For Azadeh and Saman

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Introduction

One of the major advances of science in the 20th century was the discovery of a mathematical formulation of quantum mechanics by Heisenberg in 1925 [94].[1] From a mathematical point of view, this transition from classical mechanics to quantum mechanics amounts to, among other things, passing from the *commutative algebra* of *classical observables* to the *noncommutative algebra* of *quantum mechanical observables*. To understand this better we recall that in classical mechanics an observable of a system (e.g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. Classical observables can therefore be multiplied in a pointwise manner and this multiplication is obviously commutative. Immediately after Heisenberg's work, ensuing papers by Dirac [67] and Born–Heisenberg–Jordan [16], made it clear that a quantum mechanical observable is a (selfadjoint) linear operator on a Hilbert space, called the state space of the system. These operators can again be multiplied with composition as their multiplication, but this operation is not necessarily commutative any longer.[2] In fact Heisenberg's *commutation relation*

$$pq - qp = \frac{h}{2\pi i}1$$

shows that position and momentum operators do not commute and this in turn can be shown to be responsible for the celebrated *uncertainty principle* of Heisenberg. Thus, to get a more accurate description of nature one is more or less forced to replace the commutative algebra of functions on a space by the noncommutative algebra of operators on a Hilbert space.

A little more than fifty years after these developments Alain Connes realized that a similar procedure can in fact be applied to areas of mathematics where the classical notions of space (e.g. measure space, locally compact space, or smooth space) lose its applicability and relevance [37], [35], [36], [39]. The inadequacy of the classical notion of space manifests itself for example when one deals with

[1] A rival proposal which, by the Stone–von Neumann uniqueness theorem, turned out to be essentially equivalent to Heisenberg's was arrived at shortly afterwards by Schrödinger [161]. It is however Heisenberg's *matrix mechanics* that directly and most naturally relates to noncommutative geometry.

[2] Strictly speaking selfadjoint operators do not form an algebra since they are not closed under multiplication. By an algebra of observables we therefore mean the algebra that they generate.

highly singular "*bad quotients*": spaces such as the quotient of a nice space by the ergodic action of a group, or the space of leaves of a foliation in the generic case, to give just two examples. In all these examples the quotient space is typically ill-behaved, even as a topological space. For instance it may fail to be even Hausdorff, or have enough open sets, let alone being a reasonably smooth space. The unitary dual of a discrete group, except when the group is abelian or almost abelian, is another example of an ill-behaved space.

One of Connes' key observations is that in all these situations one can define a noncommutative algebra through a universal method which we call the *noncommutative quotient construction* that captures most of the information hidden in these unwieldy quotients. Examples of this noncommutative quotient construction include the crossed product by an action of a group, or in general by an action of a groupoid. In general the noncommutative quotient is the groupoid algebra of a topological groupoid.

This new notion of geometry, which is generally known as *noncommutative geometry*, is a rapidly growing new area of mathematics that interacts with and contributes to many disciplines in mathematics and physics. Examples of such interactions and contributions include: the theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, the Standard Model of elementary particles, the quantum Hall effect, renormalization in quantum field theory, and string theory.

To understand the basic ideas of noncommutative geometry one should perhaps first come to grips with the idea of a *noncommutative space*. What is a noncommutative space? The answer to this question is based on one of the most profound ideas in mathematics, namely a *duality* or *correspondence* between algebra and geometry,[3]

$$\boxed{\text{Algebra} \longleftrightarrow \text{Geometry}}$$

according to which every concept or statement in Algebra corresponds to, and can be equally formulated by, a similar concept and statement in Geometry.

On a physiological level this correspondence is perhaps related to a division in the human brain: one computes and manipulates symbols with the left hemisphere of the brain and one visualizes things with the right hemisphere. Computations evolve in time and have a temporal character, while visualization is instant and immediate. It was for a good reason that Hamilton, one of the creators of modern algebraic methods, called his approach to algebra, e.g. to complex numbers and quaternions, the *science of pure time* [92].

We emphasize that the algebra-geometry correspondence is by no means a new observation or a new trend in mathematics. On the contrary, this duality has always existed and has been utilized in mathematics and its applications very often.

[3]For a modern and very broad point of view on this duality, close to the one adopted in this book, read the first section of Shafarevich's book [164] as well as Cartier's article [31].

The earliest example is perhaps the use of numbers in counting. It is, however, the case that throughout history each new generation of mathematicians has found new ways of formulating this principle and at the same time broadening its scope. Just to mention a few highlights of this rich history we quote Descartes (analytic geometry), Hilbert (affine varieties and commutative algebras), Gelfand–Naimark (locally compact spaces and commutative C^*-algebras), and Grothendieck (affine schemes and commutative rings). A key idea here is the well-known relation between a space and the commutative algebra of functions on that space. More precisely, there is a duality between certain categories of geometric spaces and the corresponding categories of algebras representing those spaces. Noncommutative geometry builds on, and vastly extends, this fundamental duality between classical geometry and commutative algebras.

For example, by a celebrated theorem of Gelfand and Naimark [82], one knows that the information about a compact Hausdorff space is fully encoded in the algebra of continuous complex-valued functions on that space. The space itself can be recovered as the space of maximal ideals of the algebra. Algebras that appear in this way are commutative C^*-algebras. This is a remarkable theorem since it tells us that any natural construction that involves compact spaces and continuous maps between them has a purely algebraic reformulation, and vice-versa any statement about commutative C^*-algebras and C^*-algebraic maps between them has a purely geometric-topological meaning.

Thus one can think of the category of not necessarily commutative C^*-algebras as the dual of an, otherwise undefined, category of *noncommutative locally compact spaces*. What makes this a successful proposal is, first of all, a rich supply of examples and, secondly, the possibility of extending many of the topological and geometric invariants to this new class of 'spaces' and applications thereof.

Noncommutative geometry has as its special case, in fact as its limiting case, classical geometry, but geometry expressed in algebraic terms. In some respect this should be compared with the celebrated *correspondence principle* in quantum mechanics where classical mechanics appears as a limit of quantum mechanics for large quantum numbers or small values of Planck's constant. Before describing the tools needed to study noncommutative spaces let us first briefly recall a couple of other examples from a long list of results in mathematics that put in duality certain categories of geometric objects with a corresponding category of algebraic objects.

To wit, Hilbert's Nullstellensatz states that the category of affine algebraic varieties over an algebraically closed field is equivalent to the opposite of the category of finitely generated commutative algebras without nilpotent elements (so-called reduced algebras). This is a perfect analogue of the Gelfand–Naimark theorem in the world of algebraic geometry. Similarly, Swan's (resp. Serre's) theorem states that the category of vector bundles over a compact Hausdorff space (resp. over an affine algebraic variety) X is equivalent to the category of finitely generated projective modules over the algebra of continuous functions (resp. the algebra of regular functions) on X.

A pervasive idea in noncommutative geometry is to treat certain classes of noncommutative algebras as noncommutative spaces and to try to extend tools of geometry, topology, and analysis to this new setting. It should be emphasized, however, that, as a rule, this extension is hardly straightforward and most of the times involves surprises and new phenomena. For example, the theory of the flow of weights and the corresponding modular automorphism group in von Neumann algebras [41] has no counterpart in classical measure theory, though the theory of von Neumann algebras is generally regarded as noncommutative measure theory. Similarly, as we shall see in Chapters 3 and 4 of this book, the extension of de Rham (co)homology of manifolds to cyclic (co)homology for noncommutative algebras was not straightforward and needed some highly non-trivial considerations. As a matter of fact, de Rham cohomology can be defined in an algebraic way and therefore can be extended to all commutative algebras and to all schemes. This extension, however, heavily depends on exterior products of the module of Kähler differentials and on the fact that one works with commutative algebras. In the remainder of this introduction we focus on topological invariants that have proved very useful in noncommutative geometry.

Of all topological invariants for spaces, topological K-theory has the most straightforward extension to the noncommutative realm. Recall that topological K-theory classifies vector bundles on a topological space. Motivated by the above-mentioned Serre–Swan theorem, it is natural to define, for a not necessarily commutative ring A, $K_0(A)$ as the group defined by the semigroup of isomorphism classes of finite projective A-modules. Provided that A is a Banach algebra, the definition of $K_1(A)$ follows the same pattern as for spaces, and the main theorem of topological K-theory, the Bott periodicity theorem, extends to all Banach algebras [14].

The situation was much less clear for K-homology, a dual of K-theory. By the work of Atiyah [6], Brown–Douglas–Fillmore [22], and Kasparov [106], one can say, roughly speaking, that K-homology cycles on a space X are represented by abstract elliptic operators on X and, whereas K-theory classifies the vector bundles on X, K-homology classifies the abstract elliptic operators on X. The pairing between K-theory and K-homology takes the form $\langle [D], [E] \rangle = \mathrm{index}(D_E)$, the Fredholm index of the elliptic operator D with coefficients in the 'vector bundle' E. Now one good thing about this way of formulating K-homology is that it almost immediately extends to noncommutative C^*-algebras. The two theories are unified in a single theory called KK-theory, due to Kasparov [106].

Cyclic cohomology was discovered by Connes in 1981 [36], [39] as the right noncommutative analogue of the de Rham homology of currents and as a receptacle for a noncommutative Chern character map from K-theory and K-homology. One of the main motivations was transverse index theory on foliated spaces. Cyclic cohomology can be used to identify the K-theoretic index of transversally elliptic operators which lie in the K-theory of the noncommutative algebra of the foliation. The formalism of cyclic cohomology and noncommutative Chern character maps form an indispensable part of noncommutative geometry. A very interesting recent development in cyclic cohomology theory is the *Hopf cyclic cohomology* of Hopf algebras and Hopf (co)module (co)algebras in general. Motivated by the original work in [54], [55] this theory has now been extended in [89], [90].

The following "dictionary" illustrates noncommutative analogues of some of the classical theories and concepts originally conceived for spaces. In this book we deal only with a few items of this ever expanding dictionary.

commutative	**noncommutative**
measure space	von Neumann algebra
locally compact space	C^*-algebra
vector bundle	finite projective module
complex variable	operator on a Hilbert space
infinitesimal	compact operator
range of a function	spectrum of an operator
K-theory	K-theory
vector field	derivation
integral	trace
closed de Rham current	cyclic cocycle
de Rham complex	Hochschild homology
de Rham cohomology	cyclic homology
Chern character	Connes–Chern character
Chern–Weil theory	noncommutative Chern–Weil theory
elliptic operator	K-cycle
spin^c Riemannian manifold	spectral triple
index theorem	local index formula
group, Lie algebra	Hopf algebra, quantum group
symmetry	action of Hopf algebra

Noncommutative geometry is already a vast subject. This book is an introduction to some of its basic concepts suitable for graduate students in mathematics and physics. While the idea was to write a primer for the novice to the subject, some acquaintance with functional analysis, differential geometry and algebraic topology at a first year graduate level is assumed. To get a better sense of the beauty and depth of the subject the reader can go to no better place than the authoritative book [41]. There are also several introductions to the subject, with varying lengths and attention to details, that the reader can benefit from [85], [174], [144], [97], [118], [108], [133], [135], [64], [63]. They each emphasize rather different aspects of noncommutative geometry. For the most complete account of what has happened in the subject after the publication of [41], the reader should consult [52] and references therein.

To summarize our introduction we emphasize that what makes the whole project of noncommutative geometry a viable and extremely important enterprize are the following three fundamental points:

• There is a vast repertoire of noncommutative spaces and there are very general methods to construct them. For example, consider a *bad quotient* of a nice and smooth space by an equivalence relation. Typically the (naive) quotient space is not even Hausdorff and has very bad singularities, so that it is beyond the reach of classical geometry and topology. Orbit spaces of group actions and the space of leaves of a foliation are examples of this situation. In algebraic topology one replaces such naive quotients by homotopy quotients, by using the general idea of a classifying space. This is however not good enough and not general enough, as the classifying space is only a homotopy construction and does not see any of the smooth structure. A key observation throughout [41] is that in all these situations one can attach a noncommutative space, e.g. a (dense subalgebra of a) C^*-algebra or a von Neumann algebra, that captures most of the information hidden in these quotients. The general construction starts by first replacing the equivalence relation by a groupoid and then considering the associated groupoid algebra in its various completions. We shall discuss this technique in detail in Chapter 2 of this book.

• The possibility of extending many of the tools of classical geometry and topology that are used to probe classical spaces to this noncommutative realm. The topological K-theory of Atiyah and Hirzebruch, and its dual theory known as K-homology, as well as the Bott periodicity theorem, have a natural extension to the noncommutative world [14]. Finding the right noncommutative analogue of de Rham cohomology and Chern–Weil theory was less obvious and was achieved thanks to the discovery of cyclic cohomology [36], [38]. In Chapters 3 and 4 of this book we shall give a detailed account of cyclic cohomology and its relation with K-theory and K-homology. Another big result of recent years is the local index formula of Connes and Moscovici [54]. Though we shall not discuss it in this book, it suffices to say that this result comprises a vast extension of the classical Atiyah–Singer index theorem to the noncommutative setup.

