# Path spaces of pushouts via a zigzag construction

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# Plan for the talk

- Discuss the background of the problem of understanding path spaces of pushouts
- Present the zigzag construction. First in a ‘type-theoretic’ style, then in a ‘diagrammatic’ style due to Christian Sattler.
- Discuss ‘convergence behaviour’ and applications to truncatedness of pushouts, Blakers–Massey, combinatorial group theory

# Pushouts

We work informally in HoTT with a univalent universe U closed under *pushouts*.
Given

- $A : \mathsf{U}$
- $B : \mathsf{U}$
- $R : A \to B \to \mathsf{U}$

the pushout $A +_R B : \mathsf{U}$ is freely generated by

- $\mathsf{inl} : A \to A +_R B$
- $\mathsf{inr} : B \to A +_R B$
- $\mathsf{glue} : (a : A)\,(b : B) \to R\,a\,b \to \mathsf{inl}\,a = \mathsf{inr}\,b$.

# The problem

We know that any element of $A +_R B$ is merely either of the form $\mathsf{inl}\, a$ or $\mathsf{inr}\, b$.
But what are the identity types / path spaces?

- $\mathsf{inl}\, a_0 = \mathsf{inr}\, b$ ?
- $\mathsf{inl}\, a_0 = \mathsf{inl}\, a$ ?
- ($\mathsf{inr}\, b = \mathsf{inr}\, b'$ ?)

For most type formers (Σ-, Π- W-, M-types, univalent universes, $n$-truncations, sequential colimits), identity types are easy to describe.

Identity types of pushouts are 'complicated,' like $\Omega^m S^n$.

# Some answers

Aside from the generating paths $R\,a\,b \to \mathsf{inl}\,a = \mathsf{inr}\,b$ and their inverses one also has anything built up from a zigzag of generating paths.

We have to quotient by inverse laws $(\mathsf{glue}\,r) \cdot (\mathsf{glue}\,r)^{-1} = \mathsf{refl}$, but how exactly?

Can be made precise if we are interested in set-truncation $\|\mathsf{inl}\,a = \mathsf{inr}\,b\|_0$.[1]

[1] Favonia, Shulman: The Seifert-van Kampen Theorem in Homotopy Type Theory; 2016

# Some answers

Kraus and von Raumer[2] gave a universal property for path spaces of pushouts:

For any type $X : \mathsf{U}$ with $x : X$, the type family $X \to \mathsf{U}$, $y \mapsto x = y$ is freely generated by $\mathsf{refl} : x = x$.

A type family $A +_R B \to \mathsf{U}$ is the same thing as the data

- $P : A \to \mathsf{U}$
- $Q : B \to \mathsf{U}$
- $e : (a : A)\,(b : B) \to R\,a\,b \to P\,a \simeq Q\,b$.

Thus the triple $(P, Q, e)$ freely generated by a point $p : P\,a_0$ has $P\,a \simeq (\mathsf{inl}\,a_0 = \mathsf{inl}\,a)$, $Q\,b \simeq (\mathsf{inl}\,a_0 = \mathsf{inr}\,b)$,
and $e$ corresponding to post-composition by glue.

[2] Path Spaces of Higher Inductive Types in Homotopy Type Theory; 2019

# Some answers

This is a nice description and easy to prove, but not immediately useful for many purposes.
A priori describes path spaces of non-recursive HITs (pushouts) as recursive HITs (much more complex).

There is a close analogy with the James construction.[3]

[3] Brunerie: The James Construction and $\pi_4(\mathbb{S}^3)$ in Homotopy Type Theory; 2018

# James construction

If $X : \mathsf{U}$, $x_0 : X$ is a *pointed connected* type, writing $\Sigma X :\equiv 1 +_X 1$, we have $JX :\equiv \Omega\Sigma X$ freely generated by

- a term $\varepsilon : JX$
- a map $\alpha : X \to JX \to JX$
- with $\delta : (j : JX) \to \alpha(x_0, j) = j$.

Also a recursive description, but can be 'unrecursified', so $JX = \mathrm{colim}_{n\to\infty} J_n X$.

$\alpha$ unrecursifies to $\alpha_n : X \to J_n X \to J_{n+1} X$ with a naturality condition.

