Game Semantics of Homotopy Type Theory

Norihiro Yamada

yamad041@umn.edu
University of Minnesota

Homotopy Type Theory Electronic Seminar Talks (HoTTEST)
Department of Mathematics, Western University
February 11, 2021

Background: MLTT vs. HoTT

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

- *Proofs/objects as computations* (e.g., $\mathrm{succ} : \neg \max \mathbb{N}$);

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

- *Proofs/objects as computations* (e.g., $\mathrm{succ} : \neg \max \mathbb{N}$);
- MLTT as a foundation of *constructive* maths.

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

- *Proofs/objects as computations* (e.g., $\mathrm{succ} : \neg \max \mathbb{N}$);
- MLTT as a foundation of *constructive* maths.

On the other hand, homotopy type theory (HoTT) is motivated by the *homotopical* interpretation of MLTT.

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

- *Proofs/objects as computations* (e.g., $\mathrm{succ} : \neg \max \mathbb{N}$);
- MLTT as a foundation of *constructive* maths.

On the other hand, homotopy type theory (HoTT) is motivated by the *homotopical* interpretation of MLTT.

- HoTT = MLTT + univalence + higher inductive types (HITs);

Background: MLTT vs. HoTT

Just like axiomatic set theory is explained by *sets* in an informal sense, the conceptual foundation of Martin-Löf type theory (MLTT) is *computations* in an informal sense (a.k.a. the *BHK-interpretation*).

- *Proofs/objects as computations* (e.g., $\mathrm{succ} : \neg \max \mathbb{N}$);
- MLTT as a foundation of *constructive* maths.

On the other hand, homotopy type theory (HoTT) is motivated by the *homotopical* interpretation of MLTT.

- HoTT = MLTT + univalence + higher inductive types (HITs);
- Homotopical interpretation: formulas as spaces, *proofs/objects as points*, and *higher proofs/objects as paths/homotopies*.

Motivation: computational understanding of HoTT

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths.

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths. On the other hand,

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths. On the other hand,

The topological view is orthogonal to the BHK-interpretation.

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths. On the other hand,

The topological view is orthogonal to the BHK-interpretation.

Moreover,

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths. On the other hand,

The topological view is orthogonal to the BHK-interpretation.

Moreover,

Are proofs given by working mathematicians points in spaces?

Motivation: computational understanding of HoTT

HoTT uncovers new connections between type theory, higher category theory and homotopy theory, and is a powerful foundation of maths. On the other hand,

The topological view is orthogonal to the BHK-interpretation.

Moreover,

Are proofs given by working mathematicians points in spaces?

Motivation (The BHK-interpretation of HoTT)

To extend the BHK-interpretation of MLTT to HoTT so that one can better understand *HoTT as a foundation of constructive maths*.

Main results

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);
- Identity proofs as *strategies that verify equality between strategies*.

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);
- Identity proofs as *strategies that verify equality between strategies*.

This model can be seen as a variant of the BHK-interpretation.

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);
- Identity proofs as *strategies that verify equality between strategies*.

This model can be seen as a variant of the BHK-interpretation.

Corollary (Consistency and independence)

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);
- Identity proofs as *strategies that verify equality between strategies*.

This model can be seen as a variant of the BHK-interpretation.

Corollary (Consistency and independence)

1. *Consistency of HoTT + strict univalence:* $\mathrm{Id}_U(A, B) \equiv \mathrm{Eq}(A, B)$;

Main results

Theorem (Game semantics of HoTT)

There exists game semantics of HoTT (viz., MLTT equipped with One-, Zero-, N-, Pi-, Sigma- and Id-types as well as univalent universes).

My game semantics of HoTT interprets

- Types as *games* between Player and Opponent;
- Terms as *strategies* (or algorithms for Player to play on games);
- Identity proofs as *strategies that verify equality between strategies*.

This model can be seen as a variant of the BHK-interpretation.

Corollary (Consistency and independence)

1. *Consistency of HoTT + strict univalence:* $\mathrm{Id}_U(A, B) \equiv \mathrm{Eq}(A, B)$*;*
2. *Independence of Markov's principle from this extended HoTT.*

Why game semantics? (part 1/3)

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

- Effective for the study of type theory (e.g., independence of MP);

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

- Effective for the study of type theory (e.g., independence of MP);
- Semantics of *computational effects* and *linear logic*;

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

- Effective for the study of type theory (e.g., independence of MP);
- Semantics of *computational effects* and *linear logic*;
- Applications in *program verification* and *model checking*;

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

- Effective for the study of type theory (e.g., independence of MP);
- Semantics of *computational effects* and *linear logic*;
- Applications in *program verification* and *model checking*;
- *Rich in higher structures* by its *intensionality*.

Why game semantics? (part 1/3)

Nevertheless, one might wonder why I choose game semantics among computational models such as *realisability*.
My answer is the following advantages of game semantics:

- Effective for the study of type theory (e.g., independence of MP);
- Semantics of *computational effects* and *linear logic*;
- Applications in *program verification* and *model checking*;
- *Rich in higher structures* by its *intensionality*.

The last point is new, and so let me explain it in the next few slides.

Why game semantics? (part 2/3)

Why game semantics? (part 2/3)

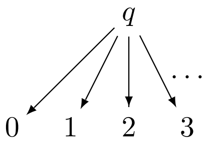
Definition (Simplified games)

A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...

Why game semantics? (part 2/3)

Definition (Simplified games)

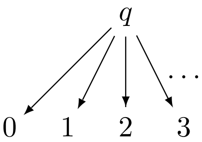
A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...



Why game semantics? (part 2/3)

Definition (Simplified games)

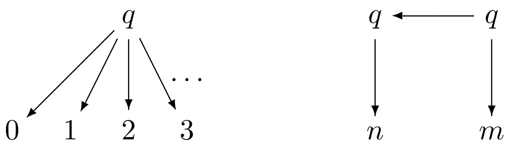
A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...



Why game semantics? (part 2/3)

Definition (Simplified games)

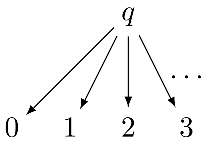
A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...

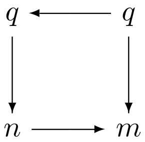


Why game semantics? (part 2/3)

Definition (Simplified games)

A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...

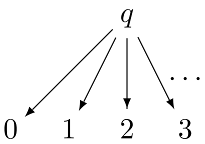


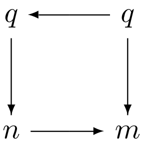


Why game semantics? (part 2/3)

Definition (Simplified games)

A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...





Definition (Simplified strategies)

A ***strategy*** σ on a game G, written $\sigma : G$, is a map

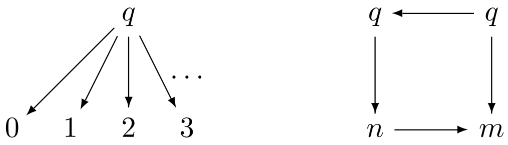
$$\{\, \text{odd-length positions } m_1 m_2 \dots m_{2i+1} \text{ in } G \,\} \to \{\, \text{P-moves } m \text{ in } G \,\}$$

s.t. $m_1 m_2 \dots m_{2i+1} m$ is a position in G.

