Univalent Foundations of Constructive Algebraic Geometry

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HoTTEST, 24.10.2024

Recent preprint:

Univalent foundations of constructive algebraic geometry

`https://arxiv.org/abs/2407.17362`

& partial formalizations of this project:

The functor of points approach to schemes in Cubical Agda

j.w.w. Matthias Hutzler

`https://doi.org/10.4230/LIPIcs.ITP.2024.38`

A univalent formalization of constructive affine schemes

j.w.w. Anders Mörtberg

`https://doi.org/10.4230/LIPIcs.TYPES.2022.14`

see my last HoTTEST: `slides`, `video`

Two notions of schemes

“Geometrical”

- as locally ringed spaces
- standard in literature and courses
- very non-constructive

“Functorial”

- as functors from rings to sets
- preferred by Grothendieck but not as widely known
- size issues

Quick recap: schemes as locally ringed spaces

A locally ringed space consists of

- a topological space X
- a sheaf of rings (comm. w/ 1) $\mathcal{O}_X$ on X
- s.t. for all $x \in X$ the stalk $\mathcal{O}_{X,x} = \mathrm{colim}_{x \in U}\, \mathcal{O}_X(U)$ is a local ring

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For ring A, the *affine scheme* $\mathrm{Spec}(A)$ is the locally ringed space

- $\{\mathfrak{p} \subseteq A \text{ prime}\}$ with Zariski topology gen. by basic opens $D(f) = \{\mathfrak{p} | f \notin \mathfrak{p}\}$ where $f \in A$
- structure sheaf given by $\mathcal{O}_A\big(D(f)\big) = A[1/f]$
- locality on stalks: $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$

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A scheme is a locally ringed space with an open affine cover, i.e. $X = \bigcup_i \operatorname{Spec}(A_i)$.

Algebraic spaces and functors

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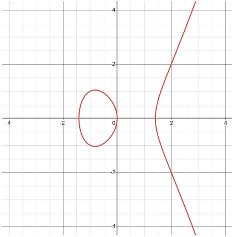
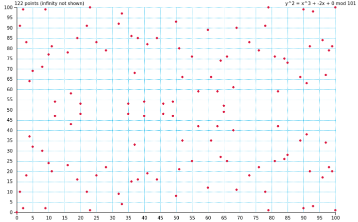
solutions in $\mathbb{Z}$

$$\{(0,0), (-1, \pm 1), (2, \pm 2), (338, \pm 6214)\}$$

solutions in $\mathbb{R}$



solutions in $\mathbb{Z}/101\mathbb{Z}$



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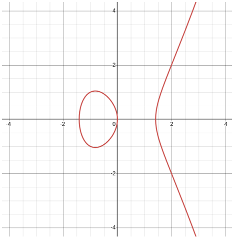

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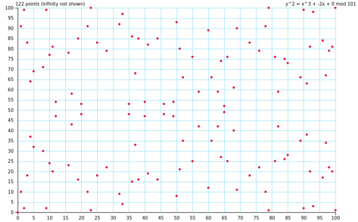

$$V_{x^3-2x-y^2} : \text{Functor}\big(\texttt{Ring},\ \texttt{Set}\big)$$

$$V_{x^3-2x-y^2}(A) := \{(a_1, a_2) \in A^2 \mid a_2^2 = a_1^3 - 2a_1\}$$

$$V_{x^3-2x-y^2}(\varphi)(a_1, a_2) := (\varphi(a_1), \varphi(a_2))$$

Algebraic spaces and functors (cont.)

$$A^2 \cong \mathrm{Hom}\left(\mathbb{Z}[x, y] \ , \ A \right)$$

$$V_{x^3-2x-y^2}(A) \cong \mathrm{Hom}\left({}^{\mathbb{Z}[x,y]}/_{\langle x^3-2x-y^2 \rangle} \ , \ A \right)$$

Algebraic spaces and functors (cont.)