# Dealing with equivalences

We would like to follow to the same strategy to 'unrecursify' Kraus–von Raumer's description of path spaces of pushouts, so $P\,a = \mathrm{colim}_{n\to\infty} P_n\,a$ and $Q\,b = \mathrm{colim}_{n\to\infty} Q_n\,b$.

But how to unrecursify $P(a) \simeq Q(b)$ for $r : R\,a\,b$?

Several ways to express equivalence constructors in HITs:
Kraus and von Raumer use biinvertible maps.
Rijke, Shulman, and Spitters[4] use *path-split* maps.

One could probably unrecursify these, but they seem to give the *wrong* sequence $P_n$.

We follow a different route.

[4] Modalities in homotopy type theory; 2020

# Interleaving sequences

## Lemma

*Given sequences* $P_0 \to P_2 \to \cdots$ *and* $Q_1 \to Q_3 \to \cdots$ *and a commutative diagram*

$$\begin{array}{ccccccccccc} P_0 & & \longrightarrow & & P_2 & & \longrightarrow & & \cdots & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ & & Q_1 & & \longrightarrow & & Q_3 & & \longrightarrow & & \cdots \end{array}$$

*we have an equivalence* $\operatorname{colim}_{n\to\infty} P_n \simeq \operatorname{colim}_{n\to\infty} Q_n$.

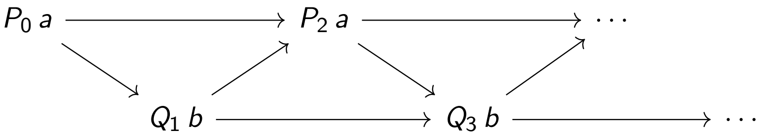
## Proof sketch.

Consider the sequence $P_0 \to Q_1 \to P_2 \to \cdots$ and its colimit.
Omitting every other term, we get the two sequences
$P_0 \to P_2 \to \cdots$ and $Q_1 \to Q_3 \to \cdots$ we started with.
Thus all three sequences have the same colimit. $\square$

# The zigzag construction

Let $A, B : \mathrm{U}$, $R : A \to B \to \mathrm{U}$, $a_0 : A$ as before.
We define $P_0, P_2, \cdots : A \to \mathrm{U}$ and $Q_1, Q_3, \cdots : B \to \mathrm{U}$ freely so that we have

- a map $P_n\, a \to P_{n+2}\, a$ for $a : A$, $n$ even,
- a map $Q_n\, b \to Q_{n+2}\, b$ for $b : B$, $n$ odd
- a term of $P_0(a_0)$,
- for $a : A$, $b : B$, $r : R\, a\, b$, an interleaving diagram.



# The zigzag construction

More concretely, we have

- $P_0\, a :\equiv (a_0 = a)$
- $Q_1\, b :\equiv R\, a_0\, b$
- $P_{n+2}\, a$ given by a pushout square

$$
\begin{array}{ccc}
(b : B) \times R\, a\, b \times P_n\, a & \longrightarrow & (b : B) \times R\, a\, b \times Q_{n+1}\, b \\
\downarrow & & \downarrow \\
P_n\, a & \longrightarrow & P_{n+2}\, a
\end{array}
$$

- $Q_{n+2}\, b$ is given by the analogous pushout square

$$
\begin{array}{ccc}
(a : A) \times R\, a\, b \times Q_n\, b & \longrightarrow & (a : A) \times R\, a\, b \times P_{n+1}\, a \\
\downarrow & & \downarrow \\
Q_n\, b & \longrightarrow & Q_{n+2}\, b
\end{array}
$$

# The zigzag construction

**Theorem**

*With notation as before,*
$(\mathsf{inl}\, a_0 = \mathsf{inl}\, a) \simeq \mathrm{colim}_{n\to\infty} P_n\, a$ *for* $a : A$ *and*
$(\mathsf{inl}\, a_0 = \mathsf{inr}\, b) \simeq \mathrm{colim}_{n\to\infty} Q_n\, b$ *for* $b : B$*.*

