Functional Analysis
Assignment 4

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1. Let T be a compact operator and $|T| = (T^*T)^{1/2}$ its absolute value as defined in class. Let

$$\mu_0(T) \geq \mu_1(T) \geq \mu_2(T) \cdots$$

denote the list of eigenvalues of $|T|$ in decreasing order.

a) Use the minimax principle to show that

$$|\mu_n(T_1) - \mu_n(T_2)| \leq ||T_1 - T_2||.$$

b) Show that

$$\mu_{m+n}(T_1 + T_2) \leq \mu_m(T_1) + \mu_n(T_2).$$

c) Let

$$\sigma_N(T) = \mu_0(T) + \cdots + \mu_{N-1}(T).$$

Show that

$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2).$$

2. Prove that $\ell_\infty = \ell_1^*$, but that $\ell_\infty^* \neq \ell_1$ by using the Hahn-Banach theorem.

3. a) Let T be a *nilpotent operator*, i.e., $T^n = 0$ for some positive integer n. Show that $1 + T$ is invertible.
b) Use a) to show that $\sigma(T) = \{0\}$, if T is nilpotent.

4. Show that for the Volterra integral operator V,

$$\sigma(V) = \{0\}.$$

Is V a nilpotent operator?

5. Show that for any two operators a and b we have

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

That is, ab and ba have the same spectrum except for one point 0. Show that $\sigma(ab)$ and $\sigma(ba)$ can be different.

6. Let $S \subset \mathbb{C}$ be a compact non-empty subset of the plane. Show that there is a bounded operator T on a Hilbert space such that

$$\sigma(T) = S.$$

7. Let T be a normal operator. Show that its spectral radius $r(T) = ||T||$.