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$\Rightarrow$ $V_{x^3-2x-y^2}$ *naturally* isomorphic to

$$\mathrm{Sp}\big(\mathbb{Z}[x,y]/\langle x^3-2x-y^2\rangle\big) := \mathrm{Hom}\big(\, \mathbb{Z}[x,y]/\langle x^3-2x-y^2\rangle\,,\; \_\,\big)$$

the *representable* under the *Yoneda* embedding

$$\mathrm{Sp} : \mathtt{Ring}^{op} \to \underbrace{\big(\mathtt{Ring} \to \mathtt{Set}\big)}_{\mathbb{Z}\text{-}\mathtt{Functor}}$$

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Definition

An *affine scheme* is a representable $\mathbb{Z}$-functor

Constructive spectrum of a ring

distributive *Zariski lattice* $\mathcal{L}_A$
(classically lattice of *compact opens* of $\mathsf{Spec}(A)$)

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generated by formal elements $D(f)$ modulo *support* relations

- $D(1) = \top \;\; \& \;\; D(0) = \bot$
- $\forall_{f,g\,:A} \;\; D(fg) = D(f) \wedge D(g)$
- $\forall_{f,g\,:A} \;\; D(f+g) \leq D(f) \vee D(g)$

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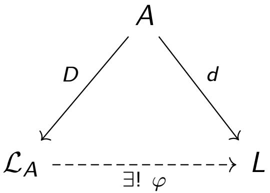
universal property in terms of supports



Constructive spectrum of a ring cont.

Alternative description: f.g. ideals modulo

$$\langle f_1, \dots, f_n \rangle \sim \langle g_1, \dots, g_m \rangle$$
$$:\Leftrightarrow \sqrt{\langle f_1, \dots, f_n \rangle} = \sqrt{\langle g_1, \dots, g_m \rangle}$$

where $\sqrt{I} = \{ f \in A \mid \exists n > 0 : f^n \in I \}$.

[1] Cor. 4.28 in https://doi.org/10.46298/lmcs-19(2:8)2023

Constructive spectrum of a ring cont.

Alternative description: f.g. ideals modulo

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where $\sqrt{I} = \{ f \in A \mid \exists n > 0 : f^n \in I \}$.

This is the right predicative notion!

- classically: $\{\text{opens of } \mathrm{Spec}(A)\} \cong \{\text{radical ideals } \sqrt{I} \subseteq A\}$
- without prop. resizing, type/frame of radicals lives in the universe above A
- result by de Jong & Escardó:[1]
  existence of a non-trivial A with small type of radicals should imply some form of prop. resizing

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Towards compact open subfunctors

$$
\begin{array}{ccc}
\mathrm{Sp}(A)_{\langle f_1,\dots,f_n\rangle} & := & \{\varphi : \mathrm{Hom}(A, \_) \mid 1 \in \langle \varphi(f_1), \dots, \varphi(f_n)\rangle\} \\
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$$

$\forall x : X(A)\ \exists f_1,\dots,f_n : A$ s.t.

$$
\begin{array}{ccc}
\mathrm{Sp}(A)_{\langle f_1,\dots,f_n\rangle} & \longrightarrow & U \\
\big\downarrow & \lrcorner & \big\downarrow \\
\mathrm{Sp}(A) & \xrightarrow[\phi_x]{} & X
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$$

Compact opens of a $\mathbb{Z}$-functor

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$$\begin{array}{ccc} [\![\, U \,]\!](A) & := & \{x : X(A) \mid U(x) = D(1)\} \\ \big\downarrow & & \\ X(A) & & \end{array}$$

Compact opens of a $\mathbb{Z}$-functor (cont.)

Let $x : X(A)$ s.t. $U(x) = D(f_1, \ldots, f_n)$

$$
\begin{array}{ccccc}
\mathrm{Sp}(A)_{\langle f_1,\ldots,f_n\rangle} & \longrightarrow & [\![\, U \,]\!] & \longrightarrow & \mathbf{1} \\
\big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle D(1)} \\
\mathrm{Sp}(A) & \xrightarrow{\ \phi_x\ } & X & \xrightarrow{\ U\ } & \mathcal{L}
\end{array}
$$

qcqs-schemes constructively

Definition

A qcqs-scheme is a *local* $X : \mathbb{Z}\texttt{-Functor}$ with a compact open affine cover: $\exists U_1, \ldots, U_n : X \Rightarrow \mathcal{L}$ s.t.