Why game semantics? (part 2/3)

Definition (Simplified games)

A ***game*** is a rooted dag whose vertices (or *moves*) have parity O/P, and paths from a root (or *positions*) have parity OPOP...



Definition (Simplified strategies)

A ***strategy*** σ on a game G, written $\sigma : G$, is a map

$$\{\, \text{odd-length positions } m_1 m_2 \ldots m_{2i+1} \text{ in } G \,\} \to \{\, \text{P-moves } m \text{ in } G \,\}$$

s.t. $m_1 m_2 \ldots m_{2i+1} m$ is a position in G. (E.g., $q \mapsto q; qqn \mapsto n+1$.)

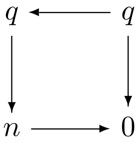
Why game semantics? (part 3/3)

Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:

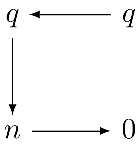
Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:



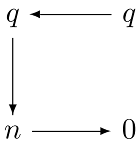
Why game semantics? (part 3/3)

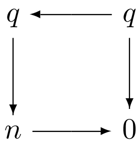
Strategies on $N \Rightarrow N$ computing *constant zero*:



Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:





Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:

Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:

They are *extensionally the same* yet *intensionally different*.

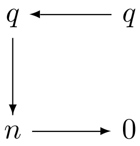
Why game semantics? (part 3/3)

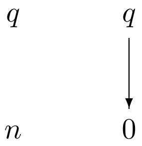
Strategies on $N \Rightarrow N$ computing *constant zero*:

They are *extensionally the same* yet *intensionally different*.
Games $\mathrm{Id}_N(\underline{n}, \underline{m})$ and $\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) := \Pi_{x:N} \mathrm{Id}_N(\phi \circ x, \psi \circ x)$:

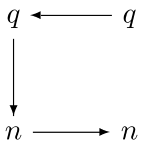
Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:





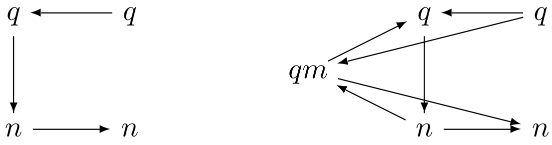
They are *extensionally the same* yet *intensionally different*.
Games $\mathrm{Id}_N(\underline{n}, \underline{m})$ and $\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) := \Pi_{x:N} \mathrm{Id}_N(\phi \circ x, \psi \circ x)$:



Why game semantics? (part 3/3)

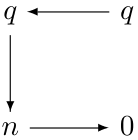
Strategies on $N \Rightarrow N$ computing *constant zero*:

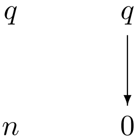
They are *extensionally the same* yet *intensionally different*.
Games $\mathrm{Id}_N(\underline{n}, \underline{m})$ and $\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) := \Pi_{x:N}\mathrm{Id}_N(\phi \circ x, \psi \circ x)$:



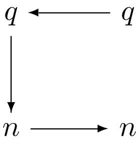
Why game semantics? (part 3/3)

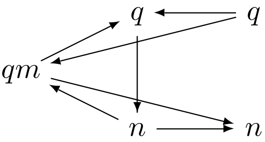
Strategies on $N \Rightarrow N$ computing *constant zero*:





They are *extensionally the same* yet *intensionally different*.
Games $\mathrm{Id}_N(\underline{n}, \underline{m})$ and $\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) := \Pi_{x:N} \mathrm{Id}_N(\phi \circ x, \psi \circ x)$:



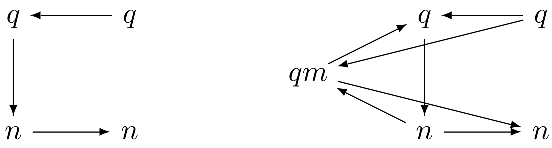


Id_N has at most one strategy, but not the case for $\mathrm{Id}_{N \Rightarrow N}$.

Why game semantics? (part 3/3)

Strategies on $N \Rightarrow N$ computing *constant zero*:

They are *extensionally the same* yet *intensionally different*.
Games $\mathrm{Id}_N(\underline{n}, \underline{m})$ and $\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) := \Pi_{x:N}\mathrm{Id}_N(\phi \circ x, \psi \circ x)$:



Id_N has at most one strategy, but not the case for $\mathrm{Id}_{N \Rightarrow N}$. In this way, the *intensionality* of games makes their higher structure *nontrivial*.

Related work and my contributions

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;
- Constructivity on the *meta-theory*.

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;
- Constructivity on the *meta-theory*.

Other related work is *cubical assemblies* proposed by Uemura and *higher dimensional meaning explanation* by Angiuli and Harper.

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;
- Constructivity on the *meta-theory*.

Other related work is *cubical assemblies* proposed by Uemura and *higher dimensional meaning explanation* by Angiuli and Harper.

- On *cubical type theory*;

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;
- Constructivity on the *meta-theory*.

Other related work is *cubical assemblies* proposed by Uemura and *higher dimensional meaning explanation* by Angiuli and Harper.

- On *cubical type theory*;
- The *unit interval*, *dimensional variables* and *path types*.

Related work and my contributions

A pioneering constructive model of HoTT is *cubical sets* initiated by Bezem, Coquand and Huber, and developed by many others.

- *Homotopical* in nature;
- Constructivity on the *meta-theory*.

Other related work is *cubical assemblies* proposed by Uemura and *higher dimensional meaning explanation* by Angiuli and Harper.

- On *cubical type theory*;
- The *unit interval*, *dimensional variables* and *path types*.

My approach: BHK-interpretation of HoTT; based on *globular* sets

Overview of my approach

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Definition (Game-semantic ∞-groupoids)

Define ***game-semantic ∞-groupoids*** to be ∞-groupoids internalised in the category $\mathcal{G}$ of games (strictly, in the subcat $\check{\mathcal{G}} \hookrightarrow \mathcal{G}$, a topos).

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Definition (Game-semantic ∞-groupoids)

Define ***game-semantic ∞-groupoids*** to be ∞-groupoids internalised in the category $\mathcal{G}$ of games (strictly, in the subcat $\check{\mathcal{G}} \hookrightarrow \mathcal{G}$, a topos).

Forenotice:

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Definition (Game-semantic ∞-groupoids)

Define ***game-semantic ∞-groupoids*** to be ∞-groupoids internalised in the category $\mathcal{G}$ of games (strictly, in the subcat $\check{\mathcal{G}} \hookrightarrow \mathcal{G}$, a topos).

Forenotice:

- Warren's *strict* ∞-groupoid model modified and internalised in $\mathcal{G}$;

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Definition (Game-semantic ∞-groupoids)

Define ***game-semantic ∞-groupoids*** to be ∞-groupoids internalised in the category $\mathcal{G}$ of games (strictly, in the subcat $\check{\mathcal{G}} \hookrightarrow \mathcal{G}$, a topos).

Forenotice:

- Warren's *strict* ∞-groupoid model modified and internalised in $\mathcal{G}$;
- Challenge: To recover Warren's method with *weak* inverses;

Overview of my approach

Throughout this work, ***∞-groupoids*** refer to *strict* ∞-categories whose morphisms are *weakly invertible* in the sense of type equivalence.

