An axiomatization of synthetic (∞-)category theory.

joint w/ D.-C. Cisinski, B. Cnossen, K. Nguyen

Roadmap

$$\begin{tikzcd}
\text{Substrate} \arrow[d] & \\
\text{Axioms} \arrow[r, "\text{established}", "\text{lectures by D.-C.}"'] & \text{basic synth. theory of } \infty\text{-cat}
\end{tikzcd}$$

- localizations
- Kan extensions, ptw formulas
- cofinality, Quillen A, B
- Yoneda embedding

advanced topics

- K-theory
  - stability
  - Bousfield loc
  - topoi
  - higher algebra

expected

- adjunctions
- all of HoTT = thy of groupoids
- Peano arithmetics, simplex cat
- un/straightenings

Overview of axioms

We can talk about synth. cats $\mathcal{C}, \mathcal{D}$,

functors between them $F, G, \ldots$

identifications $\alpha : F \simeq G$, $h : \alpha \simeq \beta$.

Axioms

(1) basic constructors

initial / terminal cat, $\emptyset$, $\{*\}$

product / coproduct $\mathcal{C} \times \mathcal{D}$, $\mathcal{C} \amalg \mathcal{D}$

pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$

functor cat $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

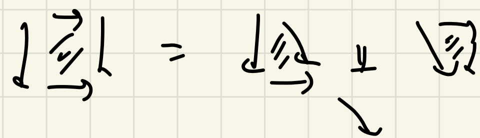
(2) have synth cat $\Delta^1 = \{0 \to 1\}$

and $\Delta^2 = \left\{ \begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\}$

and suitable function between them. (finite list)

(3) low' dim simplicial axioms ($\mathcal{C}$ synth. cat.)

- $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathrm{Fun}(\Delta^1, \mathcal{C})} \mathrm{Fun}(\Delta^2, \mathcal{C})$



- Segal axiom $\mathrm{Fun}(\Delta^2, \mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C})$

- completeness $\mathcal{C} \xrightarrow{\simeq} \mathrm{Iso}(\mathcal{C}) = \{$  $\}$

(4) Category theoretic constructors

- groupoid core $\mathcal{C} \rightsquigarrow \mathcal{C}^{\simeq}$ is a groupoid

(Def: $\mathcal{C}$ is called a groupoid if
$\mathcal{C} \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathcal{C})$ induced by $\Delta^1 \to *$ is an equiv)

equipped with $\mathcal{C}^{\simeq} \to \mathcal{C}$

- subcategory constructor

subgroupoid

$$\frac{\mathcal{C} \quad , \quad M \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}}{\langle M \rangle_{\mathcal{C}} \longrightarrow \mathcal{C}}$$ closed under comp

"the subcategory of $\mathcal{C}$ spanned by the arrows in M"

- localizations

$$\frac{\mathcal{C}\ , \quad W \subseteq \mathrm{Map}(\Delta^1, \mathcal{C}) := \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}}{\mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]}$$

$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) =$ full sub cat of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with arrows in W are inverted

special case of sub cat constructor

- join $A, B \rightsquigarrow A * B$

(5) <u>Functoriality of universals</u>

Assume $\quad x : \mathcal{C}^{\simeq} \rightsquigarrow F(x) : \mathcal{D}(x)$

construction "on objects"

If each $F(x)$ satisfies a universal property (in a suitable sense)

then we obtain

$$(x : \mathcal{C}) \xrightarrow{F} \mathcal{D}(x)$$

"functorial" assignment.

(6) <u>Limited exponentiability</u>

Not every map $T \to \mathcal{C}$ is exponentiable

($\triangleq$ Π-types along it exist) but every cartesian / cocartesian fibration is.

(7) Directed univalence

Have a hierarchy of universes $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \dots$

together with cocart fibrations $\mathcal{U}_{i,\bullet} \xrightarrow{\pi_{univ}} \mathcal{U}_i$

s.th a) every cocart. fibration $T \xrightarrow[p]{} \mathcal{C}$

is classified by some $\mathcal{C} \xrightarrow{\ulcorner p \urcorner} \mathcal{U}_i$ for $i \gg 0$

b) π_{univ} are univalent, i.e.

for $\begin{tikzcd} T \arrow[dr, "p"'] & & T' \arrow[dl, "p'"] \\ & \mathcal{C} \arrow[r, "\ulcorner p \urcorner, \ulcorner p' \urcorner"'] & \mathcal{U}_i \end{tikzcd}$ $\quad \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathcal{U}_i)}(\ulcorner p \urcorner, \ulcorner p' \urcorner) \simeq \mathrm{cocart}_{\mathcal{C}}(T, T')$

