Functional Analysis
Assignment 1

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1. a) Show that the Hilbert dimension of a Hilbert space is less than or equal to its vector space dimension

$$\text{Hilbert dimension}\, H \leq \dim_{\mathbb{C}} H.$$

It can be shown that equality holds iff H is finite dimensional (we shall see a proof later). What is the Hilbert dimension of ℓ^2? Of $L^2[0, 1]$?

b) Give an example of a non-closed subspace of a Hilbert space. Show that for any subspace M, $\overline{M} = M^{\perp\perp}$.

c) Apart from the *Pythagorean theorem* and the *parallelogram law*, there is yet a third identity in an inner product space, the *polarization identity*: show that in a complex inner product space

$$\langle x, y\rangle = \frac{1}{4}\{||x+y||^2 - ||x-y||^2 - i(||x+iy||^2 - ||x-iy||^2)\}$$

2. a) Show that the linear functional $f \mapsto f(0)$, $f \in C[0, 1]$, is not continuous with respect to L^2 metric on $C[0, 1]$, but $f \mapsto \int_0^1 f(x)dx$ is continuous.

b) Give an example of an unbounded linear functional defined on $L^2[0, 1]$. Show that for any infinite dimensional Hilbert space, one can always find unbounded linear functionals.

3. Apply the Gram-Schmidt orthonormalization process to the sequence

$$1, x, x^2, \cdots$$

in $L^2[-1,1]$ and compute the first 3 terms. The resulting sequence of polynomials

$$p_0(x), p_1(x), p_2(x), ...$$

are *Legendre polynomials*.

4. Consider the space of all measurable functions f on $\mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|^2 w(x) dx < \infty$$

where

$$w(x) = e^{-\frac{1}{2}x^2}.$$

Define an inner product on the space of all such functions and turn it into a Hilbert space. Define the *Hermite polynomials* by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, \cdots$$

Compute the first 3 Hermite polynomials. Show that they form an orthogonal set in $L^2_w(\mathbb{R})$:

$$\langle H_n, H_m \rangle = n! \sqrt{2\pi}\, \delta_{n,m}.$$

Show that H_n satisfies the *eigenvalue problem*:

$$u'' - xu' = \lambda u, \qquad \lambda = n.$$

5. Parseval's equality and Fourier series is a very powerful method to sum some remarkable series. Here are two example:
a) Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1$$

denote the *Riemann zeta function.* Let $f(x) = x$. Apply Parseval's equality

$$||f||^2 = \sum_{i=-\infty}^{\infty} |\langle f, e_i \rangle|^2$$

in $L^2[-\pi, \pi]$ to deduce

$$\zeta(2) = \frac{\pi^2}{6} \qquad \text{Euler (1739).}$$

Show that

$$\zeta(4) = \frac{\pi^4}{90} \qquad \text{Euler (1739)}$$

by a similar method applied to $f(x) = x^2$. (Euler proved that in general

$$\zeta(2n) = (-1)^n \frac{B_{2n}(2\pi)^{2n}}{2(2n)!.}$$

Here B_n denotes the n th Bernoulli number.)