Definition (Game-semantic ∞-groupoids)

Define ***game-semantic ∞-groupoids*** to be ∞-groupoids internalised in the category $\mathcal{G}$ of games (strictly, in the subcat $\check{\mathcal{G}} \hookrightarrow \mathcal{G}$, a topos).

Forenotice:

- Warren's *strict* ∞-groupoid model modified and internalised in $\mathcal{G}$;
- Challenge: To recover Warren's method with *weak* inverses;
- Mostly generalised to finitely complete CwFs $\mathcal{M}$ with an NNO such that each type $A \in \mathrm{Ty}_{\mathcal{M}}(\Gamma)$ is a map $\mathcal{M}(1, \Gamma) \to \mathrm{Ob}(\mathcal{M})$.

Game-semantic ∞-groupoids (part 1/2)

Game-semantic ∞-groupoids (part 1/2)

Explicitly, a *game-semantic ∞-category* G consists of

Game-semantic ∞-groupoids (part 1/2)

Explicitly, a *game-semantic ∞-category* G consists of

- A diagram in $\check{\mathcal{G}}$

$$\cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} G_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} G_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} G_0$$

  that satisfies $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$;

Game-semantic ∞-groupoids (part 1/2)

Explicitly, a *game-semantic ∞-category* G consists of

- A diagram in $\check{\mathcal{G}}$

$$\cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} G_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} G_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} G_0$$

  that satisfies $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$;
- A $\check{\mathcal{G}}$-morphism $*_p^{[n]} : G_n \times_{G_p} G_n \to G_n$ for each $0 \leqslant p < n$,

Game-semantic ∞-groupoids (part 1/2)

Explicitly, a *game-semantic ∞-category* G consists of

- A diagram in $\check{\mathcal{G}}$

$$\cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} G_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} G_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} G_0$$

  that satisfies $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$;
- A $\check{\mathcal{G}}$-morphism $*_p^{[n]} : G_n \times_{G_p} G_n \to G_n$ for each $0 \leqslant p < n$, where

$$\begin{array}{ccc} G_n \times_{G_p} G_n & \xrightarrow{\pi_2} & G_n \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle t^{n-p}} \\ G_n & \xrightarrow[s^{n-p}]{} & G_p \end{array}$$

Game-semantic ∞-groupoids (part 1/2)

Explicitly, a *game-semantic ∞-category* G consists of

- A diagram in $\check{\mathcal{G}}$

$$\cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} G_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} G_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} G_0$$

  that satisfies $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$;
- A $\check{\mathcal{G}}$-morphism $*_p^{[n]} : G_n \times_{G_p} G_n \to G_n$ for each $0 \leqslant p < n$, where

$$\begin{array}{ccc} G_n \times_{G_p} G_n & \xrightarrow{\pi_2} & G_n \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle t^{n-p}} \\ G_n & \xrightarrow[s^{n-p}]{} & G_p \end{array}$$

- A $\check{\mathcal{G}}$-morphism $i_n : G_n \to G_{n+1}$.

Game-semantic ∞-groupoids (part 2/2)

Game-semantic ∞-groupoids (part 2/2)

These data are required to satisfy the axioms of strict ∞-categories in terms of commutative diagrams in $\check{\mathcal{G}}$.

Game-semantic ∞-groupoids (part 2/2)

These data are required to satisfy the axioms of strict ∞-categories in terms of commutative diagrams in $\check{\mathcal{G}}$.
A game-semantic ∞-category G is a *game-semantic ∞-groupoid* if it is equipped with $\check{\mathcal{G}}$-morphisms with the ‘expected’ sources and targets

$$G_n \xrightarrow{\mathrm{inv}_n} G_n \qquad G_n \xrightarrow{\mathrm{ret}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{sec}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{tri}_n} G_{n+2}$$

that are functorial (on the nose).

Game-semantic ∞-groupoids (part 2/2)

These data are required to satisfy the axioms of strict ∞-categories in terms of commutative diagrams in $\check{\mathcal{G}}$.

A game-semantic ∞-category G is a *game-semantic ∞-groupoid* if it is equipped with $\check{\mathcal{G}}$-morphisms with the 'expected' sources and targets

$$G_n \xrightarrow{\mathrm{inv}_n} G_n \qquad G_n \xrightarrow{\mathrm{ret}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{sec}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{tri}_n} G_{n+2}$$

that are functorial (on the nose).

'Expected' sources and targets:

Game-semantic ∞-groupoids (part 2/2)

These data are required to satisfy the axioms of strict ∞-categories in terms of commutative diagrams in $\check{\mathcal{G}}$.
A game-semantic ∞-category G is a *game-semantic ∞-groupoid* if it is equipped with $\check{\mathcal{G}}$-morphisms with the 'expected' sources and targets

$$G_n \xrightarrow{\mathrm{inv}_n} G_n \qquad G_n \xrightarrow{\mathrm{ret}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{sec}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{tri}_n} G_{n+2}$$

that are functorial (on the nose).
'Expected' sources and targets: Given an n-cell $f \in \mathcal{G}(1, G_n)$, one gets

$$f^{-1} := \mathrm{inv} \circ f : t \circ f \to s \circ f \qquad \eta_f := \mathrm{ret} \circ f : f^{-1} * f \to i \circ s \circ f$$

$$\epsilon_f := \mathrm{sec} \circ f : f * f^{-1} \to i \circ t \circ f \qquad \delta_f := \mathrm{tri} \circ f : (i \circ f) * \eta_f \to \epsilon_f * (i \circ f)$$

Game-semantic ∞-groupoids (part 2/2)

These data are required to satisfy the axioms of strict ∞-categories in terms of commutative diagrams in $\check{\mathcal{G}}$.
A game-semantic ∞-category G is a *game-semantic ∞-groupoid* if it is equipped with $\check{\mathcal{G}}$-morphisms with the 'expected' sources and targets

$$G_n \xrightarrow{\mathrm{inv}_n} G_n \qquad G_n \xrightarrow{\mathrm{ret}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{sec}_n} G_{n+1} \qquad G_n \xrightarrow{\mathrm{tri}_n} G_{n+2}$$

that are functorial (on the nose).
'Expected' sources and targets: Given an n-cell $f \in \mathcal{G}(1, G_n)$, one gets

$$f^{-1} := \mathrm{inv} \circ f : t \circ f \to s \circ f \qquad \eta_f := \mathrm{ret} \circ f : f^{-1} * f \to i \circ s \circ f$$

$$\epsilon_f := \mathrm{sec} \circ f : f * f^{-1} \to i \circ t \circ f \qquad \delta_f := \mathrm{tri} \circ f : (i \circ f) * \eta_f \to \epsilon_f * (i \circ f)$$

They correspond to *type equivalence* so that we model univalence.

Game-semantic ∞-functors

Game-semantic ∞-functors

Define a (set-theoretic) ∞-groupoid $|G|$ by $|G|_n := \mathcal{G}(1, G_n)$.