• *Applications.* Even if we wanted to restrict ourselves just to classical spaces, methods of noncommutative geometry would still be very relevant and useful. For example, a very natural and general proof of the Novikov conjecture on the homotopy invariance of higher signatures of non-simply connected manifolds (with word hyperbolic fundamental groups) can be given using the machinery of noncommutative geometry [53]. The relevant noncommutative space here is the (completion of the) group ring of the fundamental group of the manifold. We also mention the geometrization of the Glashow–Weinberg–Salam Standard Model of elementary particles via noncommutative geometry (cf. [52] and references therein). Moving to more recent applications, we mention the approach to the Riemann hypothesis and the spectral realization of zeros of the zeta function via noncommutative spaces [18], [42] as well as the mathematical underpinning of renormalization in quantum field theory as a Riemann–Hilbert Correspondence [47], [48]. These results have brought noncommutative geometry much closer to central areas of modern number theory, algebraic geometry and high energy physics. We shall not follow these developments in this book. For a complete and up to date account see [52].

Let me now briefly explain the contents of this book. Chapter 1 describes some of the fundamental algebra-geometry correspondences that are vital for a better understanding of noncommutative geometry. The most basic ones, for noncommutative geometry at least, are the Gelfand–Naimark and the Serre–Swan theorems. They lead to ideas of noncommutative space and noncommutative vector bundles. We give several examples of noncommutative spaces, most notably noncommutative tori, group C^*-algebras, and quantum groups. The last section of this first chapter is a self contained introduction to Hopf algebras and quantum groups and the idea of symmetry in noncommutative geometry. Chapter 2 is about forming noncommutative quotients via groupoids and groupoid algebras. This is one of the most universal and widely used methods for constructing noncommutative spaces. Another important concept in this chapter is the idea of Morita equivalence of algebras, both at purely algebraic and C^*-algebraic levels. Among other things, Morita equivalence clarifies the relation between noncommutative quotients and classical quotients. Chapter 3 is devoted to cyclic (co)homology, its relation with Hochschild (co)homology through Connes' long sequence and spectral sequence, and its relation with de Rham (co)homology. Three different definitions of cyclic cohomology are given in this chapter, each shedding light on a different aspect of the theory. Continuous versions of cyclic and Hochschild theory for topological algebras is developed in this section. This plays an important role in applications. We also give several important examples of algebras for which these invariants are fully computed. In Chapter 4 we define the Connes–Chern character map for both K-theory and K-homology. For K-theory it is the noncommutative analogue of the classical Chern character map from K-theory to de Rham cohomology. It can also be described as a pairing between K-theory and cyclic cohomology. Fredholm modules, as cycles for K-homology, are introduced next and, for finitely summable Fredholm modules, their Connes–Chern character with values in cyclic cohomology is introduced. Then we use these pairings to prove an index formula from [39] relating the analytic Fredholm index of a finitely summable Fredholm module to its topological index. This is an example of an index formula in noncommutative geometry. The very last section of this chapter summarizes many ideas of the book into one commuting diagram which is the above mentioned index formula. In an effort to make this book as self-contained as possible, we have added four appendices covering basic material on C^*-algebras, compact and Fredholm operators, projective modules, and some basic category theory language.

This book is partly based on series of lectures I gave at the Fields Institute in Toronto, Canada, in Fall 2002 and at the second and fifth Annual Spring Institute on Noncommutative Geometry and Operator Algebras in spring 2004 and spring 2007 at Vanderbilt University, USA.

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Chapter 1

Examples of algebra-geometry correspondences

We give several examples of algebra-geometry correspondences. They all put into correspondence, or duality, certain categories of classical geometric objects with a corresponding category of commutative algebraic objects. In many cases, by relaxing the commutativity assumption we arrive at the corresponding noncommutative geometric object. Presumably, the more one knows about these duality relations the better one is prepared to pursue noncommutative geometry.

1.1 Locally compact spaces and commutative C^*-algebras

In functional analysis the celebrated *Gelfand–Naimark Theorem* [82] implies that the category of locally compact Hausdorff spaces and continuous proper maps is equivalent to the opposite of the category of commutative C^*-algebras and proper C^*-morphisms:

$$\boxed{\{\text{locally compact Hausdorff spaces}\} \simeq \{\text{commutative } C^*\text{-algebras}\}^{\text{op}}} \quad (1.1)$$

Under this correspondence, the category of compact Hausdorff spaces and continuous maps corresponds to the category of unital commutative C^*-algebras and unital C^*-morphisms:

$$\boxed{\{\text{compact Hausdorff spaces}\} \simeq \{\text{commutative unital } C^*\text{-algebras}\}^{\text{op}}} \quad (1.2)$$

By an *algebra* in this book we shall mean an associative algebra over the field of complex numbers $\mathbb{C}$. Algebras are not assumed to be commutative or unital, unless explicitly stated. An *involution* on an algebra A is a *conjugate linear* map $*\colon A \to A$, $a \mapsto a^*$, satisfying the extra relations

$$(ab)^* = b^* a^* \quad \text{and} \quad (a^*)^* = a$$

for all a and b in A. A $\mathbb{C}$-algebra endowed with an involution is called an *involutive algebra.*

By a *normed algebra* we mean an algebra A such that A is a normed vector space and

$$\|ab\| \leq \|a\| \|b\|$$

for all a, b in A. If A is unital, with its unit denoted by 1, we shall assume that $\|1\| = 1$. A *Banach algebra* is a normed algebra which is complete as a metric space in the sense that any Cauchy sequence in A is convergent.

Definition 1.1.1. A C^**-algebra* is an involutive Banach algebra A such that for all $a \in A$ the C^*-identity

$$\boxed{\|a^* a\| = \|a\|^2} \tag{1.3}$$

holds.

A *morphism* of C^*-algebras, or a C^**-morphism*, is an algebra homomorphism $f\colon A \to B$ which preserves the $*$-structure, namely

$$f(a^*) = f(a)^* \quad \text{for all } a \in A.$$

It can be shown that any C^*-morphism is automatically a *contraction* in the sense that $\|f(a)\| \leq \|a\|$ for all $a \in A$, and in particular is continuous. This *'automatic continuity'* result, and its immediate consequence that the norm of a C^*-algebra is unique, is an example of 'rigidity' of C^*-algebras. Very often, purely algebraic conditions, thanks to the mighty C^*-identity (1.3), have topological consequences. This need not be true for general Banach algebras. See Appendix A for basics of C^*-algebra theory and Appendix D for the category theory language we use in this book. Let us explain the equivalences (1.1) and (1.2).

A *character* of an algebra A is a nonzero multiplicative linear map $\chi\colon A \to \mathbb{C}$. Notice that if A has a unit then necessarily $\chi(1) = 1$. It can be shown that any character of a Banach algebra is automatically continuous and has norm one. It can also be shown that if A is a C^*-algebra then any character of A preserves the $*$-structure.

Let $\hat{A}$ denote the set of characters of the Banach algebra A. It is called the (*maximal ideal*) *spectrum* of A. We can endow $\hat{A}$ with the weak* topology, i.e., the topology of pointwise convergence, on the continuous dual, A^*, of A. By the Banach–Alaoglu theorem, the unit ball of A^* is compact in the weak* topology and, since $\hat{A} \cup \{0\}$ is a closed subset of this unit ball, we can conclude that $\hat{A}$ is a locally compact Hausdorff space. It is compact if and only if A is unital. When A is unital there is a one-to-one correspondence between characters of A and the set of maximal ideals of A: to a character χ we associate its kernel which is a maximal ideal and to a maximal ideal I we associate the character $\chi\colon A \to A/I = \mathbb{C}$. Note that, by the Gelfand–Mazur theorem, for any maximal ideal I, the skew field A/I is isomorphic to $\mathbb{C}$.

Example 1.1.1. Given a locally compact Hausdorff space X, let $C_0(X)$ denote the algebra of complex-valued continuous functions on X, vanishing at infinity. Under pointwise addition and multiplication $C_0(X)$ is obviously a commutative algebra over the field of complex numbers. It is unital if and only if X is compact, in which case it will be denoted by $C(X)$. Endowed with the *sup-norm*

$$\|f\| := \|f\|_\infty = \sup\{|f(x)|;\ x \in X\},$$

and the $*$-operation induced by complex conjugation

$$f \mapsto f^*, \quad f^*(x) = \overline{f(x)},$$

$C_0(X)$ can be easily shown to satisfy all the axioms of a C^*-algebra, including the all-important C^*-identity (1.3). Thus to any locally compact Hausdorff space we have associated a commutative C^*-algebra, and this C^*-algebra is unital if and only if the space is compact. The characters of $C_0(X)$ are easy to describe, as we show next.

For any $x \in X$ we have the *evaluation character*

$$\chi = \chi_x\colon C_0(X) \to \mathbb{C}, \quad \chi_x(f) = f(x).$$

It can be shown that all characters of $C_0(X)$ are of this form and that the map

$$X \to \widehat{C_0(X)}, \quad x \mapsto \chi_x,$$

is a homeomorphism. Thus, we can recover X, including its topology, as the space of characters of $C_0(X)$.

By a fundamental theorem of Gelfand and Naimark [82] (see below), any commutative C^*-algebra A is isomorphic to $C_0(X)$ where X is the space of characters of A. The isomorphism is implemented by the Gelfand transform, to be recalled next.

For any commutative Banach algebra A, the *Gelfand transform*

$$\Gamma\colon A \to C_0(\hat{A})$$

is defined by $\Gamma(a) = \hat{a}$, where

$$\hat{a}(\chi) = \chi(a).$$

It is a norm contractive algebra homomorphism, as can be easily seen. In general Γ need not be injective or surjective, though its image separates the points of the spectrum. The kernel of Γ is the *nilradical* of A consisting of quasi-nilpotent elements of A, that is, elements whose spectrum consists only of zero. In [82] Gelfand and Naimark, building on earlier work of Gelfand, show that under some extra assumptions on A, which is equivalent to A being a C^*-algebra, Γ is a $*$-algebra isometric isomorphism between A and $C_0(\hat{A})$.

Theorem 1.1.1 (Gelfand–Naimark Theorem [82])**.** *For any commutative C^*-algebra A with spectrum $\hat{A}$, the Gelfand transform*

$$\Gamma\colon A \to C_0(\hat{A}), \quad a \mapsto \hat{a},$$

is an isomorphism of C^*-algebras.

Theorem 1.1.1 is the main technical result needed to establish the equivalence of categories in (1.1) and (1.2). The description of these correspondences is a bit easier in the compact case (1.2) and that is what we shall do first.

Let $\mathcal{S}$ denote the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps between such spaces. Let $\mathcal{C}$ denote another category whose objects are commutative unital C^*-algebras and whose morphisms are unital C^*-algebra morphisms between such algebras.

We define contravariant functors

$$\boxed{C\colon \mathcal{S} \rightsquigarrow \mathcal{C} \quad \text{and} \quad \widehat{\ }\colon \mathcal{C} \rightsquigarrow \mathcal{S}}$$

as follows. We send a compact Hausdorff space X, to $C(X)$, the algebra of complex-valued continuous functions on X. If $f\colon X \to Y$ is a continuous map, we let

$$C(f) = f^*\colon C(Y) \to C(X), \quad f^*(g) = g \circ f,$$

be the *pullback* of f. It is clearly a C^*-algebra homomorphism which preserves the units. We have thus defined the functor C.

The functor $\widehat{\ }$, called the *functor of points* or the *maximal ideal spectrum* functor, sends a commutative unital C^*-algebra A to its space of characters, or equivalently maximal ideals, $\hat{A}$, and sends a C^*-morphism $f\colon A \to B$ to the continuous map $\hat{f}\colon \hat{B} \to \hat{A}$ defined by

$$\hat{f}(\chi) = \chi \circ f$$

for any character $\chi \in \hat{B}$.

To show that C and $\widehat{\ }$ are equivalences of categories, quasi-inverse to each other, we must show that the functor $C \circ \widehat{\ }$ is isomorphic to the identity functor of $\mathcal{C}$ and similarly the functor $\widehat{\ } \circ C$ is isomorphic to the identity functor of $\mathcal{S}$. That is, we have to show that for any compact Hausdorff space X and any commutative unital C^*-algebra A, there are natural isomorphisms $\widehat{C(X)} \simeq X$ and $C(\hat{A}) \simeq A$. But in fact we have already done this. Consider the maps

$$X \xrightarrow{\ \sim\ } \widehat{C(X)}, \quad x \mapsto \chi_x,$$
$$A \xrightarrow{\ \sim\ } C(\hat{A}), \quad a \mapsto \hat{a}.$$

Here χ_x is the *evaluation at* x map defined by $\chi_x(f) = f(x)$, and $a \mapsto \hat{a}$ is the *Gelfand transform* Γ defined above by $\hat{a}(\chi) = \chi(a)$. The first isomorphism is elementary and does not require the theory of Banach algebras. The second isomorphism is the content of Gelfand–Naimark's Theorem 1.1.1 whose proof is based on Gelfand's theory of commutative Banach algebras.

