Functional Analysis
Assignment 2

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1. (Extension of functions). Let X and Y be metric spaces and $S \subset X$ a dense subset. Show that a continuous map $f : S \to Y$ need not have an *extension* to a continuous map $X \to Y$, even if Y is complete. Show that if f is *uniformly continuous* and Y is complete then there is a unique such extension and the extension is uniformly continuous as well. (This result is useful in functional analysis since continuous linear maps between normed spaces are automatically uniformly continuous).

2. a) Show that for any $n \times n$ matrix A,

$$||A||_{HS}^2 = \mathrm{Tr}(A^* A).$$

b) Find the kernel and image of the Voltera integral operator V, and calculate its adjoint V^*. Is V a normal operator? Show that $V^n \to 0$ as $n \to \infty$.

3. a) Show that the differentiation operator

$$\frac{d}{dx} : C^1[0,1] \to L^2[0,1],$$

is *not* bounded with respect to the L^2-norms. Show that it is a bounded operator with respect to the inner product on $C^1[0,1]$ defined by

$$\langle f, g \rangle = \int_0^1 (f\bar{g} + f'\bar{g}'),$$

and the L^2 metric on its codomain.

b) Prove the *Poincaré inequality*: there is a constant C such that for all $u \in C^1[0,1]$

$$\|u - u_0\|_{L^2} \leq C\|u'\|_{L^2},$$

where

$$u_0 = \int_0^1 u(y)\,\mathrm{d}y.$$

is the*average value* of u.

4. Let $k : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic continuous function and define an integral operator $K : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ by

$$(Kf)(x) = \int_{-\pi}^{\pi} k(x-y)f(y)dy.$$

Compute the matrix of K with respect to the o.n. basis $(\frac{1}{\sqrt{2\pi)}}e^{inx})_{n\in\mathbb{Z}}$.

5. a) Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be a power series with a positive radius of convergence ρ. Show that for any bounded operator A with $||A|| < \rho$, the series $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots$ is convergent and defines a bounded operator $f(A)$. In paticular since the exponential series $exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ has infinite radius of convergence, for any bounded operator A, we can define its exponential $exp(A)$.

b) Let A and B be commuting operators. Show that

$$exp(A+B) = exp(A)exp(B).$$

Give exmaples of 2 by 2 matrices for which the above property fails.

c) Let A be a self adjoint operator. Show that $exp(iA)$ is a unitary operator.