Game-semantic ∞-functors

Define a (set-theoretic) ∞-groupoid $|G|$ by $|G|_n := \mathcal{G}(1, G_n)$.

Definition (Game-semantic ∞-functors)

Define ***game-semantic ∞-functors*** between game-semantic ∞-groupoids G and H to be strategies $\phi \in \mathrm{Tm}_{\mathcal{G}}(N, H^G)$, where $H^G(\underline{n}) := H_n^{G_n}$, s.t. $\phi^\star := (\phi_n^\star := \phi \circ \underline{n} : H_n^{G_n})_{\underline{n} \in \mathcal{G}(1,N)}$ forms ∞-functors $|G| \to |H|$ internalised in $\mathcal{G}$ that preserve the data of inverses.

Game-semantic ∞-functors

Define a (set-theoretic) ∞-groupoid $|G|$ by $|G|_n := \mathcal{G}(1, G_n)$.

Definition (Game-semantic ∞-functors)

Define ***game-semantic ∞-functors*** between game-semantic ∞-groupoids G and H to be strategies $\phi \in \mathrm{Tm}_{\mathcal{G}}(N, H^G)$, where $H^G(\underline{n}) := H_n^{G_n}$, s.t. $\phi^\star := (\phi_n^\star := \phi \circ \underline{n} : H_n^{G_n})_{\underline{n} \in \mathcal{G}(1,N)}$ forms ∞-functors $|G| \to |H|$ internalised in $\mathcal{G}$ that preserve the data of inverses.

Explicitly, the *functoriality* of the family $\phi^\star$ means: $s_n \circ \phi_{n+1}^\star = \phi_n^\star \circ s_n$, $t_n \circ \phi_{n+1}^\star = \phi_n^\star \circ t_n$, $\phi_n^\star \circ *_p^{[n]} = *_p^{[n]} \circ (\phi_n^\star \times_{B_p} \phi_n^\star)$ and $\phi_{n+1}^\star \circ i_n = i_n \circ \phi_n^\star$.

Game-semantic ∞-functors

Define a (set-theoretic) ∞-groupoid $|G|$ by $|G|_n := \mathcal{G}(1, G_n)$.

Definition (Game-semantic ∞-functors)

Define ***game-semantic ∞-functors*** between game-semantic ∞-groupoids G and H to be strategies $\phi \in \mathrm{Tm}_{\mathcal{G}}(N, H^G)$, where $H^G(\underline{n}) := H_n^{G_n}$, s.t. $\phi^\star := (\phi_n^\star := \phi \circ \underline{n} : H_n^{G_n})_{\underline{n} \in \mathcal{G}(1,N)}$ forms ∞-functors $|G| \to |H|$ internalised in $\mathcal{G}$ that preserve the data of inverses.

Explicitly, the *functoriality* of the family $\phi^\star$ means: $s_n \circ \phi_{n+1}^\star = \phi_n^\star \circ s_n$, $t_n \circ \phi_{n+1}^\star = \phi_n^\star \circ t_n$, $\phi_n^\star \circ *_p^{[n]} = *_p^{[n]} \circ (\phi_n^\star \times_{B_p} \phi_n^\star)$ and $\phi_{n+1}^\star \circ i_n = i_n \circ \phi_n^\star$.

Lemma (∞-category of game-semantic ∞-categories)

The category $\infty\mathcal{G}\mathrm{Gpd}$ *of game-semantic ∞-groupoids and ∞-functors gives rise to a (set-theoretic) ∞-category.*

Game-semantic ∞-functors

Define a (set-theoretic) ∞-groupoid $|G|$ by $|G|_n := \mathcal{G}(1, G_n)$.

Definition (Game-semantic ∞-functors)

Define ***game-semantic ∞-functors*** between game-semantic ∞-groupoids G and H to be strategies $\phi \in \mathrm{Tm}_{\mathcal{G}}(N, H^G)$, where $H^G(\underline{n}) := H_n^{G_n}$, s.t. $\phi^\star := (\phi_n^\star := \phi \circ \underline{n} : H_n^{G_n})_{\underline{n} \in \mathcal{G}(1,N)}$ forms ∞-functors $|G| \to |H|$ internalised in $\mathcal{G}$ that preserve the data of inverses.

Explicitly, the *functoriality* of the family $\phi^\star$ means: $s_n \circ \phi_{n+1}^\star = \phi_n^\star \circ s_n$, $t_n \circ \phi_{n+1}^\star = \phi_n^\star \circ t_n$, $\phi_n^\star \circ *_p^{[n]} = *_p^{[n]} \circ (\phi_n^\star \times_{B_p} \phi_n^\star)$ and $\phi_{n+1}^\star \circ i_n = i_n \circ \phi_n^\star$.

Lemma (∞-category of game-semantic ∞-categories)

The category $\infty\mathcal{G}\mathrm{Gpd}$ *of game-semantic ∞-groupoids and ∞-functors gives rise to a (set-theoretic) ∞-category.*

The map $|\_|$ extends to a functor $\infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd} := \infty\mathrm{SetGpd}$.

Game-semantic transformations

Game-semantic transformations

This functor $|\_| : \infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ sends game-semantic ∞-functors $\phi : G \to H$ to the (set-theoretic) ∞-functors $|\phi| : |G| \to |H|$ given by

$$|\phi|_n : \gamma \in \mathcal{G}(1, G_n) \mapsto \phi_n^\star \circ \gamma \in \mathcal{G}(1, H_n).$$

Game-semantic transformations

This functor $|\_| : \infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ sends game-semantic ∞-functors $\phi : G \to H$ to the (set-theoretic) ∞-functors $|\phi| : |G| \to |H|$ given by

$$|\phi|_n : \gamma \in \mathcal{G}(1, G_n) \mapsto \phi_n^\star \circ \gamma \in \mathcal{G}(1, H_n).$$

In $\infty\mathcal{G}\mathrm{Gpd}$, n-cells ($n > 1$) are *game-semantic* $(n-1)$*-transformations*:

Game-semantic transformations

This functor $|\_| : \infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ sends game-semantic ∞-functors $\phi : G \to H$ to the (set-theoretic) ∞-functors $|\phi| : |G| \to |H|$ given by

$$|\phi|_n : \gamma \in \mathcal{G}(1, G_n) \mapsto \phi_n^\star \circ \gamma \in \mathcal{G}(1, H_n).$$

In $\infty\mathcal{G}\mathrm{Gpd}$, n-cells ($n > 1$) are *game-semantic* $(n-1)$*-transformations*:

Definition (Game-semantic transformations)

Define ***game-semantic transformations*** between game-semantic ∞-functors $\phi, \psi : G \to H$ to be transformations $|\phi| \to |\psi|$ internalised in $\mathcal{G}$, and similarly ***game-semantic n-transformations*** for all $n > 0$.

