1. Summary of Dinesh's talk

2. Some remarks about enriched categories

Let $(\mathcal{V}, \otimes)$ be a symmetric monoidal category. Let's assume that that there is a unit $\mathbf{1}$. We are interested in two specific examples, simplicial sets and chain complexes.

Example 2.0.1. We will mostly be interested in the cases where $\mathcal{V}$ if the symmetric monoidal category of simplicial sets or chain complexes of abelian groups. Both of these categories have units. In the category are simplicial sets, with the Cartesian symmetric monoidal structure, the unit is Δ^0. In the category of chain complexes of abelian groups, the unit is a complex $\mathbb{Z}$ concentrated in degree zero.

If $\mathcal{C}$ is a category enriched over $\mathcal{V}$, we will write $\mathrm{Map}_{\mathcal{C}}(A, B)$ for the mapping spaces inside of the category. We will write $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for $\mathrm{Hom}_{\mathcal{V}}(\mathbf{1}, \mathrm{Map}_{\mathcal{C}}(A, B))$. This produces an ordinary category. Given $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We obtain

$$\mathrm{Map}(A, X) \otimes \mathbf{1} \to \mathrm{Map}(A, X) \otimes \mathrm{Map}(X, Y) \to \mathrm{Map}(A, Y).$$

Fix objects $V \in \mathcal{V}$ and $C \in \mathcal{C}$. If the functor

$$Y \mapsto \mathrm{Map}_{\mathcal{C}}(C, Y)^V$$

is coreprentable, the corepresenting object will be denoted by $C \otimes V$.

Example 2.0.2. Consider the complex $E(n)$ be the complex :

$$\begin{array}{ccccccc} & & n & & n-1 & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

and suppose that the additive category $\mathcal{A}$ admits direct sums. Given a complex $C_\bullet$ with values in $\mathcal{A}$ we can form the complex $C_\bullet \otimes E(n)$ in the usual way by taking the total complex associated to the double complex

$$\begin{tikzcd} \cdots \arrow[r] & C_k \arrow[r] \arrow[d, equal] & C_{k-1} \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & C_k \arrow[r] & C_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

where the top row is in degree n and the bottom in degree $n-1$.

A category $\mathcal{C}$ enriched over simplicial sets is said to be *locally fibrant* if each of its mapping spaces is a Kan complex.

A diagram in a locally fibrant simplicial category $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is said to be a homotopy pushout diagram if the diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Map}(D,Q) & \longrightarrow & \mathrm{Map}(B,Q) \\ \downarrow & & \downarrow \\ \mathrm{Map}(C,Q) & \longrightarrow & \mathrm{Map}(A,Q) \end{array}$$

is a homotopy pullback diagram of simplicial sets for each $Q \in \mathcal{C}$.

Remark 2.0.3. In our situation we have that $N_{dg}(\mathrm{Ch}(\mathcal{A})) = N(\mathrm{Ch}(\mathcal{A})_\Delta)$. The category $\mathrm{Ch}(\mathcal{A})_\Delta$ is locally fibrant. In this situation, [HTT, 4.2.4.1] tells us that the diagram produces a pushout diagram in $N_{dg}(\mathrm{Ch}(\mathcal{A}))$ upon taking nerves.

3. The derived infinity category

Throughout this section A will be an abelian category with enough projectives. If B is a full subcategory we can form the dg category of complexes in B, it will be denoted by $\mathrm{Ch}(B)$.

We denote by $\mathrm{Ch}(B)^- \subseteq \mathrm{Ch}(B)$ the full dg subcategory spanned by complexes $M_\bullet$ with $M_n = 0$ when $n << 0$.

Definition 3.0.1. Denote by A_{proj} the full subcategory of$A$ consisting of projective objects. We let $\mathcal{D}^-(A)$ denote the infinity category $N_{\mathrm{dg}}(\mathrm{Ch}^-(A_{\mathrm{proj}}))$ and call it the derived infinity category of A.

Remark 3.0.2. The homotopy category of the derived infinity category, $h\mathcal{D}^-(A)$ is the usual bounded below derived category of A. This follows from [HA, example 1.3.18] as one can construct the derived category as $K(\mathrm{Ch}^-((A_{\mathrm{proj}}))$

Remark 3.0.3. The goals of this talk are

(1) to show that the category $N_{dg}(\mathrm{Ch}(A))$ is stable
(2) to show that the derived infinity category, $\mathcal{D}^-(A)$ is stable.