Care must be applied in the more general non-compact/non-unital case. The main issue is to get the morphisms right. Let us define a category $\mathcal{S}_0$ consisting of locally compact Hausdorff spaces and *proper* continuous maps between them. Recall that a continuous map $f\colon X \to Y$ between locally compact spaces is called *proper* if for any compact $K \subset Y$, $f^{-1}(Y)$ is compact. Let $\mathcal{C}_0$ be another category whose objects are commutative not necessarily unital C^*-algebras. For morphisms between A and B we take *proper* C^*-algebra homomorphisms $f\colon A \to B$. Here proper means that for any approximate identity e_i, $i \in I$, of A, $f(e_i)$, $i \in I$ is an approximate identity for B. Equivalently, for any nonzero character χ on B, $\chi \circ f$ should be nonzero. Alternatively, f is proper if and only if $f(A)B$ is dense in B. Recall that an *approximate identity* for a C^*-algebra A is a net $e_i, i \in I$ of elements of A such that for all $a \in A$, $ae_i \to a$ and $e_i a \to a$. Thus the properness of a morphism can be seen as a replacement for being unital. In particular the zero map, while it is a C^*-map, is not proper.

Define two contravariant functors, similar to what we had before,

$$C_0\colon \mathcal{S}_0 \rightsquigarrow \mathcal{C}_0 \quad \text{and} \quad \widehat{\ }\colon \mathcal{C}_0 \rightsquigarrow \mathcal{S}_0$$

as follows. We send a locally compact Hausdorff space X to $C_0(X)$ and send a proper continuous map $f\colon X \to Y$, to its pullback $C_0(f) = f^*\colon C_0(Y) \to C_0(X)$. It is easily seen that, thanks to properness of f, $C_0(f)$ is well defined and is a proper morphism of C^*-algebras. The definition of the spectrum functor is the same as in the unital case. Notice that under a proper morphism $f\colon A \to B$, the map $f^*\colon \hat{B} \to \hat{A}$ sends a nonzero character to a nonzero character and hence is well defined. Using unitization and 1-point compactification, one can deduce the equivalence (1.2) from (1.1) as follows.

If A is a commutative C^*-algebra, its *unitization* $A^+ = A \oplus \mathbb{C}$ is obtained by adjoining a unit to A. Thus its multiplication and $*$-structure are defined by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$, $(a, \lambda)^* = (a^*, \bar{\lambda})$. The definition of its norm is less obvious and one should take the left action of A^+ on A by multiplication and the corresponding operator norm. Thus $\|(a, \lambda)\| := \sup\{\|ab + \lambda b\|;\ \|b\| \leq 1\}$. Then it can be shown that A^+ is a (unital) C^*-algebra. Let $\chi_0\colon A^+ \to \mathbb{C}$ be the character defined by $\chi_0(a, \lambda) = \lambda$. Now we have $\hat{A} = \hat{A^+} \setminus \{\chi_0\}$, which shows that, being the complement of a closed set in a compact Hausdorff space, $\hat{A}$ is a locally compact Hausdorff space. Now if $f\colon A \to B$ is a proper morphism, then one checks that $\hat{f}(\chi) := \chi \circ f$ is a non-trivial character. This defines the functor of spectrum in the non-unital case. One can now deduce the equivalence (1.1) from (1.2).

Example 1.1.2. The spectrum of a commutative C^*-algebra may not be 'visible' at first sight, unless the algebra is already in the form $C_0(X)$. For example, the algebra $C_b(\mathbb{R})$ of *bounded* continuous functions on $\mathbb{R}$ is a unital commutative C^*-algebra in a natural way. So, by Theorem 1.1.1, we know that $C_b(\mathbb{R}) = C(\beta\mathbb{R})$, where $\beta\mathbb{R}$ denotes the spectrum of $C_b(\mathbb{R})$. It is easy to see that $\beta\mathbb{R}$ is in fact the Stone–Čech compactification of $\mathbb{R}$. More generally, for a locally compact Hausdorff space X, the spectrum of $C_b(X)$ can be shown to be homeomorphic to βX, the Stone–Čech compactification of X (cf. also the next example).

For an example of a different flavor, let X be a topological space which is manifestly non-Hausdorff and let $A = C_0(X)$. Then the spectrum of A has the effect of turning X into a Hausdorff space and is in some sense the 'Hausdorffization' of X. For yet a different type of example, the reader should try to describe the spectrum of $L^\infty[0, 1]$, the algebra of essentially bounded measurable functions on $[0, 1]$.

Example 1.1.3 (Essential ideals and compactifications)**.** Let X be a locally compact Hausdorff space. Recall that a *Hausdorff compactification* of X is a compact Hausdorff space Y, where X is homeomorphic to a dense subset of Y. We consider X as a subspace of Y. Then X is open in Y and its boundary $Y \setminus X$ is compact. We have an exact sequence

$$0 \to C_0(X) \to C(Y) \to C(Y \setminus X) \to 0,$$

where $C_0(X)$ is an *essential ideal* of $C(Y)$. (An ideal $I \subset A$ is called essential if whenever $aI = 0$, then $a = 0$.) Conversely, any extension

$$0 \to C_0(X) \to A \to B \to 0,$$

where A, and hence B, is a commutative unital C^*-algebra and $C_0(X)$ is an essential ideal of A, defines a Hausdorff compactification of X. Thus, we have a one-to-one correspondence between Hausdorff compactifications of X and (isomorphism classes of) essential extensions of $C_0(X)$. In particular, the 1-point compactification and the Stone–Čech compactification correspond to

$$0 \to C_0(X) \to C_0(X)^+ \to \mathbb{C} \to 0,$$

and

$$0 \to C_0(X) \to C_b(X) \to C(\beta X \setminus X) \to 0.$$

Example 1.1.4. Under the correspondence (1.1), constructions on spaces have their algebraic counterparts and vice versa. We list a few of these correspondences. The *disjoint union* of spaces $X \cup Y$ corresponds to the *direct sum* of algebras $A \oplus B$; the *Cartesian product* of spaces $X \times Y$ corresponds to a certain topological tensor product $A \hat{\otimes} B$ of algebras. Closed subspaces $Y \subset X$ of a space correspond to closed ideals; A compact Hausdorff space X is connected if and only if the algebra $C(X)$ has no non-trivial idempotent. Recall that an idempotent in an algebra is an element e such that $e^2 = e$. It is called non-trivial if $e \neq 0, 1$. The 1-point compactification of a locally compact Hausdorff space corresponds to unitization, i.e., the operation of adding a unit to the algebra. We record some of these correspondences in the following table.

Space	**Algebra**
compact	unital
1-point compactification	unitization
Stone–Čech compactification	multiplier algebra
closed subspace; inclusion	closed ideal; quotient algebra
surjection	injection
injection	surjection
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state
disjoint union	direct sum
Cartesian product	minimal tensor product

Theorem 1.1.1 and the correspondence (1.1) form the foundation of the idea that the category of noncommutative C^*-algebras may be regarded as the dual of an, otherwise undefined, category of noncommutative (NC) spaces. Thus formally one can propose a category of noncommutative spaces as the dual of the category of C^*-algebras and C^*-morphisms:

$$\boxed{\{\text{NC locally compact spaces}\} := \{\text{NC } C^*\text{-algebras and } C^*\text{-morphisms}\}^{\text{op}}} \tag{1.4}$$

Notice that while (1.1) is a theorem, (1.4) is a proposal and at the moment there is no other way to define its left-hand side by any other means. Various operations and concepts for spaces can be paraphrased in terms of algebras of functions on spaces and then one can try to generalize them to noncommutative spaces. This is the rather easy part of noncommutative geometry. The more interesting and harder part is to find properties and phenomena that have no commutative counterpart.

While (1.4) is a useful definition, it is by far not enough and one should take a broader perspective on the nature of a noncommutative space. For example, (1.4) captures only the topological aspects, and issues like smooth or complex structures, metric and Riemannian structures, etc. are totally left out. One can probe a space with a hierarchy of classes of functions:

$$\boxed{\text{polynomial} \subset \text{analytic} \subset \text{smooth} \subset \text{continuous} \subset \text{measurable}}$$

There is a similar hierarchy in the noncommutative realm, though it is much less well understood at the moment.

Our working definition of a noncommutative space is a noncommutative algebra, possibly endowed with some extra structure. Operator algebras, i.e., algebras of bounded operators on a Hilbert space, provided the first really deep insights into this noncommutative realm.

Example 1.1.5. The quintessential example of a noncommutative C^*-algebra is the algebra $\mathcal{L}(H)$ of all bounded linear operators on a complex Hilbert space H. The adjoint T^* of a bounded linear operator $T\colon H \to H$ is defined by the usual equation $\langle Tx,\, y\rangle = \langle x,\, T^*y\rangle$. It is easy to see that, endowed with this $*$-structure and the *operator norm*

$$\|T\| := \sup\{\|T(x)\|;\ \|x\| \leq 1\},$$

$\mathcal{L}(H)$ is a C^*-algebra. If H is *finite dimensional* then $\mathcal{L}(H)$ is isomorphic to the algebra $M_q(\mathbb{C})$ of $q\times q$ matrices over $\mathbb{C}$, where $q = \dim_{\mathbb{C}} H$. A direct sum of matrix algebras

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

is a C^*-algebra as well and it can be shown that any finite dimensional C^*-algebra is indeed a direct sum of matrix algebras [66]. Thus, finite dimensional C^*-algebras are semi-simple. The most interesting examples of C^*-algebras are clearly infinite dimensional ones. Here is a very general method of defining them.

It is clear that any subalgebra $A \subset \mathcal{L}(H)$ which is *selfadjoint* in the sense that if $T \in A$ then $T^* \in A$, and is closed under the norm topology, is a C^*-algebra. (More generally, any selfadjoint norm closed subalgebra of a C^*-algebra is itself a C^*-algebra). An example is the algebra $\mathcal{K}(H)$ of *compact operators* on H. By a fundamental theorem of Gelfand and Naimark any C^*-algebra is defined in this way. That is, it can be realized as a closed $*$-subalgebra of the algebra of bounded operators on a Hilbert space.

A *von Neumann algebra* is a unital selfadjoint subalgebra of $\mathcal{L}(H)$ which is closed in the weak operator topology. By definition the *weak operator topology* on $\mathcal{L}(H)$ is the weakest topology for which the maps $T \mapsto \langle Tx, y\rangle$ are continuous for al $x, y \in H$. Clearly a von Neumann algebra is a C^*-algebra. The relation between *commutative von Neumann algebras* and *measure spaces* is similar to the relationship between commutative C^*-algebras and locally compact Hausdorff spaces. Given a measure space (X, μ), let $L^\infty(X, \mu)$ denote the $*$-algebra of essentially bounded, measurable and complex valued functions on X. This algebra acts on the Hilbert space $H = L^2(X, \mu)$ as multiplication operators and its image in $\mathcal{L}(H)$ can be shown to be closed in the weak operator topology, hence is a commutative von Neumann algebra. Conversely, any commutative von Neumann algebra can be shown to be algebraically isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ). Because of this correspondence the theory of von Neumann algebras is often regarded as *noncommutative measure theory.*

The paper of Gelfand and Naimark [82] is the birthplace of the theory of C^*-algebras. Together with Murray–von Neumann's series of papers on von Neumann algebras [147], they form the foundation stone of operator algebras. The following two fundamental results on the structure of C^*-algebras are proved in this paper. We have already discussed the first part.

Theorem 1.1.2 (Gelfand–Naimark [82])**.** a) *For any commutative C^*-algebra A with spectrum $\hat{A}$ the Gelfand transform*

$$A \to C_0(\hat{A}), \quad a \mapsto \hat{a}, \tag{1.5}$$

defines an isomorphism of C^*-algebras.

b) *Any C^*-algebra is isomorphic to a C^*-subalgebra of the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H.*

In Appendix A we shall sketch a proof of both parts.

Example 1.1.6 (Noncommutative spaces from groups)**.** To any locally compact topological group G one can associate two C^*-algebras, the *full* and the *reduced group* C^*-algebras of G, denoted by $C^*(G)$ and $C^*_r(G)$, respectively. Both algebras are completions of the group algebra (convolution algebra) of G, but under different norms. Their universal properties are as follows: there is a one-to-one correspondence between unitary representations of G and representations of $C^*(G)$, and a one-to-one correspondence between unitary representations of G which are equivalent to a sub-representation of its left regular representation and representations of $C^*_r(G)$. There is always a surjective C^*-morphism $C^*(G) \to C^*_r(G)$, which is injective if and only if the group G is *amenable*. We describe these C^*-algebras for discrete groups first.

