 0$

$n \Vdash 1$ " $n = 1$

$(X, \Vdash)$

$n \Vdash x$ iff $n = 0$

$X = 2$

For example, write $\varphi_e(n)$ for the output of the Turing machine coded by e on input n. We define the halting set $K : \mathbb{N} \to \nabla_0 2$ as

$$K(n) := \begin{cases} 1 & \varphi_n(n) \downarrow = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that K is non computable by construction.
We can define K constructively, even in realizability models of HoTT, where all functions are computable. In that case K *cannot* extend to a function $\mathbb{N} \to 2$.

- $\neg\neg$-stable h-propositions are much better behaved than general h-propositions in cubical sets constructively.
  - They are interpreted as monomorphisms.
  - If $\Omega_{\neg\neg}$ classifies $\neg\neg$-stable h-propositions in **Set**, then $\Delta(\Omega_{\neg\neg})$ does so in $\mathbf{Set}^{\square^{\mathrm{op}}}$.
- Despite this they are useful in type theory for
  - Constructing $\mathbb{R}_D$
  - Formulating **ECT** and showing its consistency
  - $\neg\neg$-sheafification
- Moreover, each of these constructions matches closely with the corresponding notion in whatever metatheory we working in when constructing cubical sets.

Thank you for your attention!