Game-semantic transformations

This functor $|\_| : \infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ sends game-semantic ∞-functors $\phi : G \to H$ to the (set-theoretic) ∞-functors $|\phi| : |G| \to |H|$ given by

$$|\phi|_n : \gamma \in \mathcal{G}(1, G_n) \mapsto \phi_n^\star \circ \gamma \in \mathcal{G}(1, H_n).$$

In $\infty\mathcal{G}\mathrm{Gpd}$, n-cells ($n > 1$) are *game-semantic* $(n-1)$*-transformations*:

Definition (Game-semantic transformations)

Define ***game-semantic transformations*** between game-semantic ∞-functors $\phi, \psi : G \to H$ to be transformations $|\phi| \to |\psi|$ internalised in $\mathcal{G}$, and similarly ***game-semantic n-transformations*** for all $n > 0$.

Explicitly, a *game-semantic* 1*-transformation* $\phi \to \psi$ is a $\mathcal{G}$-morphism $\alpha : G_0 \to H_1$ with $s \circ \alpha = \phi_0^\star$, $t \circ \alpha = \psi_0^\star$ and naturality.

Game-semantic transformations

This functor $|\_| : \infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ sends game-semantic ∞-functors $\phi : G \to H$ to the (set-theoretic) ∞-functors $|\phi| : |G| \to |H|$ given by

$$|\phi|_n : \gamma \in \mathcal{G}(1, G_n) \mapsto \phi_n^\star \circ \gamma \in \mathcal{G}(1, H_n).$$

In $\infty\mathcal{G}\mathrm{Gpd}$, n-cells ($n > 1$) are *game-semantic* $(n-1)$-*transformations*:

Definition (Game-semantic transformations)

Define ***game-semantic transformations*** between game-semantic ∞-functors $\phi, \psi : G \to H$ to be transformations $|\phi| \to |\psi|$ internalised in $\mathcal{G}$, and similarly ***game-semantic n-transformations*** for all $n > 0$.

Explicitly, a *game-semantic* 1-*transformation* $\phi \to \psi$ is a $\mathcal{G}$-morphism $\alpha : G_0 \to H_1$ with $s \circ \alpha = \phi_0^\star$, $t \circ \alpha = \psi_0^\star$ and naturality.
The functor $|\_|$ extends to an ∞-functor $\infty\mathcal{G}\mathrm{Gpd} \to \infty\mathrm{Gpd}$ by

$$|\alpha|_\gamma := \alpha \circ \gamma \in \mathcal{G}(1, H_{n+1}) \quad (\gamma \in \mathcal{G}(1, G_0)).$$

Overview of the interpretation

,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}$Gpd:

,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma \ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;

,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;
- $\Gamma \vdash q : p_1 =_{a_1 =_A a_2} p_2 \mapsto [\![q]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_3([\![p_1]\!]_{\mathcal{G}}, [\![p_2]\!]_{\mathcal{G}})$,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;
- $\Gamma \vdash q : p_1 =_{a_1 =_A a_2} p_2 \mapsto [\![q]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_3([\![p_1]\!]_{\mathcal{G}}, [\![p_2]\!]_{\mathcal{G}})$, and so on.

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;
- $\Gamma \vdash q : p_1 =_{a_1 =_A a_2} p_2 \mapsto [\![q]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_3([\![p_1]\!]_{\mathcal{G}}, [\![p_2]\!]_{\mathcal{G}})$, and so on.

We interpret One-, Zero- and N-types by discrete game-semantic ∞-groupoids,

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;
- $\Gamma \vdash q : p_1 =_{a_1 =_A a_2} p_2 \mapsto [\![q]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_3([\![p_1]\!]_{\mathcal{G}}, [\![p_2]\!]_{\mathcal{G}})$, and so on.

We interpret One-, Zero- and N-types by discrete game-semantic ∞-groupoids, and Id-type by $\mathrm{Id}_A(\gamma, \alpha_1, \alpha_2) := A(\gamma)(\alpha_1, \alpha_2) \hookrightarrow A(\gamma)$.

Overview of the interpretation

Now, one can sketch the game semantics $[\![\_]\!]_{\mathcal{G}}$ of HoTT in $\infty\mathcal{G}\mathrm{Gpd}$:

- $\vdash \Gamma\ \mathsf{ctx} \mapsto [\![\Gamma]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_0$;
- $\Gamma \vdash A\ \mathsf{ty} \mapsto [\![A]\!]_{\mathcal{G}} \in \infty\mathrm{Gpd}_1(|[\![\Gamma]\!]_{\mathcal{G}}|, \infty\mathcal{G}\mathrm{Gpd})$, where we later extend the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an *∞-groupoid*;
- $\Gamma \vdash a : A \mapsto [\![a]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_1([\![\Gamma]\!]_{\mathcal{G}}, [\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}})$ that is a section of the first projection $[\![\Gamma]\!]_{\mathcal{G}}.[\![A]\!]_{\mathcal{G}} \to [\![\Gamma]\!]_{\mathcal{G}}$ (of Grothendieck const.);
- $\Gamma \vdash p : a_1 =_A a_2 \mapsto [\![p]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_2([\![a_1]\!]_{\mathcal{G}}, [\![a_2]\!]_{\mathcal{G}})$;
- $\Gamma \vdash q : p_1 =_{a_1 =_A a_2} p_2 \mapsto [\![q]\!]_{\mathcal{G}} \in \infty\mathcal{G}\mathrm{Gpd}_3([\![p_1]\!]_{\mathcal{G}}, [\![p_2]\!]_{\mathcal{G}})$, and so on.

We interpret One-, Zero- and N-types by discrete game-semantic ∞-groupoids, and Id-type by $\mathrm{Id}_A(\gamma, \alpha_1, \alpha_2) := A(\gamma)(\alpha_1, \alpha_2) \hookrightarrow A(\gamma)$. In the rest of the talk, I focus on Pi-type and univalent universes.

Game semantics of Pi-type (part 1/2)

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;
- The composition $*_p : \Pi(\Gamma, A)_n \times_{\Pi(\Gamma, A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ internalises the following algorithm in $\mathcal{G}$:

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;
- The composition $*_p : \Pi(\Gamma, A)_n \times_{\Pi(\Gamma,A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ internalises the following algorithm in $\mathcal{G}$:

$$\left(\Gamma_0 \xrightarrow{\sigma,\tau} (\Gamma.A)_n\right) \mapsto \left(\Gamma_0 \xrightarrow{\Delta} \Gamma_0^2 \xrightarrow{\sigma\times\tau} (\Gamma.A)_n^2 \xrightarrow{*_p} (\Gamma.A)_n\right);$$

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;
- The composition $*_p : \Pi(\Gamma, A)_n \times_{\Pi(\Gamma,A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ internalises the following algorithm in $\mathcal{G}$:

$$\left(\Gamma_0 \xrightarrow{\sigma,\tau} (\Gamma.A)_n\right) \mapsto \left(\Gamma_0 \xrightarrow{\Delta} \Gamma_0^2 \xrightarrow{\sigma\times\tau} (\Gamma.A)_n^2 \xrightarrow{*_p} (\Gamma.A)_n\right);$$

- The identity $i_n : \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_{n+1}$ internalises the following algorithm in $\mathcal{G}$:

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;
- The composition $*_p : \Pi(\Gamma, A)_n \times_{\Pi(\Gamma,A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ internalises the following algorithm in $\mathcal{G}$:

$$\Big(\Gamma_0 \xrightarrow{\sigma,\tau} (\Gamma.A)_n\Big) \mapsto \Big(\Gamma_0 \xrightarrow{\Delta} \Gamma_0^2 \xrightarrow{\sigma\times\tau} (\Gamma.A)_n^2 \xrightarrow{*_p} (\Gamma.A)_n\Big);$$