Proposition 3.0.4. *Consider a morphism*

$$f : A_\bullet \to B_\bullet$$

of simplicial abelian groups. Then f is a Kan vibration if and only if

$$N_n(A) \to N_n(B)$$

is surjective for every $n > 0$.

Proof. Let $E(n)$ be the complex

$$\begin{array}{ccccccc} & & n & & n-1 & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

One checks

(1) $\mathrm{Hom}(E(n), M_n) = M_n$
(2) The exists a canonical morphism

$$\theta : E(n) \to N_\bullet(\Delta^n)$$

$(N_n(\Delta^n) = \mathbb{Z})$

(3) if $n > 0$ then for every $0 \leq i \leq n$ we have an induced isomorphism

$$N_\bullet(\Lambda^n_i) \oplus E(n)_\bullet \to N_\bullet(\Delta^n).$$

The result now follows easily from the Dold-Kan correspondence. □

Corollary 3.0.5. *Let $\mathcal{A}$ be an additive category, and suppose that we have a pushout diagram*

$$\begin{tikzcd} M_\bullet \arrow[r, "f"] \arrow[d] & M'_\bullet \arrow[d] \\ N_\bullet \arrow[r] & N'_\bullet \end{tikzcd}$$

in $\mathrm{Ch}(\mathcal{A})$ *. If f is degreewise split, then the corresponding diagram in the simplicial category* $\mathrm{Ch}(\mathcal{A})_\Delta$ *is a homotopy pushout diagram.*

Proof. We must show that for every complex $Q_\bullet \in \mathrm{Ch}(\mathcal{A})$, the diagram of simplicial abelian groups, obtained by mapping into $Q_\bullet$,

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(N'_\bullet, Q_\bullet) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(N_\bullet, Q_\bullet) \arrow[d] \\ \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M'_\bullet, Q_\bullet) \arrow[r, "g"] & \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(M_\bullet, Q_\bullet). \end{tikzcd}$$

is a homotopy Cartesian. Firstly the given diagram is a Cartesian square. To see this, observed that pullbacks are computed degree wise and hence it suffices to show that we obtain pullbacks squares overview in groups upon applying the functor $\mathrm{Hom}(E(n), -)$. The result will then follow from the fact that $E(n)\otimes$ preserves push out diagrams.

To complete the proof, it suffices to show that the bottom map labelled g is a fibration. This follows from the proposition, by a calculation using the degreewise splitness. □

Remark 3.0.6. We can always satisfy the hypothesis of the corollary upto homotopy. Given $f : M_\bullet \to M'_\bullet$. Recall the mapping cylinder, it is the complex in degree n given by

$$M_n \oplus M_{n-1} \oplus M'_n \quad \text{and differential} \quad \begin{bmatrix} d_M & 1_{M_{n-1}} & 0 \\ 0 & -d_M & 0 \\ 0 & -f & d' \end{bmatrix}.$$

One can show that $M'_\bullet \hookrightarrow \mathrm{cyl}(f)$ is has a homotopy inverse given by $\phi(m_1, m_2, m') = f(m_1) + m'$.

Remark 3.0.7. A related construction is the mapping cone. It is the complex that is in degree n given by

$$M_{n-1} \oplus M'_n \quad \text{and differential} \quad \begin{bmatrix} -d_M & 0 \\ -f & d_{M'} \end{bmatrix}$$

We have a pushout diagram :

$$\begin{tikzcd} M_\bullet \arrow[r] \arrow[d] & E(1) \otimes M_\bullet \arrow[d] \\ M'_\bullet \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

So mapping cones are cofibers.

Theorem 3.0.8. *The category $N_{dg}(\mathrm{Ch}(\mathcal{A}))$ is stable.*

Proof. Pushouts can be constructed using the mapping cylinder, previous corollary and [HTT, 4.2.4.1].

There is a zero object.

There is a pushout diagram

$$\begin{tikzcd} M_\bullet \arrow[r] \arrow[d] & E(1) \otimes M_\bullet \arrow[d] \\ 0 \arrow[r] & M_{\bullet-1} \end{tikzcd}$$

Hence the shift is the suspension which is easily seen to be an equivalence. By James' lecture, we are done, this was one of his equivalent characterisations. □

Corollary 3.0.9. *The category $D^-(\mathcal{A})$ is stable.*

References

[HA] Higher algebra, Lurie, Jacob, 2012 pages

[HTT] Higher topos theory, Lurie, Jacob, 2009 pages

[SP] The Stacks Project Authors, *Stacks project*, http://stacks.math.columbia.edu, 2016 pages