**Proof sketch.**

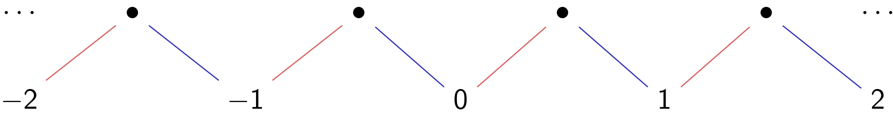
Write $P_\infty\, a :\equiv \mathrm{colim}_{n\to\infty} P_n\, a$.
Then $e\, r : P_\infty\, a \simeq Q_\infty\, b$ for $r : R\, a\, b$ by the interleaving diagram.
Now $(P_\infty, Q_\infty, e)$ is freely generated by a term of $P_\infty\, a_0$ essentially by construction.
So we can appeal to Kraus–von Raumer's characterisation of the path spaces. □

# An example: $\Omega S^1$

We have $S^1 \simeq 1 +_2 1$ where $2 = \{B, R\}$.[5]

Writing $N = \mathsf{inl}\,\star$ and $S = \mathsf{inr}\,\star$, the construction describes $N = N$ and $N = S$ as sequential colimits. How?

We picture $N = N$ as the bottom row above and $N = S$ as the top row. The filtrations (i.e. types $P_n$ and $Q_n$) are given by intervals centred on 0.



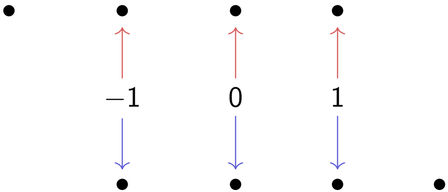

[5]Note that a relation $R : 1 \to 1 \to \mathsf{U}$ is just a type.

# An example: $\Omega S^1$

The following pushout diagram describes $P_4$:

$$\begin{array}{ccc} 2 \times P_2 & \longrightarrow & 2 \times Q_3 \\ \downarrow & & \downarrow \\ P_2 & \longrightarrow & P_4 \end{array}$$

This expresses that $\{-2, -1, 0, 1, 2\}$ is given by gluing two four-element sets along $\{-1, 0, 1\}$.



# A diagrammatic perspective

An alternative presentation of the construction due to Christian Sattler avoids type theory-style indexing.

Say given a span of spaces $A \leftarrow R \rightarrow B$.
Given a map $Y \rightarrow A +_R B$ (e.g. $\mathsf{inl} : A \rightarrow A +_R B$) we seek to understand the pullback of $A \leftarrow R \rightarrow B$ along $Y \rightarrow A +_R B$.

In general suppose we have a span $P_0 \leftarrow T_0 \rightarrow Q_0$ *over* the first one. We describe the pullback of the first span along the induced map $P_0 +_{T_0} Q_0 \rightarrow A +_R B$.

To this end we construct a sequence of overspans $(P_n \leftarrow T_n \rightarrow Q_n)_{n:\mathbb{N}}$ and take the colimit $P_\infty \leftarrow T_\infty \rightarrow Q_\infty$.

# A diagrammatic perspective

*Descent for pushouts* means that if both squares below are pullback squares then the top span is the pullback of the bottom one along $P_\infty +_{T_\infty} Q_\infty \to A +_R B$.

$$\begin{array}{ccccc} P_\infty & \longleftarrow & T_\infty & \longrightarrow & Q_\infty \\ \downarrow & \llcorner & \downarrow & \lrcorner & \downarrow \\ A & \longleftarrow & R & \longrightarrow & B \end{array}$$

We can in turn ensure that these squares are pullback squares – in short that $P_\infty \leftarrow T_\infty$ and $T_\infty \to Q_\infty$ are cartesian – by ensuring that $P_n \leftarrow T_n$ and $T_n \to Q_n$ are each cartesian for infinitely many $n$.
(This uses commutativity of pullbacks and sequential colimits.)