- $\exists A_1, \ldots, A_n$ s.t. $[\![U_i]\!] \cong \mathrm{Sp}(A_i)$
- $\forall x \in X(A)$ we have $\bigvee_{i=1}^{n} U_i(x) = D(1)$

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⚠ local = sheaf wrt. the Zariski coverage (not today)

⚠ size issues: Fix universe $\mathcal{U}$,

$$\mathbb{Z}\text{-}\mathtt{Functor}_{\mathcal{U}} := \mathtt{Functor}\big(\mathtt{Ring}_{\mathcal{U}},\ \mathtt{Set}_{\mathcal{U}}\big)$$

$$\underbrace{(X \Rightarrow \mathcal{L})}_{(A:\mathrm{Ring}_{\mathcal{U}}) \to X(A) \to \mathcal{L}_A \quad \text{natural}} \quad : \quad \mathcal{U}^{+}$$

Theorem (Stone's representation theorem for distributive lattices)

There is a contravariant equivalence of categories

$$\begin{array}{rcll} \texttt{CohSp} & \xrightarrow{\simeq} & \texttt{DistLattice} & \\ X & \longmapsto & \mathbf{K}^o(X) & \textit{(lattice of compact opens)} \end{array}$$

Where CohSp *is the category of*

- *coherent spaces (=quasi-compact, quasi-separated & sober)*
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- $\mathrm{Spec}(A)$ coherent and $\mathbf{K}^o(\mathrm{Spec}(A)) \cong \mathcal{L}_A$
- Can (with *univalence*!) equip Zariski lattice with structure sheaf

$$\begin{array}{rcl} \mathcal{O}_A : \mathcal{L}_A^{op} & \to & \texttt{Ring} \\ D(f) & \mapsto & A[1/f] \end{array}$$

- More morphisms of ringed lattices $(\mathcal{L}_A, \mathcal{O}_A) \to (\mathcal{L}_B, \mathcal{O}_B)$ than ring morphisms $A \to B$. *Need morphisms of locally ringed lattices*

A classical lemma

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subseteq X$ be open and $s \in \mathcal{O}_X(U)$, then

$$\mathcal{D}_U(s) = \{x \in U \mid s_x \in \mathcal{O}_{X,x}^\times\}$$

is the biggest open of U where s becomes invertible.

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TFAE:

- $(X, \mathcal{O}_X)$ is a locally ringed space
- For all U, $\mathcal{D}_U$ is a support, i.e.
  - $\mathcal{D}_U(1) = U$ and $\mathcal{D}_U(0) = \emptyset$
  - $\mathcal{D}_U(st) = \mathcal{D}_U(s) \cap \mathcal{D}_U(t)$
  - $\mathcal{D}_U(s + t) \subseteq \mathcal{D}_U(s) \cup \mathcal{D}_U(t)$

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A morphism of ringed spaces $(f, f^\sharp) : X \to Y$ is a morphism of locally ringed spaces iff $f^{-1}\big(\mathcal{D}_U(s)\big) = \mathcal{D}_{f^{-1}(U)}(f^\sharp(s))$

Locally ringed lattices

Let $(L, \mathcal{O}_L)$ be a ringed lattice with a (dependent) function

$$\mathcal{D} : (u : L) \to \mathcal{O}_L(u) \to \downarrow u$$

such that

- for $u : L$ and $s : \mathcal{O}_L(u)$, $\mathcal{D}_u(s)$ is the largest element $\leq u$, where s becomes invertible.
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Then the triple $(L, \mathcal{O}_L, \mathcal{D})$ is called a locally ringed lattice.

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Then the triple $(L, \mathcal{O}_L, \mathcal{D})$ is called a locally ringed lattice.