- The identity $i_n : \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_{n+1}$ internalises the following algorithm in $\mathcal{G}$:

$$\Big(\Gamma_0 \xrightarrow{\sigma} (\Gamma.A)_n\Big) \mapsto \Big(\Gamma_0 \xrightarrow{\sigma} (\Gamma.A)_n \xrightarrow{i_n} (\Gamma.A)_{n+1}\Big),$$

Game semantics of Pi-type (part 1/2)

In the following, I omit the semantic bracket $[\![\_]\!]_{\mathcal{G}}$.
I interpret Pi-type by $\Pi(\Gamma, A) \in \infty\mathcal{G}\mathrm{Gpd}_0$:

- 0-cells are sections $\Gamma \to \Gamma.A$ of the projection $\Gamma.A \to \Gamma$;
- n-cells $\sigma \to \tau$ $(n > 0)$ are game-semantic n-transformations $\sigma \to \tau$;
- The composition $*_p : \Pi(\Gamma, A)_n \times_{\Pi(\Gamma, A)_p} \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_n$ internalises the following algorithm in $\mathcal{G}$:

$$\left(\Gamma_0 \xrightarrow{\sigma, \tau} (\Gamma.A)_n\right) \mapsto \left(\Gamma_0 \xrightarrow{\Delta} \Gamma_0^2 \xrightarrow{\sigma \times \tau} (\Gamma.A)_n^2 \xrightarrow{*_p} (\Gamma.A)_n\right);$$

- The identity $i_n : \Pi(\Gamma, A)_n \to \Pi(\Gamma, A)_{n+1}$ internalises the following algorithm in $\mathcal{G}$:

$$\left(\Gamma_0 \xrightarrow{\sigma} (\Gamma.A)_n\right) \mapsto \left(\Gamma_0 \xrightarrow{\sigma} (\Gamma.A)_n \xrightarrow{i_n} (\Gamma.A)_{n+1}\right),$$

  and similarly for the data of inverses: inv_n, ret_n, sec_n and tri_n.

Game semantics of Pi-type (part 2/2)

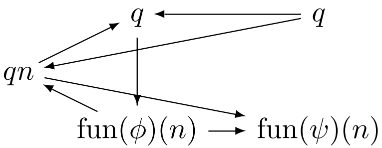
Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

$$\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) \quad = $$

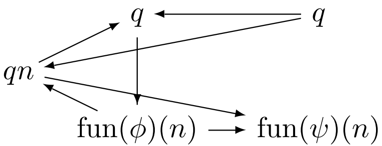


where $\mathrm{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\mathrm{fun}(\psi)$.

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

$$\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) \quad = $$



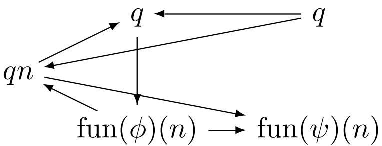
where $\mathrm{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\mathrm{fun}(\psi)$.

- Validates *strict* funext: $\mathrm{Id}_{\Pi(\Gamma,A)}(\phi, \psi) = \Pi_{x:\Gamma}\mathrm{Id}_{A(x)}(\phi(x), \psi(x))$;

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

$$\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) \quad = $$

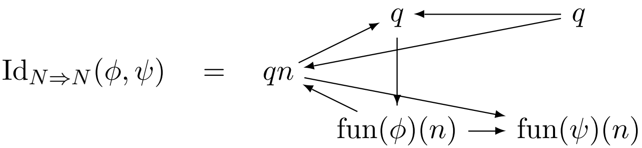


where $\mathrm{fun}(\phi) := \{\, (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \,\}$, and similarly for $\mathrm{fun}(\psi)$.

- Validates *strict* funext: $\mathrm{Id}_{\Pi(\Gamma,A)}(\phi, \psi) = \Pi_{x:\Gamma}\mathrm{Id}_{A(x)}(\phi(x), \psi(x))$;
- *Nontrivial* computations that verify $\mathrm{fun}(\phi) = \mathrm{fun}(\psi)$;

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.



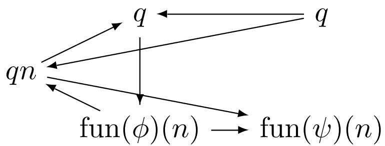
where $\text{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\text{fun}(\psi)$.

- Validates *strict* funext: $\text{Id}_{\Pi(\Gamma,A)}(\phi, \psi) = \Pi_{x:\Gamma}\text{Id}_{A(x)}(\phi(x), \psi(x))$;
- *Nontrivial* computations that verify $\text{fun}(\phi) = \text{fun}(\psi)$;
- The *oracle* O is crucial to handle the *infinitary* domain N.

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

$\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) \quad = \quad$



where $\mathrm{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\mathrm{fun}(\psi)$.

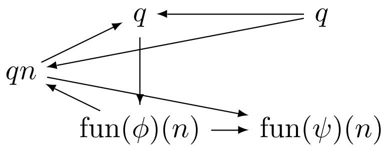
- Validates *strict* funext: $\mathrm{Id}_{\Pi(\Gamma, A)}(\phi, \psi) = \Pi_{x:\Gamma} \mathrm{Id}_{A(x)}(\phi(x), \psi(x))$;
- *Nontrivial* computations that verify $\mathrm{fun}(\phi) = \mathrm{fun}(\psi)$;
- The *oracle* O is crucial to handle the *infinitary* domain N.

Some higher cells in $\Pi(\Gamma, A)$ are only weakly invertible, and so we must weaken the strict invertibility anyway (not only for univalence).

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.

$\mathrm{Id}_{N \Rightarrow N}(\phi, \psi) \quad = $



where $\mathrm{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\mathrm{fun}(\psi)$.

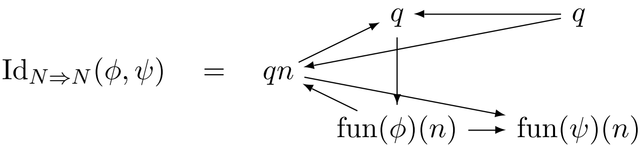
- Validates *strict* funext: $\mathrm{Id}_{\Pi(\Gamma,A)}(\phi, \psi) = \Pi_{x:\Gamma}\mathrm{Id}_{A(x)}(\phi(x), \psi(x))$;
- *Nontrivial* computations that verify $\mathrm{fun}(\phi) = \mathrm{fun}(\psi)$;
- The *oracle* O is crucial to handle the *infinitary* domain N.

Some higher cells in $\Pi(\Gamma, A)$ are only weakly invertible, and so we must weaken the strict invertibility anyway (not only for univalence).

- Identities in $\Pi(\Gamma, A)$ never visit the domain N;

Game semantics of Pi-type (part 2/2)

Game-semantic trans. make sense as *identity proofs in Pi-type*.



where $\mathrm{fun}(\phi) := \{ (n, m) \in \mathbb{N}^2 \mid \phi \circ \underline{n} = \underline{m} \}$, and similarly for $\mathrm{fun}(\psi)$.