Let Γ be a discrete group and let $H = \ell^2(\Gamma)$ denote the Hilbert space of square summable complex-valued functions on Γ. It has a canonical orthonormal basis consisting of delta functions $\{\delta_g\}$, $g \in \Gamma$. Let $\mathbb{C}\Gamma$ denote the *group algebra* of Γ over $\mathbb{C}$. Elements of $\mathbb{C}\Gamma$ consist of functions $\xi\colon \Gamma \to \mathbb{C}$ with *finite support*. Its multiplication is defined by the *convolution product*:

$$\xi\eta(g) = \sum_{hk=g} \xi(h)\eta(k).$$

It is a $*$-algebra under the operation $(\xi^*)(g) = \bar{\xi}(g^{-1})$. The *left regular representation* of Γ is the unitary representation $\pi\colon \Gamma \to \mathcal{L}(\ell^2(\Gamma))$ defined by

$$(\pi g)\xi(h) = \xi(g^{-1}h).$$

There is a unique linear extension of π to an (injective) $*$-algebra homomorphism

$$\pi\colon \mathbb{C}\Gamma \to \mathcal{L}(H).$$

The *reduced group C^*-algebra* of Γ, denoted by $C^*_r\Gamma$, is the norm closure of $\pi(\mathbb{C}\Gamma)$ in $\mathcal{L}(H)$. It is obviously a unital C^*-algebra. The linear functional

$$\tau(a) = \langle a\delta_e, \delta_e\rangle, \quad a \in C^*_r\Gamma,$$

defines a *positive* and *faithful trace* $\tau\colon C^*_r\Gamma \to \mathbb{C}$. This means, for all a, b,

$$\tau(ab) = \tau(ba), \quad \text{and} \quad \tau(aa^*) > 0 \quad \text{if } a \neq 0.$$

Checking the faithfulness of the trace τ on the dense subalgebra $\mathbb{C}\Gamma$ is straightforward. We refer to [66] for its faithfulness on $C_r^*\Gamma$ in general.

The *full* group C^*-algebra of Γ is the norm completion of $\mathbb{C}\Gamma$ under the norm

$$\|\xi\| = \sup\{\|\pi(\xi)\|;\ \pi \text{ is a } *\text{-representation of } \mathbb{C}\Gamma\},$$

where by a $*$-representation we mean a $*$-representation on a Hilbert space. Note that $\|\xi\|$ is finite since for any $*$-representation π we have

$$\|\pi(\xi)\| \leq \sum \|\xi(g)\pi(g)\| = \sum |\xi(g)|.$$

By its very definition it is clear that there is a one-to-one correspondence between unitary representations of Γ and C^* representations of $C^*\Gamma$. Since the identity map $\mathrm{id}\colon (\mathbb{C}\Gamma, \|\ \|) \to (\mathbb{C}\Gamma, \|\ \|_r)$ is continuous, we obtain a surjective C^*-algebra homomorphism

$$C^*\Gamma \to C_r^*\Gamma.$$

It is known that this map is an isomorphism if and only if Γ is an amenable group [15]. Abelian groups, compact groups and solvable groups are amenable. Non-abelian free groups, on the other hand, are not amenable.

Let now Γ be an abelian group. Then $C^*\Gamma$ $(= C_r^*\Gamma)$ is a unital commutative C^*-algebra and so by the Gelfand–Naimark theorem it is isomorphic to the algebra of continuous functions on a compact Hausdorff space X. It is easy to describe X directly in terms of Γ. Let $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{T})$ be the group of unitary *characters* of Γ, also known as the *Pontryagin dual* of Γ. Under pointwise multiplication and with the compact-open topology, it is a compact topological group and it is easy to see that it is homeomorphic to the space of characters of the commutative C^*-algebra $C^*\Gamma$. Thus the Gelfand transform (1.5) coincides with the *Fourier transform* and defines a C^*-algebra isomorphism

$$C^*\Gamma \simeq C(\hat{\Gamma}). \tag{1.6}$$

Under this isomorphism, the canonical trace τ on the left-hand side can be identified with the normalized Haar measure on $C(\hat{\Gamma})$. As a special case, for $\Gamma = \mathbb{Z}^n$, we obtain an isomorphism of C^*-algebras

$$C^*\mathbb{Z}^n \simeq C(\mathbb{T}^n).$$

The isomorphism (1.6) identifies the group C^*-algebra of an abelian group Γ with the 'algebra of coordinates' on the unitary dual of Γ. When Γ is noncommutative, the unitary dual is a badly behaved space in general, but the noncommutative dual $C^*\Gamma$ is a perfectly legitimate noncommutative space (see the unitary dual of the infinite dihedral group in [41] and its noncommutative replacement).

We look at another special case. When the group Γ is finite the group C^*-algebra coincides with the group algebra $\mathbb{C}\Gamma$. From basic representation theory we know that the group algebra $\mathbb{C}\Gamma$ decomposes as a direct sum of matrix algebras

$$C^*\Gamma \simeq \mathbb{C}\Gamma \simeq \oplus M_{n_i}(\mathbb{C}),$$

where the summation is over the set of conjugacy classes of Γ.

We extend the definition of group C^*-algebras to topological groups. Let G be a locally compact topological group and μ a left Haar measure on G. For $f, g \in L^1(G, \mu)$, their *convolution product* $f * g$ is defined by

$$(f * g)(t) = \int_G f(s) g(s^{-1}t)\, d\mu(s).$$

Under this convolution product, $L^1(G, \mu)$ is a Banach $*$-algebra. The $*$-structure is defined by $f^*(t) := \Delta_G(t^{-1})\bar{f}(t^{-1})$ for all $f \in L^1(G, \mu)$. Here $\Delta_G \colon G \to \mathbb{R}^\times$ is the modular character of G. The *left regular representation* λ of $L^1(G)$ on $L^2(G)$ is defined by

$$(\lambda(f)\xi)(s) = \int_G f(t)\xi(t^{-1}s)\, d\mu(t).$$

It is an (injective) $*$-representation of $L^1(G)$. The reduced group C^*-algebra of G, $C_r^*(G)$, is, by definition, the C^*-algebra generated by the image of λ in $\mathcal{L}(L^2(G))$. The full group C^*-algebra of G, $C^*(G)$, is defined as in the discrete case as the completion of $L^1(G)$ under the norm

$$\|f\| = \sup\{\|\pi(f)\|;\ \pi \text{ is a } *\text{-representation of } L^1(G)\}.$$

The estimate $\|\pi(f)\| \leq \|f\|_1$ shows that $\|f\| < \infty$ for all $f \in L^1(G)$.

Now if G is a locally compact *abelian group*, the Gelfand–Naimark theorem shows that $C^*(G) \simeq C_0(\hat{G})$, where $\hat{G}$, the Pontryagin dual of G, is the locally compact abelian group of continuous characters of G. Notice that in this case the Gelfand transform coincides with the Fourier transform for locally compact abelian groups.

Example 1.1.7 (From quantum mechanics to noncommutative spaces; noncommutative tori)**.** One of the most intensively studied noncommutative spaces is a class of algebras known as *noncommutative tori*. They provide a testing ground for many ideas and techniques of noncommutative geometry. As we shall gradually see in this book, these algebras can be defined in a variety of ways, e.g. as the C^*-algebra of the Kronecker foliation of the two-torus by lines of constant slope $dy = \theta dx$; as the crossed product algebra $C(S^1) \rtimes \mathbb{Z}$ associated to the automorphism of the circle by rotating by an angle $2\pi\theta$; as strict deformation quantization; as a twisted group algebra; or by generators and relations as we define them now. First, a connection with quantum mechanics.

The so called *canonical commutation relation* of quantum mechanics $pq - qp = \frac{h}{2\pi i}1$ relates the position q and momentum p operators. It can be realized by unbounded selfadjoint operators but has no representation by bounded operators. In fact the selfadjoint unbounded operators $q, p \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(qf)(x) = xf(x)$ and $(pf) = \frac{h}{2\pi i}\frac{d}{dx}(f)$ are easily seen to satisfy this relation. Weyl proposed an *integrated* bounded form of the commutation relation which, among other things, provides an opening to one of the most intensively studied noncommutative spaces, that is, noncommutative tori. Let $U_t = e^{itp}$ and $V_s = e^{isq}$, be the one parameter groups of unitary operators generated by the selfadjoint operators p and q. Then, using the canonical commutation relation, it is easy to check that these unitary operators satisfy the relation

$$V_s U_t = e^{2\pi i \hbar st} U_t V_s,$$

where $\hbar = \frac{h}{2\pi}$.

Let $\theta \in \mathbb{R}$ and $\lambda = e^{2\pi i\theta}$. The *noncommutative torus* A_θ is the *universal* unital C^*-algebra generated by unitaries U and V subject to the following relation:

$$\boxed{VU = \lambda UV} \tag{1.7}$$

By universality we mean the following property: given any unital C^*-algebra B with two unitaries u and v satisfying $vu = \lambda uv$, there exists a unique unital C^*-morphism $A_\theta \to B$ sending U to u and V to v.

Unlike the purely algebraic case where any set of generators and relations automatically defines a universal algebra, this is not the case for universal C^*-algebras. Care must be applied in defining a norm satisfying the C^*-identity, and in general the universal problem does not have a solution (cf. [15] and Exercise 1.1.6 at the end of this section). For the noncommutative torus we proceed as follows. Consider the unitary operators $U, V \colon L^2(S^1) \to L^2(S^1)$ defined by

$$(Uf)(x) = e^{2\pi ix} f(x), \quad (Vf)(x) = f(x + \theta). \tag{1.8}$$

Here we think of S^1 as $\mathbb{R}/\mathbb{Z}$ with its canonical normalized Haar measure. They clearly satisfy the relation $VU = \lambda UV$. Let A_θ be the unital C^*-subalgebra of $\mathcal{L}(L^2(S^1))$ generated by U and V. It is not difficult to show that it satisfies the required universal property.

Let $\mathcal{O}(\mathbb{T}^2_\theta) := \mathbb{C}\langle U, V\rangle/(VU - \lambda UV)$ denote the unital $*$-algebra generated by unitaries U and V subject to the relation $VU = \lambda UV$. We think of $\mathcal{O}(\mathbb{T}^2_\theta)$ as the coordinate ring of an algebraic noncommutative torus. We often think of it as the dense subalgebra of A_θ generated by U and V.

Let $e_n = e^{2\pi inx}$, $n \in \mathbb{Z}$. They form an orthonormal basis for $L^2(S^1)$. It can be shown ([151]) that the formula

$$\tau(a) = \langle ae_0, e_0\rangle \tag{1.9}$$

defines a *positive* and *faithful trace* $\tau \colon A_\theta \to \mathbb{C}$. That is, for all a, b we have

$$\tau(ab) = \tau(ba), \quad \text{and} \quad \tau(aa^*) > 0 \quad \text{if } a \neq 0.$$

Using the relations $Ue_n = e_{n+1}$ and $Ve_n = e^{2\pi i\theta} e_n$ one checks that on the dense subalgebra $\mathcal{O}(\mathbb{T}^2_\theta)$ we have

$$\tau\Big(\sum_{m,n} a_{mn} U^m V^n\Big) = a_{00}.$$

The structure of A_θ strongly depends on θ. Of course, for any integer n, $A_{\theta+n} \simeq A_\theta$, and for $\theta = 0$ simple Fourier theory shows that A_0 is isomorphic to the algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. We also have the isomorphism $A_\theta \simeq A_{1-\theta}$ induced by the map sending U to V and V to U. Thus we can restrict the range of θ to $[0, \frac{1}{2}]$. It is known that for distinct θ in this range the algebras A_θ are mutually non-isomorphic. It is also known that for *irrational* θ, A_θ is a simple C^*-algebra, i.e., it has no proper closed two-sided ideal. In particular it has no finite dimensional representation.