# A diagrammatic perspective

Given $P \leftarrow T \rightarrow Q$ a span over $A \leftarrow R \rightarrow B$ we construct a span $P' \leftarrow T' \rightarrow Q'$ in between the above two such that

- $P \rightarrow P'$ is an equivalence,
- $P' \leftarrow T'$ is cartesian over $A \leftarrow R$,
- $P +_T Q \rightarrow P' +_{T'} Q'$ is an equivalence,

as follows.

$$\begin{array}{ccccc}
P & \longleftarrow & T & \longrightarrow & Q \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \ulcorner & \downarrow \\
P' & \longleftarrow & T' & \longrightarrow & Q' \\
\downarrow & \llcorner & \downarrow & & \downarrow \\
A & \longleftarrow & R & \longrightarrow & B
\end{array}$$

# A diagrammatic perspective

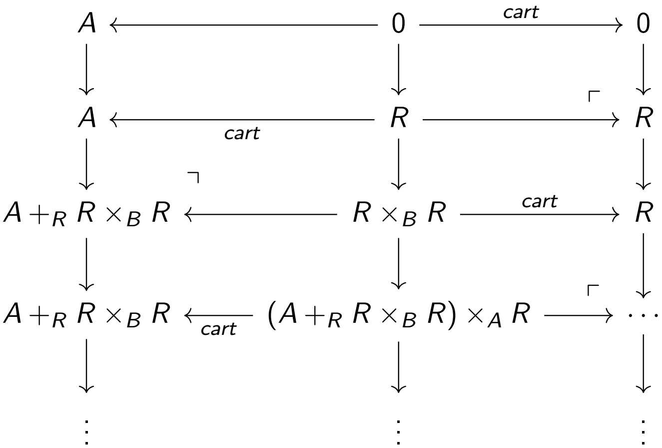
Now starting from $P_0 \leftarrow T_0 \rightarrow Q_0$ one can iterate the previous construction, alternating between making $P_n \leftarrow T_n$ cartesian and making $T_n \rightarrow Q_n$ cartesian.

The pushout is unchanged in each step so is unchanged also in the colimit as $n \rightarrow \infty$.

Thus $P_\infty \leftarrow T_\infty \rightarrow Q_\infty$ is precisely the pullback of $A \leftarrow R \rightarrow B$ along $P_0 +_{T_0} Q_0 \rightarrow A +_R B$.

We have e.g. $A = A +_0 0$, and $1 = 1 +_0 0$.

# The first few steps



# Convergence behaviour

It will be fruitful to analyse the convergence behaviour of the construction.

How well does $P_n$ approximate $P_\infty$?

How 'far' is the map $P_n \to P_{n+2}$ from being an equivalence?

We are particularly interested in $P_0\, a \to P_\infty\, a$, corresponding to
$\mathsf{ap}_{\mathsf{inl}} : (a_0 = a) \to (\mathsf{inl}\, a_0 = \mathsf{inl}\, a)$,
and in $Q_1\, b \to Q_\infty b$, corresponding to
$\mathsf{glue} : R\, a_0\, b \to (\mathsf{inl}\, a_0 = \mathsf{inr}\, b)$.

# Informal explanation of convergence behaviour

The fibres of the map $P_n \to P_{n+2}$ express when a zigzag of length at most $n+2$ has length at most $n$.

This happens when one can *reduce* an pair of adjacent edges.

This is controlled by the identity types of $(a : A) \times R\,a\,b$ for $b : B$ and of $(b : B) \times R\,a\,b$ for $a : A$,

or equivalently by the diagonals of $R \to B$ and $R \to A$.

It is enough to reduce *some* adjacent pair in the zigzag.
This is why *joins* of these identity types show up.

# Formal analysis of convergence behaviour

We want to understand the map $P_n\,a \to P_{n+2}\,a$.
It is defined as a pushout of
$(b : B) \times R\,a\,b \times P_n\,a \to (b : B) \times R\,a\,b \times Q_{n+1}\,b$.

So suffices to understand $P_n\,a \to Q_{n+1}\,b$ given $r : R\,a\,b$.

**Theorem**

*For $r : R\,a\,b$ and $n$ even, the map $P_n\,a \to Q_{n+1}\,b$ is a pushout of a map $f$ such that all fibres of $f$ are of the form $X * Y$ where $X$ is the fibre of a map $Q_{n-1}\,b' \to P_n\,a$ given by $r' : R\,a\,b'$ and $Y$ is $(b, r) = (b', r')$.*
*Similarly, for $r : R\,a\,b$ and $n$ odd, the map $Q_n\,b \to P_{n+1}\,a$ is a pushout of a map $g$ such that all fibres of $g$ are of the form $X * Y$ where $X$ is the fibre of a map $P_{n-1}\,a' \to Q_n\,b$ given by $r' : R\,a'\,b$ and $Y$ is $(a, r) = (a', r')$.*