A morphism of locally ringed lattices $X \to Y$ consists of

- a lattice hom $\pi : L_X \to L_Y$
- a nat. trans. $\pi^\sharp : \mathcal{O}_X \Rightarrow \pi_* \mathcal{O}_Y$
- such that $\pi\big(\mathcal{D}_u(s)\big) = \mathcal{D}_{\pi(u)}(\pi^\sharp(s))$

qcqs-schemes as locally ringed lattices

- $(\mathcal{L}_A, \mathcal{O}_A)$ can be given loc. ringed lattice structure induced by support D. This is the constructive Spec(A)
- $\texttt{LRDL}\big(\mathrm{Spec}(A), \mathrm{Spec}(B)\big) \cong \mathrm{Hom}(A, B)$ (unit of an adjunction)
- Classically: the category of qcqs-schemes embeds into $\texttt{LRDL}^{op}$ (fact: any morphism between two qcqs-schemes is qc/coherent)

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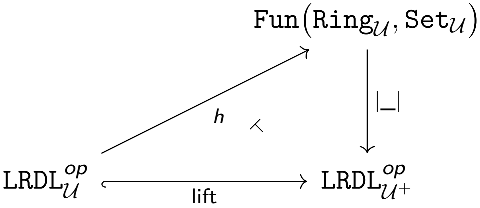
Definition

A qcqs-scheme is a locally ringed lattice $X = (L_X, \mathcal{O}_X, \mathcal{D})$ such that there merely $\exists u_1, \dots, u_n : L_X$

- that cover X: $u_1 \vee \dots \vee u_n = \top$.
- each u_i is affine: $(\downarrow u_i, \mathcal{O}_X\restriction_{\downarrow u_i}) \cong \mathrm{Spec}\big(\mathcal{O}_X(u_i)\big)$
(as locally ringed lattices)

Comparison


$\mathtt{Fun}(\mathtt{Ring}_{\mathcal{U}}, \mathtt{Set}_{\mathcal{U}})$
h
$|\_|$
$\mathtt{LRDL}_{\mathcal{U}}^{op}$
lift
$\mathtt{LRDL}_{\mathcal{U}^+}^{op}$

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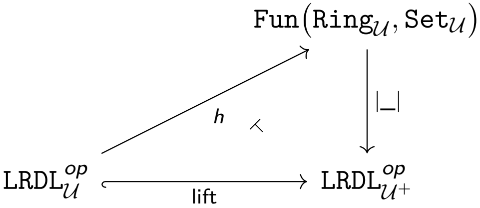

- Functor of points: $h_X = \mathtt{LRDL}^{op}(\mathrm{Spec}(\_), X)$

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$$
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 & \overset{h}{\nearrow} \quad \swarrow & \Big\downarrow |\_| \\
\mathtt{LRDL}^{op}_{\mathcal{U}} & \underset{\text{lift}}{\hookrightarrow} & \mathtt{LRDL}^{op}_{\mathcal{U}^+}
\end{array}
$$

- Functor of points: $h_X = \mathtt{LRDL}^{op}\big(\mathrm{Spec}(\_), X\big)$
- Realization of a $\mathbb{Z}$-functor: $|X|$ is loc. ringed lattice
  - lattice $X \Rightarrow \mathcal{L}$
  - sheaf $\mathcal{O}_{|X|}(U) = [\![\, U]\!] \Rightarrow \mathbb{A}^1$ (nat. trans. to forgetful functor)
  - canonical support

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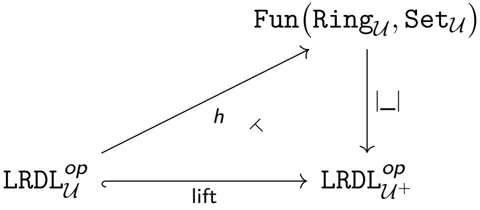

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  - canonical support
- Relative adjunction

$$X \Rightarrow h_Y \; \cong \; \mathtt{LRDL}^{op}_{\mathcal{U}^+}(|X|, \mathsf{lift}(Y))$$

Theorem (Comparison thm. in MLTT (2 universes) + ua + set quot.)