Let $\theta = \frac{p}{q}$ be a rational number, where we assume that p and q are relatively prime and $q > 0$. Then A_θ has a finite dimensional representation and in fact we have the following

Proposition 1.1.1. *There is a flat rank q complex vector bundle E on the 2-torus such that $A_{\frac{p}{q}}$ is isomorphic to the algebra of continuous sections of the endomorphism bundle of E:*

$$A_{\frac{p}{q}} \simeq C(\mathbb{T}^2, \mathrm{End}(E)).$$

Proof. The required bundle E is obtained as the quotient of the trivial bundle $\mathbb{T}^2 \times \mathbb{C}^q$ by a *free* action of the abelian group $G = \mathbb{Z}_q \times \mathbb{Z}_q$. Consider the unitary $q \times q$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 0 & \lambda^{q-1} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}. \tag{1.10}$$

They satisfy the relations

$$vu = \lambda uv \quad \text{and} \quad u^q = v^q = 1. \tag{1.11}$$

The formulae

$$\mathbb{T}^2 \times \mathbb{C}^q \ni (z_1,\, z_2,\, \xi) \mapsto (\lambda z_1,\, z_2,\, u\xi),$$
$$\mathbb{T}^2 \times \mathbb{C}^q \ni (z_1,\, z_2,\, \xi) \mapsto (z_1,\, \lambda z_2,\, v\xi)$$

define a pair of *commuting* automorphisms of order q of the trivial vector bundle $\mathbb{T}^2 \times \mathbb{C}^q$ and hence an action of $\mathbb{Z}_q \times \mathbb{Z}_q$. This action is clearly free. Moreover the quotient of the base space is again the torus and in this way we obtain a flat bundle E over $\mathbb{T}^2$. From its definition it is clear that the space of sections of $\mathrm{End}(E))$ is the *fixed point algebra* of the induced action of G on $C(\mathbb{T}^2,\, M_q(\mathbb{C}))$ under the action of G. Using the basis $u^i v^j$, $1 \leq i, j \leq q$ for $M_q(\mathbb{C})$, we can write a section of this bundle as $\sum_{i,j=1}^q f_{ij}(z_1, z_2) \otimes u^i v^j$. It is then easy to see that such a section is G-invariant if and only if its coefficients are of the form $f_{ij}(z_1^q, z_2^q)$.

Now within $A_{\frac{p}{q}}$, we have $U^qV = VU^q$ and $V^qU = UV^q$ which show that U^q and V^q are in the *center* of $A_{\frac{p}{q}}$. Also, any element of $A_{\frac{p}{q}}$ has a unique expression

$$a = \sum_{i,j=1}^{q} f_{ij}(U^q, V^q)\, U^i V^j$$

with $f_{ij} \in C(\mathbb{T}^2)$. Now the required isomorphism is defined by the map

$$\sum_{i,j=1}^{q} f_{ij}(U^q, V^q)\, U^i V^j \longmapsto \sum_{i,j=1}^{q} f_{ij}(z_1^q, z_2^q) \otimes u^i v^j. \qquad \square$$

Notice that the above proof shows that the closed subalgebra generated by U^q and V^q is in fact all of the center of $A_{\frac{p}{q}}$:

$$Z(A_{\frac{p}{q}}) \simeq C(\mathbb{T}^2).$$

There is a dense $*$-subalgebra $\mathcal{A}_\theta \subset A_\theta$ that deserves to be called the algebra of *'smooth functions'* on the noncommutative torus. By definition $a \in \mathcal{A}_\theta$ if it is of the form

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n,$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is a rapidly decreasing *Schwartz class* sequence. By definition this means

$$\sup_{m,n\in\mathbb{Z}} (1 + m^2 + n^2)^k \, |a_{mn}| < \infty \quad \text{for all } k \in \mathbb{N}. \tag{1.12}$$

Notice that for $\theta = 0$, condition (1.12) on the Fourier coefficients of a function $f \in C(\mathbb{T}^2)$ is necessary and sufficient for f to be in $C^\infty(\mathbb{T}^2)$. This is one rationale to call $\mathcal{A}_\theta$ the algebra of smooth functions on the noncommutative torus. Let us denote the algebras $\mathcal{A}_\theta$ and A_θ by $C^\infty(\mathbb{T}^2_\theta)$ and $C(\mathbb{T}^2_\theta)$, respectively. So we have the hierarchy of algebras

$$\mathcal{O}(\mathbb{T}^2_\theta) \subset C^\infty(\mathbb{T}^2_\theta) \subset C(\mathbb{T}^2_\theta)$$

resembling the hierarchy of algebras of functions *algebraic* $\subset$ *smooth* $\subset$ *continuous* in the noncommutative world. If $\theta = \frac{p}{q}$ is rational, the proof of Theorem 1.1.1 shows that $\mathcal{A}_{\frac{p}{q}}$ is isomorphic to the space of smooth sections of the bundle $\mathrm{End}(E)$ over $\mathbb{T}^2$:

$$\mathcal{A}_{\frac{p}{q}} \simeq C^\infty(\mathbb{T}^2, \mathrm{End}(E)).$$

A *derivation* on an algebra A is a $\mathbb{C}$-linear map $\delta\colon A \to A$ such that $\delta(ab) = a\delta(b)+\delta(a)b$ for all $a, b \in A$. (Cf. Chapter 3 for a more general notion of derivation.) Notice that a derivation is determined by its values on a set of generators for the algebra. It is easy to see that the following formulae define derivations $\delta_1, \delta_2 \colon \mathcal{A}_\theta \to \mathcal{A}_\theta$:

$$\delta_1\Big(\sum a_{mn}U^mV^n\Big) = 2\pi i \sum m a_{mn}U^mV^n,$$
$$\delta_2\Big(\sum a_{mn}U^mV^n\Big) = 2\pi i \sum n a_{mn}U^mV^n.$$

They are uniquely defined by the following relations:

$$\delta_1(U) = 2\pi i U, \qquad \delta_1(V) = 0,$$
$$\delta_2(U) = 0, \qquad \delta_2(V) = 2\pi i V.$$

The trace τ defined by (1.9) has a beautiful invariance property which is the noncommutative analogue of the invariance property of the Haar measure for the torus. Indeed, it is easy to check that

$$\tau(\delta_i(a)) = 0 \quad \text{for all } a \in \mathcal{A}_\theta \text{ and } i = 1, 2.$$

Note that δ_1 and δ_2 are $*$-derivations in the sense that $\delta_i(a^*) = \delta_i(a)^*$ for all $a \in \mathcal{A}_\theta, i = 1, 2$. These derivations generate commuting one-parameter groups of C^*-automorphisms of A_θ and in fact a continuous action of the 2-torus $\mathbb{T}^2$ on A_θ which on generators is defined by

$$(z_1, z_2)U = z_1 U, \quad (z_1, z_2)V = z_2 V.$$

Exercise 1.1.1. Show that the map

$$X \to \widehat{C_0(X)}, \quad x \mapsto \chi_x,$$

in Example 1.1.1 is a homeomorphism of topological spaces.

Exercise 1.1.2. Show that the category of commutative C^*-algebras and C^*-morphisms is anti-equivalent to the category of *pointed* compact Hausdorff spaces and base point preserving continuous maps.

Exercise 1.1.3. Show that the C^*-algebra generated by two unitaries u and v subject to relations (1.11) is isomorphic to $M_q(\mathbb{C})$.

Exercise 1.1.4. Let θ be an irrational number. Show that $Z(\mathcal{A}_\theta) = \mathbb{C}1$, where Z denotes the center, and that any trace on $\mathcal{A}_\theta$ is a multiple of the canonical trace τ.

Exercise 1.1.5. Assume H is infinite dimensional. Show that any trace on the algebra $\mathcal{L}(H)$ of bounded operators on H vanishes identically. The same holds for traces on the algebra $\mathcal{K}(H)$ of compact operators on H.

Exercise 1.1.6. An element u in an involutive unital algebra is called a *unitary* if $u^*u = uu^* = 1$. Show that $C(S^1)$ is the universal C^*-algebra generated by a unitary. Give similar descriptions for $C(S^n)$ (continuous functions on the n-sphere) for all n. Show that there is *no* universal C^*-algebra generated by a single selfadjoint element. (Hint: In $C[0, a]$ the function $f(x) = x$ is selfadjoint and has norm $\|f\| = a$).

1.2 Vector bundles, finite projective modules, and idempotents

Swan's theorem [166] states that the category of complex vector bundles on a compact Hausdorff space X is equivalent to the category of finite (i.e., finitely generated) projective modules over the algebra $C(X)$ of continuous complex valued functions on X:

$$\boxed{\{\text{vector bundles on } X\} \simeq \{\text{finite projective } C(X)\text{-modules}\}} \tag{1.13}$$

There are similar results for real and quaternionic vector bundles [166]. Swan's theorem was motivated and in fact is the topological counterpart of an analogous earlier result, due to Serre [163], which characterizes algebraic vector bundles over an affine algebraic variety as finite projective modules over the coordinate ring of the variety. The two theorems are collectively referred to as the *Serre–Swan theorem.*

Motivated by these results, we can think of a finite projective module E over a not necessarily commutative algebra A as a *noncommutative vector bundle* over the noncommutative space represented by A:

$$\boxed{\{\text{NC vector bundles on A}\} := \{\text{finite projective } A\text{-modules}\}} \tag{1.14}$$

That this is a useful point of view is completely justified, as will be indicated in this book, by a rich source of examples, a powerful topological K-theory based on noncommutative vector bundles, the existence of a noncommutative Chern–Weil theory, and a viable Yang–Mills theory on noncommutative vector bundles.

Let us explain the Serre–Swan correspondence (1.13) between vector bundles and finite projective modules. Recall that a right module P over a unital algebra A is called *projective* if there exists a right A-module Q such that

$$P \oplus Q \simeq A^I$$

is a *free* A-module. Equivalently, P is projective if every A-module surjection $P \to Q \to 0$ splits as a right A-module map. P is called *finite* if there exists a surjection $A^n \to P \to 0$ for some integer n. Thus P is finite projective if and only if there is an integer n and a module Q such that

$$P \oplus Q \simeq A^n.$$

Given a vector bundle $p\colon E \to X$, let

$$P = \Gamma(E) = \{s\colon X \to E;\ ps = \mathrm{id}_X\}$$

be the set of all continuous *global sections* of E. It is clear that, under fiberwise scalar multiplication and addition, P is a $C(X)$-module. If $f\colon E \to F$ is a bundle map, we define a module map $\Gamma(f)\colon \Gamma(E) \to \Gamma(F)$ by $\Gamma(f)(s)(x) = f(s(x))$ for all $s \in \Gamma(E)$ and $x \in X$. We have thus defined a functor Γ, called the global section functor, from the category of vector bundles over X and continuous bundle maps to the category of $C(X)$-modules and module maps.

Using compactness of X and a partition of unity one shows that there is a vector bundle F on X such that $E \oplus F \simeq X \times \mathbb{C}^n$ is a trivial bundle. Let Q be the space of global sections of F. We have

$$P \oplus Q \simeq A^n,$$

which shows that P is finite projective.

To show that all finite projective $C(X)$-modules arise in this way we proceed as follows. Given a finite projective $C(X)$-module P, let Q be a $C(X)$-module such that $P \oplus Q \simeq A^n$, for some integer n. Let $e\colon A^n \to A^n$ be the right A-linear map corresponding to the projection onto the first coordinate: $(p, q) \mapsto (p, 0)$. It is obviously an idempotent in $M_n(C(X))$. Since the rank of an idempotent in $M_n(\mathbb{C})$ is equal to its trace, the rank of the family $e(x)$ is continuous in x and hence is locally constant. This shows that one can define a vector bundle E as the image of this idempotent e, and as a subbundle of the trivial bundle $X \times \mathbb{C}^n$:

$$E = \{(x, v);\ e(x)v = v \text{ for all } x \in X,\ v \in \mathbb{C}^n\} \subset X \times \mathbb{C}^n.$$

Now it is easily shown that $\Gamma(E) \simeq P$. With some more work it is shown that the functor Γ is full and faithful and hence defines an equivalence of categories. This finishes the proof of Swan's theorem and now we start looking at (1.14) in earnest.

Where do finite projective modules come from? We can show that they are all constructed from *idempotents* in matrix algebras over the given algebra. Let A be a unital algebra and let $M_n(A)$ denote the algebra of n by n matrices with entries in A. If we think of A^n as a right A-module then clearly $M_n(A) = \mathrm{End}_A(A^n)$. Let

$$e \in M_n(A), \quad e^2 = e,$$

be an idempotent. Left multiplication by e defines a right A-module map

$$e\colon A^n \to A^n, \quad \xi \mapsto e\xi,$$

where we think of A^n as the space of column matrices. Let

$$P = eA^n \quad \text{and} \quad Q = (1 - e)A^n$$

be the image and kernel of this map. Then, using the idempotent condition $e^2 = e$, we obtain a direct sum decomposition

$$P \oplus Q = A^n,$$

which shows that both P and Q are projective modules. Moreover, they are obviously finitely generated. It follows that both P and Q are finite projective modules.

Conversely, given any finite projective right A-module P, let Q be a module such that $P \oplus Q \simeq A^n$ for some integer n. Let $e\colon A^n \to A^n$ be the right A-module map that corresponds to the projection map

$$(p, q) \mapsto (p, 0).$$

Then it is easily seen that we have an isomorphism of A-modules

$$P \simeq eA^n.$$

We have shown that any idempotent $e \in M_n(A)$ defines a finite projective A-module and that all finite projective A-modules are obtained from an idempotent in some matrix algebra over A.