# The key theorem

### Theorem

*Let $C_{-1}, C_0, C_1, \cdots$ be classes of maps of types such that*

- *Each class is determined fibrewise: there is a class $T_n$ of types such that $C_n$ consists of all maps whose fibres are all in $T_n$.*
- *Each class is closed under pushouts.*
- *$T_{-1}$ contains the empty type.*
- *For each $n \geq 0$, $T_n$ contains any type that is a join of a type in $T_{n-1}$ with an identity type in $(a : A) \times R\, a\, b$ if $n$ is even, $b : B$ or an identity type in $(b : B) \times R\, a\, b$ if $n$ is odd, $a : A$.*

*Then the maps $P_n\, a \to Q_{n+1}\, b$ and $P_n\, a \to P_{n+2}\, a$ lie in $C_n$ for $n$ even and $Q_n\, b \to P_{n+1}\, a$, $Q_n\, b \to Q_{n+2}\, b$ lies in $C_n$ for $n$ odd. If moreover $C_{n+1} \subseteq C_n$ for all $n$ and $C_n$ is closed under transfinite composition, then the same holds for $P_n\, a \to P_\infty\, a$ and $Q_n\, b \to Q_\infty\, b$.*

# Pushouts of embeddings

Suppose $R \to B$ is an embedding.
Then we can take $C_n$ to be the class of all equivalences for $n \geq 0$.
This shows that $\mathsf{ap}_{\mathsf{inl}}$ is an equivalence i.e. $\mathsf{inl} : A \to A +_R B$ is an embedding i.e. embeddings are closed under pushouts.
Also glue is an equivalence so the pushout square is a pullback square.

If $R \to A$ is an embedding we can take $C_n$ to be all equivalences for $n \geq 1$ to see that $P_2\, a \simeq (\mathsf{inl}\, a_0 = \mathsf{inl}\, a)$.

# Pushouts of 0-truncated spans

**Theorem**

*Suppose $R \to A$ and $R \to B$ are both 0-truncated i.e. their diagonals are embeddings. Then the same holds for* $\mathsf{inl}$ *and* $\mathsf{inr}$*, and* $\mathsf{glue}$ *is an embedding.*

**Proof.**

Take each $C_n$ to consist of all embeddings. □

The same result holds if we replace ‘embedding’ with ‘complemented’ / ‘decidable embedding’ throughout.

# Truncatedness of pushouts

## Corollary

*If $R \to A$ and $R \to B$ are both 0-truncated and $A$, $B$ are both $n$-truncated with $n \geq 1$ then $A +_R B$ is also $n$-truncated.*

## Proof.

To be $n$-truncated means that $\Omega^{n+1}$ is contractible at each point, and inl, inr induce equivalences already on $\Omega^2$. □

So the suspension of a set, or any other pushout of sets, is 1-truncated. This resolves an open question from the HoTT book.

# Some group theory

The following observation is due to Buchholtz, de Jong, and Rijke.

## Theorem

*Given a parallel pair of group embeddings $H \rightrightarrows G$, we have that $G$ embeds in the associated HNN extension $G{*_H}$.*

## Proof.

The coequaliser of $BH \rightrightarrows BG$ is a delooping of $G{*_H}$. Equivalently this is a pushout:

$$\begin{array}{ccc} BH + BH & \longrightarrow & BH \\ \downarrow & & \downarrow \\ BG & \longrightarrow & B(G{*_H}) \end{array}$$

□

The proof is directly constructive and avoids combinatorial reasoning about words.

# The Blakers–Massey theorem

**Theorem**

*Let $k, l \geq 0$ be integers such that the diagonal of $R \to A$ is $k$-connected and the diagonal of $R \to B$ is $l$-connected. Then* glue *is $(k + l + 2)$-connected.*

**Proof.**

Take $C_n$ to be the class of $((l + 2) + (k + 2) + (l + 2) + \ldots - 2)$-connected maps, where the sum contains $n + 1$ terms.

Then glue lies in $C_1$ which is the class of $(l + 2 + k + 2 - 2)$-connected maps. □

This directly generalises a corresponding argument for the James construction.

A rough preprint with some more details is available online at

dwarn.se/po-paths.pdf

Thanks for listening!