- Validates *strict* funext: $\mathrm{Id}_{\Pi(\Gamma,A)}(\phi, \psi) = \Pi_{x:\Gamma} \mathrm{Id}_{A(x)}(\phi(x), \psi(x))$;
- *Nontrivial* computations that verify $\mathrm{fun}(\phi) = \mathrm{fun}(\psi)$;
- The *oracle* O is crucial to handle the *infinitary* domain N.

Some higher cells in $\Pi(\Gamma, A)$ are only weakly invertible, and so we must weaken the strict invertibility anyway (not only for univalence).

- Identities in $\Pi(\Gamma, A)$ never visit the domain N;
- Compositions in $\Pi(\Gamma, A)$ cannot undo a visit to the domain N.

Game semantics of univalent universes (part 1/3)

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid, in which higher cells are *type equivalence*;

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid, in which higher cells are *type equivalence*;
2. Internalise $\mathcal{U} \hookrightarrow \infty\mathcal{G}\mathrm{Gpd}$ in $\mathcal{G}$, where plays on $\mathcal{U}_0$ *encode* freely generated objects in $\infty\mathcal{G}\mathrm{Gpd}_0$, and $\mathcal{U}_n$ $(n > 0)$ restricts $\infty\mathcal{G}\mathrm{Gpd}_n$.

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid, in which higher cells are *type equivalence*;
2. Internalise $\mathcal{U} \hookrightarrow \infty\mathcal{G}\mathrm{Gpd}$ in $\mathcal{G}$, where plays on $\mathcal{U}_0$ *encode* freely generated objects in $\infty\mathcal{G}\mathrm{Gpd}_0$, and $\mathcal{U}_n$ $(n > 0)$ restricts $\infty\mathcal{G}\mathrm{Gpd}_n$.

Lemma (∞-groupoid of game-semantic ∞-groupoids)

The following extends the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid.

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid, in which higher cells are *type equivalence*;
2. Internalise $\mathcal{U} \hookrightarrow \infty\mathcal{G}\mathrm{Gpd}$ in $\mathcal{G}$, where plays on $\mathcal{U}_0$ *encode* freely generated objects in $\infty\mathcal{G}\mathrm{Gpd}_0$, and $\mathcal{U}_n$ $(n > 0)$ restricts $\infty\mathcal{G}\mathrm{Gpd}_n$.

Lemma (∞-groupoid of game-semantic ∞-groupoids)

The following extends the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid.

- 0-cells are game-semantic ∞-groupoids;

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}$Gpd to an ∞-groupoid, in which higher cells are *type equivalence*;
2. Internalise $\mathcal{U} \hookrightarrow \infty\mathcal{G}\mathrm{Gpd}$ in $\mathcal{G}$, where plays on $\mathcal{U}_0$ *encode* freely generated objects in $\infty\mathcal{G}\mathrm{Gpd}_0$, and $\mathcal{U}_n$ $(n > 0)$ restricts $\infty\mathcal{G}\mathrm{Gpd}_n$.

Lemma (∞-groupoid of game-semantic ∞-groupoids)

The following extends the ∞-category $\infty\mathcal{G}$Gpd *to an ∞-groupoid.*

- 0-cells are game-semantic ∞-groupoids;
- A 1-cell $(\phi, \psi, \sigma, \tau, \mu) : G \to H$ consists of game-semantic ∞-functors $\phi : G \to H$ and $\psi : H \to G$, trans. $\sigma : \psi * \phi \to i(G)$ and $\tau : \phi * \psi \to i(H)$, and 2-trans. $\mu : i(\phi) * \sigma \to \tau * i(\psi)$;

Game semantics of univalent universes (part 1/3)

1. Extend the (set-theoretic) ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid, in which higher cells are *type equivalence*;
2. Internalise $\mathcal{U} \hookrightarrow \infty\mathcal{G}\mathrm{Gpd}$ in $\mathcal{G}$, where plays on $\mathcal{U}_0$ *encode* freely generated objects in $\infty\mathcal{G}\mathrm{Gpd}_0$, and $\mathcal{U}_n$ $(n > 0)$ restricts $\infty\mathcal{G}\mathrm{Gpd}_n$.

Lemma (∞-groupoid of game-semantic ∞-groupoids)

The following extends the ∞-category $\infty\mathcal{G}\mathrm{Gpd}$ to an ∞-groupoid.

- 0-cells are game-semantic ∞-groupoids;
- A 1-cell $(\phi, \psi, \sigma, \tau, \mu) : G \to H$ consists of game-semantic ∞-functors $\phi : G \to H$ and $\psi : H \to G$, trans. $\sigma : \psi * \phi \to i(G)$ and $\tau : \phi * \psi \to i(H)$, and 2-trans. $\mu : i(\phi) * \sigma \to \tau * i(\psi)$;
- $(n > 1)$ An n-cell $(\alpha, \beta, \eta, \epsilon, \delta) : (\phi, \psi, \sigma, \tau, \mu) \to (\phi', \psi', \sigma', \tau', \mu')$ consists of game-semantic $(n-1)$-trans. $\alpha : \phi \to \phi'$ and $\beta : \phi' \to \phi$, n-trans. $\eta : \beta * \alpha \to i(\phi)$ and $\epsilon : \alpha * \beta \to i(\phi')$, and an $(n+1)$-trans. $\delta : i(\alpha) * \eta \to \epsilon * i(\alpha)$;

Game semantics of univalent universes (part 2/3)

Game semantics of univalent universes (part 2/3)

- The inverse of each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quadruple

$$\mathrm{inv}(\alpha, \beta, \eta, \epsilon, \delta) := (\beta, \alpha, \epsilon, \eta, \mathrm{dual}(\delta)),$$

where $\mathrm{dual}(\mu) : i(\beta) * \epsilon \to \eta * i(\beta)$ is too technical to present here;

Game semantics of univalent universes (part 2/3)

- The inverse of each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quadruple

$$\mathrm{inv}(\alpha, \beta, \eta, \epsilon, \delta) := (\beta, \alpha, \epsilon, \eta, \mathrm{dual}(\delta)),$$

  where $\mathrm{dual}(\mu) : i(\beta) * \epsilon \to \eta * i(\beta)$ is too technical to present here;
- The identity on each 0-cell G is the quintuple

$$i(G) := (i(G), i(G), i^2(G), i^2(G), i^3(G));$$

Game semantics of univalent universes (part 2/3)

- The inverse of each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quadruple

$$\mathrm{inv}(\alpha, \beta, \eta, \epsilon, \delta) := (\beta, \alpha, \epsilon, \eta, \mathrm{dual}(\delta)),$$

  where $\mathrm{dual}(\mu) : i(\beta) * \epsilon \to \eta * i(\beta)$ is too technical to present here;
- The identity on each 0-cell G is the quintuple

$$i(G) := (i(G), i(G), i^2(G), i^2(G), i^3(G));$$

- $(n > 0)$ The identity on each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quintuple

$$i(\alpha, \beta, \eta, \epsilon, \delta) := (i(\alpha), i(\alpha), i^2(\alpha), i^2(\alpha), i^3(\alpha));$$