The idempotent $e \in M_n(A)$ associated to a finite projective A-module P depends of course on the choice of the splitting $P \oplus Q \simeq A^n$. Let $P \oplus Q' \simeq A^m$ be another splitting and $f \in M_m(A)$ the corresponding idempotent. Define the operators $u \in \mathrm{Hom}_A(A^m, A^n)$, $v \in \mathrm{Hom}_A(A^n, A^m)$ as compositions

$$u\colon A^m \xrightarrow{\sim} P \oplus Q \to P \to P \oplus Q' \xrightarrow{\sim} A^n,$$

$$v\colon A^n \xrightarrow{\sim} P \oplus Q' \to P \to P \oplus Q \xrightarrow{\sim} A^m.$$

We have

$$uv = e, \quad vu = f.$$

In general, two idempotents satisfying the above relations are called *Murray–von Neumann equivalent*. Conversely, it is easily seen that Murray–von Neumann equivalent idempotents define isomorphic finite projective modules.

Here are a few examples starting with a commutative one.

Example 1.2.1. The *Hopf line bundle* on the two-sphere S^2, also known as the *magnetic monopole bundle*, can be defined in various ways. (It was discovered, independently, by Hopf and Dirac in 1931, motivated by very different considerations.) Here is an approach that lends itself to noncommutative generalizations. Let σ_1, σ_2, σ_3, be three matrices in $M_2(\mathbb{C})$ that satisfy the *canonical anticommutation relations*:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

for all $i, j = 1, 2, 3$. Here δ_{ij} is the Kronecker symbol. A canonical choice is the so called *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define a function

$$F\colon S^2 \to M_2(\mathbb{C}), \quad F(x_1, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where x_1, x_2, x_3 are coordinate functions on S^2, so that $x_1^2 + x_2^2 + x_3^2 = 1$. Then $F^2(x) = 1$, the identity matrix, for all $x \in S^2$, and therefore

$$e = \frac{1 + F}{2}$$

is an idempotent in $C(S^2, M_2(\mathbb{C})) \simeq M_2(C(S^2))$. It thus defines a complex vector bundle on S^2. We have

$$e(x_1, x_2, x_3) = \frac{1}{2}\begin{pmatrix} 1 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & 1 - x_3 \end{pmatrix}.$$

Since

$$\operatorname{rank} e(x) = \operatorname{trace} e(x) = 1$$

for all $x \in S^2$, we have in fact a complex line bundle over S^2. It can be shown that it is the line bundle associated to the Hopf fibration

$$S^1 \to S^3 \to S^2.$$

Incidentally, e induces a map $f\colon S^2 \to P^1(\mathbb{C})$, where $f(x)$ is the 1-dimensional subspace defined by the image of $e(x)$, which is one-to-one and onto. Our line bundle is just the pull back of the canonical line bundle over $P^1(\mathbb{C})$.

This example can be generalized to higher-dimensional spheres. One can construct matrices $\sigma_1, \ldots, \sigma_{2n+1}$ in $M_{2^n}(\mathbb{C})$ satisfying the *Clifford algebra* relations [103]

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

for all $i, j = 1, \ldots, 2n+1$. Define a matrix-valued function F on the $2n$-dimensional sphere S^{2n}, $F \in M_{2^n}(C(S^{2n}))$, by

$$F = \sum_{i=1}^{2n+1} x_i \sigma_i. \tag{1.15}$$

Then $F^2(x) = 1$ for all $x \in S^{2n}$, so that $e = \frac{1+F}{2}$ is an idempotent and defines a vector bundle over S^{2n}.

Example 1.2.2 (Hopf line bundle on quantum spheres)**.** The Podleś quantum sphere S_q^2 is the C^*-algebra generated by the elements a, a^* and b subject to the relations

$$aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1, \quad ab = q^{-2}ba, \quad a^*b = q^2ba^*.$$

The quantum analogue of the Dirac (or Hopf) monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$ [91] (cf. also [24]):

$$\mathbf{e}_q = \frac{1}{2}\begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

It can be directly checked that $\mathbf{e}_q^2 = \mathbf{e}_q$. Similar to the commutative case for S^2, for any integer $n \in \mathbb{Z}$ there is a quantum 'line bundle' with 'topological charge' n over S_q^2. We refer to [91] for its explicit description in terms of projections.

Is there a noncommutative analogue of the Hopf 2-plane bundle over the 4-sphere S^4, associated to the principal SU(2)-bundle $\mathrm{SU}(2) \to S^7 \to S^4$? The answer is positive and we refer to the survey [119] and references therein for its description.

Example 1.2.3 (Projective modules on noncommutative tori)**.** The noncommutative torus A_θ and its dense subalgebra $\mathcal{A}_\theta$ were defined in Example 1.1.7. We saw that when θ is rational A_θ (resp. $\mathcal{A}_\theta$) is isomorphic to the algebra of continuous (resp. smooth) sections of a bundle of full matrix algebras on the torus $\mathbb{T}^2$. This in particular implies that A_θ (resp. $\mathcal{A}_\theta$) is Morita equivalent to the commutative algebra $C(\mathbb{T}^2)$ (resp. $C^\infty(\mathbb{T}^2)$). Morita equivalent algebras, to be defined in the next chapter, have equivalent categories of modules, projective modules, and finite projective modules. Using Swan's theorem, this implies that, for rational θ, there is a one-to-one correspondence between (isomorphism classes of) finite projective modules on A_θ and vector bundles on $\mathbb{T}^2$.

Notice that, since $\mathbb{T}^2$ is connected, for $\theta \in \mathbb{Z}$, $A_\theta = C(\mathbb{T}^2)$ does not contain a non-trivial idempotent. For $\theta \notin \mathbb{Z}$, the existence of a non-trivial idempotent in A_θ was eventually settled by the following *Powers–Rieffel projection*, as we briefly recall next. Let $0 < \theta \leq \frac{1}{2}$. Consider the following element of A_θ:

$$p = f_{-1}(U)V^{-1} + f_0 + f_1(U)V,$$

where $f_{-1},\ f_0,\ f_1$ are in $C^\infty(\mathbb{R}/\mathbb{Z})$. Using the concrete realization of U and V given by (1.8) one can show that the conditions $p^2 = p = p^*$ are equivalent to the following relations

$$\begin{aligned} f_1(t)f_1(t-\theta) &= 0, \\ f_1(t)f_0(t-\theta) &= (1 - f_0(t))f_1(t), \\ |f_1(t)|^2 + |f_1(t+\theta)|^2 &= f_0(t)(1 - f_0(t)). \end{aligned}$$

It is rather elementary to see that these equations have (many) solutions and in this way one obtains a non-trivial projection in $\mathcal{A}_\theta$ (cf. [41], [85] for concrete examples). Computing the trace of this projection is instructive. This is carried out in [85] and we just cite the final result:

$$\tau(p) = \int_0^1 f_0(t)\,dt = \int_0^\theta f_0(t)\,dt + \int_0^\theta (1 - f_0(t))\,dt = \theta.$$

Let $E = \mathcal{S}(\mathbb{R})$ be the *Schwartz space* of *rapidly decreasing functions* on $\mathbb{R}$, where a function f is called rapidly decreasing if for all its derivatives $f^{(n)}$, and all $k \in \mathbb{N}$, there is a constant C, depending on n and k, such that

$$|f^{(n)}(x)|(1 + x^2)^k < C \quad \text{for all } x \in \mathbb{R}.$$

The following formulas define a left $\mathcal{A}_\theta$-module structure on E:

$$(Uf)(x) = f(x - \theta), \quad (Vf)(x) = e^{2\pi i x} f(x).$$

It can be shown that E is finitely generated and projective [35]. Using E and the following observation we can construct more $\mathcal{A}_\theta$-modules.

Let E_1 (resp. E_2) be left $\mathcal{A}_{\theta_1}$- (resp. $\mathcal{A}_{\theta_2}$-) modules, where the generators U

and V of $\mathcal{A}_\theta$ act by U_1 and V_1 (resp. U_2 and V_2). The following formulas define a left action of $\mathcal{A}_{\theta_1+\theta_2}$ on $E_1 \otimes E_2$:

$$U(\xi_1 \otimes \xi_2) = U_1\xi_1 \otimes U_2\xi_2, \quad V(\xi_1 \otimes \xi_2) = V_1\xi_1 \otimes V_2\xi_2. \tag{1.16}$$

For each pair of integers p, q with $q > 0$, the $q \times q$ matrices u and v defined by (1.10) define a finite dimensional representation of $\mathcal{A}_{\frac{p}{q}}$ on the vector space $E'_{p,q} = \mathbb{C}^n$. Now we can take $\theta_1 = \theta - \frac{p}{q}$ and $\theta_2 = \frac{p}{q}$ in (1.16) and obtain a sequence of $\mathcal{A}_\theta$-modules

$$E_{p,q} = E\left(\theta - \frac{p}{q}\right) \otimes E'_{p,q}.$$

We give an equivalent definition of $E_{p,q}$ [35], [59]. Let $E_{p,q} = \mathcal{S}(\mathbb{R} \times \mathbb{Z}_q)$, where $\mathbb{Z}_q$ is the cyclic group of order q. The following formulas define an $\mathcal{A}_\theta$-module structure on $E_{p,q}$:

$$(Uf)(x, j) = f\left(x + \theta - \frac{p}{q}, \, j - 1\right),$$
$$(Vf)(x, j) = e^{2\pi i (x - j\frac{p}{q})} f(x, \, j).$$

It can be shown that for $p - q\theta \neq 0$, the module $E_{p,q}$ is finite and projective. In particular for irrational θ it is always finite and projective.

For more examples of noncommutative vector bundles see [46], [85], [119].

Exercise 1.2.1. Let A be a unital algebra. Show that idempotents e and f in A are Murray–von Neumann equivalent if and only if the finite projective modules eA and fA are isomorphic (as right A-modules).

Exercise 1.2.2. Idempotents e and f in a unital algebra A are called *similar* if there is an invertible $u \in A$ such that $e = ufu^{-1}$. This is an equivalence relation, and clearly similarity implies Murray–von Neumann equivalence. Give examples of idempotents which are Murray–von Neumann equivalent but not similar. Find necessary and sufficient conditions for idempotents in $\mathcal{L}(H)$ to be Murray–von Neumann equivalent or similar.

Exercise 1.2.3. Compute the rank of the vector bundle defined by (1.15).

Exercise 1.2.4. Verify that (1.16) defines a left $\mathcal{A}_{\theta_1+\theta_2}$-module.

1.3 Affine varieties and finitely generated commutative reduced algebras

In algebraic geometry, *Hilbert's Nullstellensatz* [93], [33] immediately implies that the category of affine algebraic varieties over an algebraically closed field $\mathbb{F}$ is equivalent to the opposite of the category of finitely generated commutative reduced unital $\mathbb{F}$-algebras:

$$\boxed{\begin{array}{c}\{\text{affine algebraic varieties}\}\\ \simeq \\ \{\text{finitely generated commutative reduced algebras}\}^{\text{op}}\end{array}} \tag{1.17}$$

This is a perfect analogue of the Gelfand–Naimark correspondence (1.1) in the world of affine algebraic geometry.

An *affine algebraic variety* (sometimes called an *algebraic set*) over an algebraically closed field $\mathbb{F}$ is a subset $V \subset \mathbb{F}^n$ of an affine space which is the set of zeros of a collection I of polynomials in n variables over $\mathbb{F}$:

$$V = V(I) = \{z \in \mathbb{F}^n;\ p(z) = 0 \text{ for all } p \in I\}. \tag{1.18}$$

Without loss of generality we can assume that I is an ideal in $\mathbb{F}[x_1, \ldots, x_n]$. A morphism between affine varieties $V \subset \mathbb{F}^n$ and $W \subset \mathbb{F}^m$ is a map $f\colon V \to W$ which is the restriction of a polynomial map $\mathbb{F}^n \to \mathbb{F}^m$. It is clear that affine varieties and morphisms between them form a category. This is the category that appears on the left hand side of (1.17).

A *reduced algebra* is by definition an algebra with no *nilpotent elements*, i.e., if $x^n = 0$ for some n, then $x = 0$. Consider the category of unital finitely generated commutative and reduced algebras and unital algebra homomorphisms. This is the category that appears on the right-hand side of (1.17).

The equivalence (1.17) is defined by a functor that associates to a variety $V \subset \mathbb{F}^n$ its *coordinate ring* $\mathcal{O}[V]$ defined by

$$\mathcal{O}[V] := \mathrm{Hom}(V, \mathrm{pt}) \simeq \mathbb{F}[x_1, \ldots, x_n]/I,$$

where I is the *vanishing ideal* of V defined by

$$I = \{p \in \mathbb{F}[x_1, \ldots, x_n];\ p(x) = 0 \text{ for all } x \in V\}.$$

Obviously $\mathcal{O}[V]$ is a finitely generated commutative unital reduced algebra. Moreover, given a morphism of varieties $f\colon V \to W$, its pullback defines an algebra homomorphism $f^*\colon \mathcal{O}[W] \to \mathcal{O}[V]$. We have thus defined a contravariant functor

$$V \rightsquigarrow \mathcal{O}[V]$$

from affine varieties to finitely generated reduced commutative unital algebras.

