Functional Analysis
Assignment 3

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1. Consider the Sturm-Liouville problems
$$-u'' + q_i u = \lambda u, \quad i = 1, 2, \quad u(a) = u(b) = 0.$$
Let λ_n^1, λ_n^2 be the strictly increasing sequence of eigenvalues of these two boundary value problems. Show that if $q_1 \leq q_2$, then
$$\lambda_n^1 \leq \lambda_n^2, \qquad \forall n,$$
and if $|q_1(x) - q_2(x)| \leq M$ for all x, then
$$|\lambda_n^1 - \lambda_n^2| \leq M, \quad \forall n.$$
(Hint: use the MaxMin principle.)

2. Conclude from the above result that, for any q there is a constant C such that
$$|\lambda_n - \frac{n^2\pi^2}{\ell^2}| \leq C, \quad \forall n,$$
where $\ell = b - a$. (Hint: compare the eigenvalues with the eigenvalues of a Sturm-Liouville problem with constant q.) Conclude that
$$\lim_{n\to\infty} \frac{\lambda_n}{n^2} = \frac{\pi^2}{\ell^2}.$$

3. Show that the eigenvalues of a Sturm-Liouville problem satisfy
$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \int_a^b g(x,x)dx,$$
where $g(x, y)$ is the Green function we constructed in class.

4. *(The Fredholm alternative).* Let K be a compact operator. Show that the operator $T = I + K$ has a finite dimensional kernel and cokernel (cokernl $T := H/im(T)$) and

$$\dim(kerT) - \dim(cokerT) = 0.$$

This implies the *Fredholm alternative:* The inhomogeneous equation

$$Tx + x = y$$

has a solution for all $y \in H$ iff the homogeneous equation $Tx + x = 0$ has only the trivial solution $x = 0$.

5. *(Uncertainty Principle).* Let $A : H \to H$ be a selfadjoint linear operator and $x \in H$ a unit vector. The *expectation* value (or *mean value*) of A in the state x is defined as

$$\langle A \rangle_x := \langle Ax, x \rangle,$$

and the *standard deviation* or *dispersion* of A in the state x is defined as

$$\Delta_x A := \sqrt{\langle (A - \langle A \rangle_x)^2 \rangle} = \sqrt{\langle A^2 \rangle_x - \langle A \rangle_x^2}.$$

It gives the 'error' involved in measuring a quantum mechanical observable represented by the selfadjoint operator A when the system is in the state x. Let A and B be selfadjoint operators and let $[A, B] = AB - BA$ denote their commutators. Use the Cauchy-Schwartz inequality to prove *Heisenberg's uncertainty principle*:

$$\Delta_x A \Delta_x B \geq \frac{1}{2} |\langle [A, B] \rangle_x|$$