$= \left\{ \begin{tikzcd} T \arrow[r] \arrow[dr] & T' \arrow[d] \\ & \mathcal{C} \end{tikzcd}, \text{ preserve cocart. arrows} \right\}$

arrows $\Delta^1 \xrightarrow{f} \mathrm{Fun}(\mathcal{C}, \mathcal{U}_i)$

s.th. $f_0 = \ulcorner p \urcorner$, $f_1 = \ulcorner p' \urcorner$

Substrates for these axiomatics

① naive synth. cat theory

② within classical mathematics, 1-category theory using Joyal's tribes

③ embedded as fragment of simplicial HoTT

④ directed type theory

② (our original formulation)

A tribe $\mathcal{E}$ is a 1-cat with a class of maps $\twoheadrightarrow$ "fibrations"

+ axioms

e.g. every map $A \longrightarrow B$ factors

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, tail, "\sim"'] & & B \\ & B' \arrow[ur, two heads] & \end{tikzcd}$$

$A \overset{\sim}{\rightarrowtail} B$ anodyne (i.e. left lifts against fibr)

- synth cat $\mathcal{C}$ $\triangleq$ object of $\mathcal{E}$
- functor $\mathcal{C} \to \mathcal{D}$ $\triangleq$ morph in $\mathcal{E}$
- $\alpha : F \Rightarrow G$ $\triangleq$ homotopy $h : F \simeq G$
  $F, G : A \to B$

i.e.

$$
\begin{tikzcd}
 & & B \\
A \arrow[r, dashed, "h"] \arrow[urr, bend left, "F"] \arrow[drr, bend right, "C"'] & P(B) \arrow[ur] \arrow[dr] & \\
 & & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, tail] \arrow[r, bend right, "B"'] & P(B) \arrow[r, two heads] & B \times B
\end{tikzcd}
$$

- pullback $\hat{=}$ homotopy pullbacks

<u>Adv</u>: · fully rigorous within classical math.

· allows for interpretations, e.g. in quasicats,

<u>Con</u>: Not fully invariant, e.g. "fibration"

① Naive cat theory (in analogy to naive set theory)

We take concepts of synth. cat $\mathcal{C}, \mathcal{D}$,

functors $F, G, \dots$

isos $\alpha : F \cong G$, $h : \alpha \cong \beta$

as primitives

and impose some behavior on them

minimal amount of structure and coherence

$$\xrightarrow{F} \xrightarrow{G} \quad \rightsquigarrow \quad GF$$

$$\overset{\alpha}{\cong} \ \overset{\beta}{\cong} \quad \rightsquigarrow \quad \beta\alpha$$

$$F(GH) \cong (FG)H$$

In practice we only need very low layers of coherence.

<u>Example</u>: terminal cat $\{*\}$ would be introduced with

- for each T have $T \to \{*\}$
- for each $F, G : T \to \{*\}$ have $F \cong G$

<u>Features</u>:
- Can be taught with no prerequisites
- Can be used by beginners & non-experts

<u>Drawback</u>:
- Some low-level structure / coherence has to be put in by hand.

③ + ⑤ Type theoretic

③ embed within (versions of) [Riehl-Shulman]

[Gratzer-Weinberger-Buchholtz]

one can select the types that behave like cats

and establish axioms for those.

<u>Feature</u>: • full MLTT with id-types, Σ-types, <u>Π-types</u>

• well suited for formalization

<u>Con</u>: Not every type is a category.

(4) <u>directed type theory</u> (w.i.p, conjectural)

synth cat = types

version of simplicial type theory, crisp in the sense of Shulman

not all Π-types exist.

Functoriality of universals

Given $x : A \vdash B(x)$ cocartesian

$(\triangleq A \to U)$

$$\frac{x : A^{\simeq} \vdash t(x) : B(x) \ \text{terminal}}{x : A \vdash \bar{t}(x) : B(x), \quad x : A^{\simeq} \vdash t(x) \equiv \bar{t}(x)}$$

$$\begin{tikzcd}
 & B \arrow[d] \\
A^{\simeq} \arrow[ur, bend left, "\text{terminal}"] \arrow[r] & A \arrow[u, dashed, bend left]
\end{tikzcd}$$