Given a finitely generated commutative unital algebra A with n generators we can obviously write it as a quotient:

$$A \simeq \mathbb{F}[x_1, \ldots, x_n]/I.$$

Notice that A is a reduced algebra, i.e., it has no nilpotent elements, if and only if the ideal I is a *radical ideal* in the sense that if $x^n \in I$ then $x \in I$. One of the classical forms of the Nullstellensatz [93] states that if $\mathbb{F}$ is algebraically closed and

I is a radical ideal, then A can be recovered as the coordinate ring of the variety V defined by (1.18):

$$\mathcal{O}[V] \simeq A = \mathbb{F}[x_1, \ldots, x_n]/I.$$

This shows that the coordinate ring functor $V \mapsto \mathcal{O}[V]$ is essentially surjective in the sense of Appendix D and is the main step in establishing (1.17). Showing that the functor is full and faithful is much easier. In Appendix A we sketch a proof of the Nullstellensatz when $\mathbb{F}$ is the field of complex numbers.

Much as in the Gelfand–Naimark correspondence, under the correspondence (1.17) geometric constructions can be translated into algebraic terms and vice-versa. Thus, for example we have $\mathcal{O}[V_1] \oplus \mathcal{O}[V_2] \simeq \mathcal{O}[V_1 \cup V_2]$ (disjoint union); $\mathcal{O}[V_1] \otimes \mathcal{O}[V_2] \simeq \mathcal{O}[V_1 \times V_2]$; and V is irreducible if and only if $\mathcal{O}[V]$ is an integral domain. There are also various equivalent ways of characterizing smooth (nonsingular) varieties in terms of their coordinate rings.

Unlike the Gelfand–Naimark correspondence, at present the correspondence (1.17) does not seem to indicate what is the right notion of a *noncommutative affine variety*, or noncommutative (affine) algebraic geometry in general. There seems to be a lot remains to be done in this area, but we indicate one possible approach that has been pursued at least in the smooth case.

A particularly important characterization of non-singularity that lends itself to noncommutative generalization is the following result of Grothendieck explained in [124]: a variety V is smooth if and only if its coordinate ring $A = \mathcal{O}[V]$ has the *lifting property* with respect to *nilpotent extensions.* More precisely, V is smooth if and only if for any pair (C, I), where C is a commutative algebra and I is a nilpotent ideal, the map

$$\mathrm{Hom}_{\mathrm{alg}}(A, C) \to \mathrm{Hom}_{\mathrm{alg}}(A, C/I) \tag{1.19}$$

is surjective.

Motivated by this characterization of smoothness, in [61] a not necessarily commutative algebra A over $\mathbb{C}$ is called *NC smooth* (or *quasi-free*) by Cuntz and Quillen, if the lifting property (1.19) holds in the category of *all* algebras, i.e., C is now allowed to be noncommutative. Obviously a *free algebra* (also known as tensor algebra, or algebra of noncommutative polynomials) $A = T(W)$ is smooth in this sense. The bad news is that algebras which are smooth in the commutative world need not be smooth in this new sense. In fact one can show that an algebra is NC smooth if and only if it has Hochschild cohomological dimension one [61]. In particular polynomials in $n \geq 2$ variables and in general coordinate rings of smooth varieties of dimension $n \geq 2$ are not NC smooth. Nevertheless this notion of NC smoothness has played a very important role in the development of a version of *noncommutative algebraic geometry* in [113], [123].

An alternative approach to noncommutative algebraic geometry is proposed in [5] and references therein. One of the underlying ideas here is the *projective Nullstellensatz* theorem [93] that characterizes the graded coordinate ring of a projective variety defined as sections of powers of an ample line bundle over the variety. Thus in this approach a noncommutative variety is represented by a noncommutative graded ring with certain extra properties.

1.4 Affine schemes and commutative rings

The above correspondence (1.17) between finitely generated reduced commutative algebras and affine varieties is not an ideal result. One is naturally interested in larger classes of algebras, like algebras with nilpotent elements as well as algebras over fields which are not algebraically closed or algebras over arbitrary commutative rings; this last case is particularly important in number theory. In general one wants to know what kind of geometric objects correspond to a commutative ring and how this correspondence goes. *Affine schemes* are exactly defined to address this question. We follow the exposition in [93].

Let A be a commutative unital ring. The *prime spectrum* (or simply the *spectrum*) of A is a pair $(\operatorname{Spec} A, \mathcal{O}_A)$ where $\operatorname{Spec} A$ is a topological space and $\mathcal{O}_A$ is a sheaf of rings on $\operatorname{Spec} A$ defined as follows. As a set $\operatorname{Spec} A$ consists of all *prime ideals* of A (an ideal $I \subset A$ is called *prime* if $I \neq A$, and for all a, b in A, $ab \in A$ implies that either $a \in I$ or $b \in I$). Given an ideal $I \subset A$, let $V(I) \subset \operatorname{Spec} A$ be the set of all prime ideals which contain I. We can define a topology on $\operatorname{Spec} A$, called the *Zariski topology*, by declaring sets of the type $V(I)$ to be closed (this makes sense since the easily established relations $V(IJ) = V(I) \cup V(J)$ and $V\big(\sum I_i\big) = \bigcap V(I_i)$ show that the intersection of a family of closed sets is closed and the union of two closed sets is closed as well). One checks that $\operatorname{Spec} A$ is always compact but is not necessarily Hausdorff.

For each prime ideal $p \subset A$, let A_p denote the *localization* of A at p. For an open set $U \subset \operatorname{Spec} A$, let $\mathcal{O}_A(U)$ be the set of all continuous sections $s\colon U \to \bigcup_{p \in U} A_p$. (By definition a section s is called continuous if locally around any point $p \in U$ it is of the form $\frac{f}{g}$, with $g \notin p$). One checks that $\mathcal{O}_A$ is a sheaf of commutative rings on $\operatorname{Spec} A$.

Now $(\operatorname{Spec} A, \mathcal{O}_A)$ is a so-called *ringed space* and $A \mapsto (\operatorname{Spec} A, \mathcal{O}_A)$ is a functor called the *spectrum functor*. A unital ring homomorphism $f\colon A \to B$ defines a continuous map $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ by $f^*(p) = f^{-1}(p)$ for all prime ideals $p \subset B$. Note that if I is a maximal ideal $f^{-1}(I)$ is not necessarily maximal. This is one of the reasons one considers, for arbitrary rings, the prime spectrum and not the maximal spectrum, as we did in the case of commutative C^*-algebras.

An *affine scheme* is a ringed space $(X, \mathcal{O})$ such that X is homeomorphic to $\operatorname{Spec} A$ for a commutative ring A and $\mathcal{O}$ is isomorphic to $\mathcal{O}_A$. The spectrum functor defines an equivalence of categories:

$$\boxed{\{\text{affine schemes}\} \simeq \{\text{commutative rings}\}^{\text{op}}}$$

The inverse equivalence is given by the *global section functor* that sends an affine scheme to the ring of its global sections.

In the same vein categories of modules over a ring can be identified with categories of sheaves of modules over the spectrum of the ring. Let A be a commutative ring and let M be an A-module. We define a sheaf of modules $\mathcal{M}$ over $\operatorname{Spec} A$ as follows. For each prime ideal $p \subset A$, let M_p denote the localization of M at p. For any open set $U \subset \operatorname{Spec} A$ let $\mathcal{M}(U)$ denote the set of continuous sections $s\colon U \to \bigcup_p M_p$ (this means that s is locally a fraction $\frac{m}{f}$ with $m \in M$ and $f \in A_p$). One can recover M from $\mathcal{M}$ by showing that $M \simeq \Gamma\mathcal{M}$ is the space of global sections of $\mathcal{M}$. Sheaves of $\mathcal{O}_A$-modules on Spec A obtained in this way are called *quasi-coherent sheaves*. They are local models for a more general notion of quasi-coherent sheaves on arbitrary schemes. The functors $M \mapsto \mathcal{M}$ and $\mathcal{M} \mapsto \Gamma\mathcal{M}$ define an equivalence of categories [93]:

$$\{\text{modules over } A\} \simeq \{\text{quasi-coherent sheaves on Spec } A\}$$

Based on this correspondence, given a, not necessarily commutative, algebra A, we can think of the category of A-modules as a replacement for the category of quasi-coherent sheaves over the noncommutative space represented by A. This is a very fruitful idea in the development of the subject of noncommutative algebraic geometry, about which we shall say nothing in this book (see [5], [113], [123]).

1.5 Compact Riemann surfaces and algebraic function fields

It can be shown that the category of compact Riemann surfaces is equivalent to the opposite of the category of algebraic function fields:

$$\{\text{compact Riemann surfaces}\} \simeq \{\text{algebraic function fields}\}^{\text{op}}$$

For a proof of this correspondence see, e.g., [78], Section IV.11.

A *Riemann surface* is a complex manifold of complex dimension one. A morphism between Riemann surfaces X and Y is a holomorphic map $f\colon X \to Y$. An *algebraic function field* is a finite extension of the field $\mathbb{C}(x)$ of rational functions in one variable. A morphism of function fields is simply an algebra map.

To a compact Riemann surface one associates the field $M(X)$ of meromorphic functions on X. For example the field of meromorphic functions on the Riemann sphere is the field of rational functions $\mathbb{C}(x)$. In the other direction, to a finite extension of $\mathbb{C}(x)$ one associates the compact Riemann surface of the algebraic function $p(z, w) = 0$. Here w is a generator of the field over $\mathbb{C}(x)$. This correspondence is essentially due to Riemann. Despite its depth and beauty, this correspondence so far has not revealed any way of finding the noncommutative analogue of complex geometry.

1.6 Sets and Boolean algebras

Perhaps the simplest notion of space of any kind, free of any extra structure, is the notion of a set. In a sense set theory can be regarded as the geometrization of logic. There is a duality between the category of sets and set maps, and the category of complete atomic Boolean algebras [7]:

$$\boxed{\{\text{sets}\} \simeq \{\text{complete atomic Boolean algebras}\}^{\text{op}}}$$

A *Boolean algebra* is a unital ring B in which $x^2 = x$ for all x in B. A Boolean algebra is necessarily commutative as can be easily shown. One defines an order relation on B by declaring $x \leq y$ if there is an y' such that $x = yy'$. It can be checked that this is in fact a partial order relation on B. An *atom* in a Boolean algebra is an element x such that there is no y with $0 < y < x$. A Boolean algebra is *atomic* if every element x is the supremum of all the atoms smaller than x. A Boolean algebra is *complete* if every subset has a supremum and infimum. A morphism of complete Boolean algebras is a unital ring map which preserves all infs and sups. (Of course, any unital ring map between Boolean algebras preserves finite sups and infs).

Now, given a set S let

$$B = \mathbf{2}^S = \{f : S \to \mathbf{2}\},$$

where $\mathbf{2} := \{0, 1\}$. Note that B is a complete atomic Boolean algebra. Any map $f : S \to T$ between sets defines a morphism of complete atomic Boolean algebras via pullback: $f^*(g) := g \circ f$, and

$$S \rightsquigarrow \mathbf{2}^S$$

is a contravariant functor from the category of sets to the category of complete atomic Boolean algebras.

In the opposite direction, given a Boolean algebra B, one defines its *spectrum* $\hat{B}$ by

$$\hat{B} = \mathrm{Hom}_{\mathrm{Boolean}}(B, \mathbf{2}),$$

where we now think of $\mathbf{2}$ as a Boolean algebra with two elements. Any algebra map $f : B \to C$ induces a set map $\hat{f} : \hat{C} \to \hat{B}$ by $\hat{f}(\chi) = \chi \circ f$ for all $\chi \in \hat{C}$. It can be shown that the two functors that we have just defined are anti-equivalences of categories, quasi-inverse to each other. Thus once again we have a duality between a certain category of geometric objects, namely sets, and a category of commutative algebras, namely complete atomic Boolean algebras. This result is a special case of the *Stone duality* between Boolean algebras and a certain class of topological spaces [102].

This result, unfortunately, does not indicate a way of extending the notion of a set to some kind of 'noncommutative set'. As was mentioned before, the commutativity of a Boolean algebra is automatic and hence a naive approach to 'quantizing set theory' via 'noncommutative Boolean algebras' is doomed to fail.

1.7 From groups to Hopf algebras and quantum groups

The game that we have been playing so far in this chapter should be familiar by now. We encode geometric or topological structures on a space in terms of a suitable algebra of functions on that space and then try to see how much of this structure makes sense without the commutativity hypothesis on the part of the algebra. If we are lucky we can then find a noncommutative analogue of the given structure. Let us apply this idea to one last example.