Game semantics of univalent universes (part 2/3)

- The inverse of each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quadruple

$$\mathrm{inv}(\alpha, \beta, \eta, \epsilon, \delta) := (\beta, \alpha, \epsilon, \eta, \mathrm{dual}(\delta)),$$

where $\mathrm{dual}(\mu) : i(\beta) * \epsilon \to \eta * i(\beta)$ is too technical to present here;

- The identity on each 0-cell G is the quintuple

$$i(G) := (i(G), i(G), i^2(G), i^2(G), i^3(G));$$

- $(n > 0)$ The identity on each n-cell $(\alpha, \beta, \eta, \epsilon, \delta)$ is the quintuple

$$i(\alpha, \beta, \eta, \epsilon, \delta) := (i(\alpha), i(\alpha), i^2(\alpha), i^2(\alpha), i^3(\alpha));$$

- $\mathrm{ret}(\alpha, \beta, \eta, \epsilon, \delta) := (\eta, \mathrm{inv} \circ \eta, \mathrm{ret} \circ \eta, \mathrm{sec} \circ \eta, \mathrm{tri} \circ \eta)$, and similarly for $\mathrm{sec}(\alpha, \beta, \eta, \epsilon, \delta)$ and $\mathrm{tri}(\alpha, \beta, \eta, \epsilon, \delta)$;

Game semantics of univalent universes (part 3/3)

Game semantics of univalent universes (part 3/3)

- The $*_p$-composition of composable n-cells $(\alpha, \beta, \eta, \epsilon, \delta)$ and $(\alpha', \beta', \eta', \epsilon', \delta')$ is the quintuple

$$(\alpha', \beta', \eta', \epsilon', \delta') *_p (\alpha, \beta, \eta, \epsilon, \delta)$$
$$:= \begin{cases} (\alpha' *_n \alpha, \beta *_n \beta', \eta *_{n+1} (i(\beta) *_n \eta' *_n i(\alpha)), \dots) & \text{if } p = n-1; \\ (\alpha' *_p \alpha, \beta' *_p \beta, \eta' *_p \eta, \epsilon' *_p \epsilon, \delta' *_p \delta) & \text{if } p < n-1, \end{cases}$$

Game semantics of univalent universes (part 3/3)

- The $*_p$-composition of composable n-cells $(\alpha, \beta, \eta, \epsilon, \delta)$ and $(\alpha', \beta', \eta', \epsilon', \delta')$ is the quintuple

$$
\begin{aligned}
&(\alpha', \beta', \eta', \epsilon', \delta') *_p (\alpha, \beta, \eta, \epsilon, \delta) \\
:= &\begin{cases} (\alpha' *_n \alpha, \beta *_n \beta', \eta *_{n+1} (i(\beta) *_n \eta' *_n i(\alpha)), \dots) & \text{if } p = n-1; \\ (\alpha' *_p \alpha, \beta' *_p \beta, \eta' *_p \eta, \epsilon' *_p \epsilon, \delta' *_p \delta) & \text{if } p < n-1, \end{cases}
\end{aligned}
$$

For $\big(\mu : \Gamma_0 \to \mathcal{U}_0\big) \Leftrightarrow \big(\mathrm{fun}(\mu) \in \mathrm{Ty}_{\mathcal{G}}(\Gamma)_0\big)$, define the game $\mathcal{U}_0$ by

Game semantics of univalent universes (part 3/3)

- The $*_p$-composition of composable n-cells $(\alpha, \beta, \eta, \epsilon, \delta)$ and $(\alpha', \beta', \eta', \epsilon', \delta')$ is the quintuple

$$
\begin{aligned}
&(\alpha', \beta', \eta', \epsilon', \delta') *_p (\alpha, \beta, \eta, \epsilon, \delta) \\
:=& \begin{cases} (\alpha' *_n \alpha, \beta *_n \beta', \eta *_{n+1} (i(\beta) *_n \eta' *_n i(\alpha)), \dots) & \text{if } p = n-1; \\ (\alpha' *_p \alpha, \beta' *_p \beta, \eta' *_p \eta, \epsilon' *_p \epsilon, \delta' *_p \delta) & \text{if } p < n-1, \end{cases}
\end{aligned}
$$

For $\big(\mu : \Gamma_0 \to \mathcal{U}_0\big) \Leftrightarrow \big(\mathrm{fun}(\mu) \in \mathrm{Ty}_{\mathcal{G}}(\Gamma)_0\big)$, define the game $\mathcal{U}_0$ by

$\underline{N}$

q

$\downarrow$

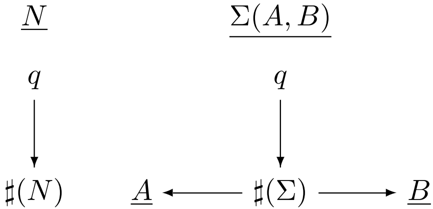
$\sharp(N)$

Game semantics of univalent universes (part 3/3)

- The $*_p$-composition of composable n-cells $(\alpha, \beta, \eta, \epsilon, \delta)$ and $(\alpha', \beta', \eta', \epsilon', \delta')$ is the quintuple

$$
\begin{aligned}
&(\alpha', \beta', \eta', \epsilon', \delta') *_p (\alpha, \beta, \eta, \epsilon, \delta) \\
:= &\begin{cases} (\alpha' *_n \alpha, \beta *_n \beta', \eta *_{n+1} (i(\beta) *_n \eta' *_n i(\alpha)), \dots) & \text{if } p = n-1; \\ (\alpha' *_p \alpha, \beta' *_p \beta, \eta' *_p \eta, \epsilon' *_p \epsilon, \delta' *_p \delta) & \text{if } p < n-1, \end{cases}
\end{aligned}
$$

For $\big(\mu : \Gamma_0 \to \mathcal{U}_0\big) \Leftrightarrow \big(\mathrm{fun}(\mu) \in \mathrm{Ty}_{\mathcal{G}}(\Gamma)_0\big)$, define the game $\mathcal{U}_0$ by



Game semantics of univalent universes (part 3/3)

- The $*_p$-composition of composable n-cells $(\alpha, \beta, \eta, \epsilon, \delta)$ and $(\alpha', \beta', \eta', \epsilon', \delta')$ is the quintuple

$$
\begin{aligned}
&(\alpha', \beta', \eta', \epsilon', \delta') *_p (\alpha, \beta, \eta, \epsilon, \delta) \\
:= &\begin{cases} (\alpha' *_n \alpha, \beta *_n \beta', \eta *_{n+1} (i(\beta) *_n \eta' *_n i(\alpha)), \dots) & \text{if } p = n-1; \\ (\alpha' *_p \alpha, \beta' *_p \beta, \eta' *_p \eta, \epsilon' *_p \epsilon, \delta' *_p \delta) & \text{if } p < n-1, \end{cases}
\end{aligned}
$$

For $\big(\mu : \Gamma_0 \to \mathcal{U}_0\big) \Leftrightarrow \big(\mathrm{fun}(\mu) \in \mathrm{Ty}_{\mathcal{G}}(\Gamma)_0\big)$, define the game $\mathcal{U}_0$ by