1. *If X is a qcqs-scheme ($\mathtt{LRDL}$), then h_X is a functorial qcqs-scheme*
2. *If X is a functorial qcqs-scheme, then $|X|$ is a big qcqs-scheme & there (merely) exists a small qcqs-scheme Y s.t. $|X| \cong \mathrm{lift}(Y)$*
3. *This induces an adjoint equivalence of categories between qcqs-schemes as locally ringed lattices and functorial qcqs-schemes.*

Proof of comparison theorem

A qcqs-scheme X is (merely) a finite colimit in $\mathtt{LRDL}^{op}$

$$X \cong \mathrm{colim}\big\{\mathrm{Spec}(A_i) \leftarrow \mathrm{Spec}(A_{ijk}) \rightarrow \mathrm{Spec}(A_j)\big\}$$

A functorial qcqs-scheme Y is a (merely) a finite colimit in *local* $\mathbb{Z}$-functors

$$Y \cong \mathrm{colim}\big\{\mathrm{Sp}(A_i) \leftarrow \mathrm{Sp}(A_{ijk}) \rightarrow \mathrm{Sp}(A_j)\big\}$$

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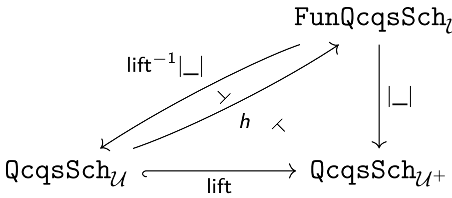
Idea:

- $h_{\operatorname{Spec}(A)} \cong \operatorname{Sp}(A)$
- $|\operatorname{Sp}(A)| \cong \operatorname{lift}\big(\operatorname{Spec}(A)\big)$
- both h and $|\_|$ respect colimits of the above shape (modulo the Sp/Spec correspondence)

Proof of comparison theorem cont.



- For existence of $\text{lift}^{-1}|\_|$ need only

$$\forall_{X:\texttt{FunQcqsSch}_{\mathcal{U}}} \; \exists_{Y:\texttt{QcqsSch}_{\mathcal{U}}} \; |X| \cong \text{lift}(Y)$$

  as

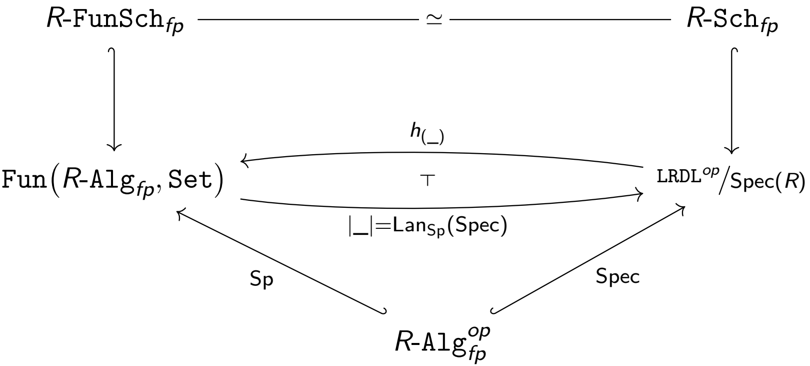
$$\text{isProp} \left(\Sigma_{Y:\texttt{QcqsSch}_{\mathcal{U}}} \; |X| \cong \text{lift}(Y)\right)$$

- For adjunction to be an equivalence it suffices to prove: h and $|\_|$ are both fully faithful

Definition

A is *finitely presented* if merely $A \cong {}^{R[x_1,\ldots,x_n]}/_{\langle p_1,\ldots,p_m\rangle}$ as R-algebras.
small category:

- objects: lists of polynomials $p_1, \ldots, p_m$ with $p_i : R[x_1, \ldots, x_n]$
- arrows: R-algebra morphisms



Formalizing schemes in `Cubical Agda`

- Zariski lattice: `850 loc`
- Structure sheaf on Zariski lattice
  - lift sheaf from basis: `2000 loc`
  - univalent and other auxillary lemmas: `300 loc`
  - putting it all together: `400 loc`
- Functorial qcqs-schemes
  - $\mathbb{Z}$-functors and Zariski sheaves: `450 loc`
  - Compact opens and def. qcqs-scheme: `450 loc`
  - Compact opens of affines are qcqs-schemes: `200 loc`
- A lot of commutative algebra and category theory
  e.g. localizations of rings: `2000 loc`

Thank You