Let G be a group. Can we encode the group structure on G in terms of the algebra of functions on G? The answer is yes and by relaxing the commutativity assumption on the resulting structure we obtain an object which in many ways behaves like a group but is not a group. It is called a *Hopf algebra*. *Quantum groups* are closely related objects. They contain a dense Hopf subalgebra, but typically, as in Woronowicz' compact quantum groups, they are not Hopf algebras in the strict algebraic sense of this concept (see Example 1.7.1 iv) for more on this).

We start with a simple example. Let G be a finite group and let $H = C(G)$ denote the commutative algebra of complex-valued functions on G. Notice that the algebra structure on H has nothing to do with the group structure on G. The group structure on G is usually defined via the multiplication, inversion and unit maps

$$\begin{aligned} p&\colon G \times G \to G, \\ i&\colon G \to G, \\ u&\colon * \to G, \end{aligned}$$

where $*$ denotes a set with one element. These maps are assumed to satisfy the associativity, inverse, and unit axioms. By dualizing these maps, we obtain algebra homomorphisms

$$\begin{aligned} \Delta = p^*&\colon H \to H \otimes H, \\ S = i^*&\colon H \to H, \\ \varepsilon = u^*&\colon H \to \mathbb{C}, \end{aligned}$$

called the *comultiplication*, *antipode*, and *counit* of H respectively. Notice that we have identified $C(G \times G)$ with $C(G) \otimes C(G)$, which is fine since G is finite. Let $m\colon C(G) \otimes C(G) \to C(G)$ and $\eta\colon \mathbb{C} \to C(G)$ denote the multiplication and unit maps of $C(G)$. The associativity, inverse, and unit axioms for groups are dualized and in fact are easily seen to be equivalent to the following *coassociativity, antipode, and counit axioms* for H:

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta\colon H \to H \otimes H \otimes H, \tag{1.20}$$

$$(\varepsilon \otimes I)\Delta = (I \otimes \varepsilon)\Delta = I\colon H \to H, \tag{1.21}$$

$$m(S \otimes I) = m(I \otimes S) = \eta\varepsilon\colon H \to H, \tag{1.22}$$

where I denotes the identity map. In short, we have shown that $H = C(G)$ is a *commutative Hopf algebra*. Now the group G can be recovered from the algebra H as the set of *grouplike elements* of H, i.e., as those $h \in H$ that satisfy

$$\Delta(h) = h \otimes h, \quad h \neq 0.$$

The general definition of a Hopf algebra is as follows. Let H be a unital algebra and let $m\colon H \otimes H \to H$ and $\eta\colon \mathbb{C} \to H$ denote its multiplication and unit maps, respectively.

Definition 1.7.1. A unital algebra (H, m, η) endowed with unital algebra homomorphisms $\Delta\colon H \to H \otimes H$, $\varepsilon\colon H \to \mathbb{C}$ and a linear map $S\colon H \to H$ satisfying axioms (1.20)–(1.22) is called a Hopf algebra.

We call Δ the *comultiplication*, ε the *counit*, and S the *antipode* of H. If existence of an antipode is not assumed, then we say we have a *bialgebra*. For example, if G is only a finite monoid then $C(G)$ is a bialgebra. A Hopf algebra is called commutative if it is commutative as an algebra, and is called *cocommutative* if $\tau\Delta = \Delta$, where $\tau\colon H \otimes H \to H \otimes H$ is the flip map defined by $\tau(x \otimes y) = y \otimes x$. Thus $H = C(G)$ is cocommutative if and only if G is a commutative group.

Example 1.7.1. An important idea in Hopf algebra theory is that commutative or cocommutative Hopf algebras are closely related to groups and Lie algebras. We have already seen one example in $H = C(G)$ above. We give a few more examples to indicate this connection.

1. Let G be a discrete group (it need not be finite) and let $H = \mathbb{C}G$ denote the *group algebra* of G. A typical element of $\mathbb{C}G$ is a *finite* formal linear combination $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{C}$. Group multiplication in G then defines the multiplication of $\mathbb{C}G$. Let

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1$$

for all $g \in G$, and linearly extend them to H. It is easy to check that $(H, \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if G is commutative. Note that when G is finite we have already attached another Hopf algebra $C(G)$ to G. These two Hopf algebras are *dual* to each other in a sense to be defined below.

2. Let $\mathfrak{g}$ be a Lie algebra and let $H = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. It is an associative algebra and the canonical map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ is universal in the sense that for any other associative algebra A, any linear map $\alpha\colon \mathfrak{g} \to A$ satisfying $\alpha([x, y]) = \alpha(x)\alpha(y) - \alpha(y)\alpha(x)$ uniquely factorises through i. Using the universal property of $U(\mathfrak{g})$ one checks that there are uniquely defined algebra homomorphisms $\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\varepsilon\colon U(\mathfrak{g}) \to \mathbb{C}$ and an anti-algebra map $S\colon U(\mathfrak{g}) \to U(\mathfrak{g})$, determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad \text{and} \quad S(X) = -X$$

for all $X \in \mathfrak{g}$. One then checks that $(U(\mathfrak{g}), \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if $\mathfrak{g}$ is an abelian Lie algebra, in which case $U(\mathfrak{g}) = S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

3. Let H be a Hopf algebra. A *group-like* element of H is a nonzero element $h \in H$ such that

$$\Delta h = h \otimes h.$$

We have, using the axioms for the antipode, $hS(h) = S(h)h = 1_H$, which shows that a group-like element is invertible. It is easily seen that group-like elements of H form a subgroup of the multiplicative group of H. For example, for $H = \mathbb{C}G$ the set of grouplike elements coincides with the group G itself. A *primitive element* of a Hopf algebra is an element $h \in H$ such that

$$\Delta h = 1 \otimes h + h \otimes 1.$$

It is easily seen that the bracket $[x, y] := xy - yx$ of two primitive elements is again a primitive element. It follows that primitive elements form a Lie algebra. For $H = U(\mathfrak{g})$ any element of $\mathfrak{g}$ is primitive and in fact using the *Poincaré–Birkhoff–Witt* theorem, one can show that the set of primitive elements of $U(\mathfrak{g})$ coincides with the Lie algebra $\mathfrak{g}$.

4. (Compact groups) Let G be a compact topological group and let $C(G)$ denote the algebra of continuous complex-valued functions on G. Unless G is a finite group, $C(G)$ cannot be turned into a Hopf algebra in the sense that we defined above. The problem is with defining the coproduct Δ as the dual of the multiplication of G and is caused by the fact that $C(G) \otimes C(G)$ is only dense in $C(G \times G)$ and the two are different if G is not a finite group. There are basically two methods to get around this problem: one can either restrict to an appropriate dense subalgebra of $C(G)$ and define the coproduct just on that subalgebra, or one can broaden the notion of Hopf algebras by allowing completed topological tensor products as opposed to algebraic ones. The two approaches are essentially equivalent and eventually lead to Woronowicz' theory of compact quantum groups [180]. We start with the first approach.

A continuous function $f : G \to \mathbb{C}$ is called a *representative function* if the set of left translations of f by all elements of G forms a finite dimensional subspace of $C(G)$. It is easy to see that f is representative if and only if it appears as a matrix entry of a finite dimensional complex representation of G. Let $H = \text{Rep}(G)$ denote the linear span of representative functions on G (we hope this is not confused with the *representation ring* of G). It is a subalgebra of $C(G)$ which is closed under complex conjugation. By the *Peter–Weyl Theorem* (see, for example, [21]), $\text{Rep}(G)$ is a dense $*$-subalgebra of $C(G)$. Now let $p : G \times G \to G$ denote the multiplication of G and let

$$p^* : C(G) \to C(G \times G), \quad p^* f(x, y) = f(xy),$$

denote its dual map. One checks that [21], [85] if f is representative, then

$$p^* f \in \text{Rep}(G) \otimes \text{Rep}(G) \subset C(G \times G).$$

Let e denote the identity of G. The formulas

$$\Delta f = p^* f, \quad \varepsilon f = f(e), \quad \text{and} \quad (Sf)(g) = f(g^{-1})$$

define a Hopf algebra structure on $\mathrm{Rep}(G)$. Alternatively, one can describe $\mathrm{Rep}(G)$ as the linear span of *matrix coefficients* of isomorphism classes of all irreducible finite dimensional complex representations of G. The coproduct can also be defined as

$$\Delta(f_{ij}) = \sum_{k=1}^{n} f_{ik} \otimes f_{kj}.$$

This algebra is finitely generated (as an algebra) if and only if G is a (compact) Lie group.

For a concrete example, let $G = U(1)$ be the group of complex numbers of absolute value 1. Irreducible representations of G are all 1-dimensional and are parameterized by integers $n \in \mathbb{Z}$. With a little work one can show that $H = \mathrm{Rep}(G)$ is the Laurent polynomial algebra

$$H = \mathbb{C}[u, u^{-1}]$$

with u a unitary ($uu^* = u^*u = 1$) and with comultiplication and counit given by

$$\Delta(u^n) = u^n \otimes u^n, \quad \varepsilon(u^n) = 1, \quad S(u^n) = u^{-n}$$

for all $n \in \mathbb{Z}$.

A more interesting example is when $G = \mathrm{SU}(2)$ is the group of unitary 2 by 2 complex matrices with determinant 1. The algebra $C(\mathrm{SU}(2)) = C(S^3)$ is the algebra of continuous functions on the three-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1\}$. Let α and β denote the coordinate functions defined by $\alpha(z_1, z_2) = z_1$ and $\beta(z_1, z_2) = z_2$. They satisfy the relation $\alpha\alpha^* + \beta\beta^* = 1$. It can be shown that $C(S^3)$ is the universal unital commutative C^*-algebra generated by two generators α and β with relation

$$\alpha\alpha^* + \beta\beta^* = 1.$$

Notice that this relation amounts to saying that

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

is a unitary matrix, i.e., $UU^* = U^*U = I$. All irreducible unitary representations of $\mathrm{SU}(2)$ are tensor products of the fundamental representation whose matrix is U [21]. Though it is by no means obvious, it can be shown that $\mathrm{Rep}(\mathrm{SU}(2))$ is the $*$-subalgebra of $C(\mathrm{SU}(2))$ generated by α and β. Its coproduct, counit, and antipode are uniquely induced by their values on the generators:

$$\Delta \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

$$\varepsilon\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$S(\alpha)=\alpha^*,\quad S(\beta)=-\beta,\quad S(\beta^*)=-\beta^*,\quad S(\alpha^*)=\alpha.$$

Here the notation $\dot{\otimes}$ is meant to imply $\Delta(\alpha)=\alpha\otimes\alpha+\beta\otimes(-\beta^*)$, etc.

5. (Affine group schemes) An *affine algebraic group*, say over $\mathbb{C}$, is an affine algebraic variety G such that G is a group and its multiplication and inversion maps, $p\colon G\times G\to G$ and $i\colon G\to G$, are morphisms of varieties. The coordinate ring $H=\mathcal{O}[G]$ of an affine algebraic group G is a commutative Hopf algebra. The maps Δ, ε, and S are the duals of the multiplication, unit, and inversion maps of G, similar to what we did with finite and compact groups. Here is a concrete example. Let $G=\mathrm{GL}_n(\mathbb{C})$ be the general linear groups consisting of all invertible $n\times n$ complex matrices. As an algebra, $H=\mathcal{O}[\mathrm{GL}_n(\mathbb{C})]$ is generated by pairwise commuting elements x_{ij}, D for $i,j=1,\ldots,n$ and the relation

$$\det(x_{ij})D=1.$$

The coproduct, counit and antipode of H are given by

$$\Delta(x_{ij})=\sum_{k=1}^{k}x_{ik}\otimes x_{kj},\quad \Delta(D)=D\otimes D,$$

$$\varepsilon(x_{ij})=\delta_{ij},\quad \varepsilon(D)=1,$$

$$S(x_{ij})=D\,\mathrm{Adj}(x_{ij}),\quad S(D)=D^{-1}.$$

These formulas are obtained by dualizing the usual linear algebra formulas for matrix multiplication, the identity matrix, and the adjoint formula for the inverse of a matrix.

More generally, an *affine group scheme* over a commutative ring k is a commutative Hopf algebra over k. This definition can be cast in the language of *representable functors* à la Grothendieck [176]. In fact, given such a Hopf algebra H, it is easy to see that for any commutative k-algebra A, the set

$$G=\mathrm{Hom}_{\mathrm{Alg}}(H,\,A)$$

of algebra maps from H to A is a group under the *convolution product*. The convolution product of any two linear maps $f,g\colon H\to A$, denoted by $f*g$, is defined as the composition

$$H\xrightarrow{\Delta}H\otimes H\xrightarrow{f\otimes g}A\otimes A\overset{m}{\to}A,$$

or, equivalently, by

$$(f*g)(h)=\sum f(h^{(1)})g(h^{(2)}),$$

if we use Sweedler notation $\Delta(h)=\sum h^{(1)}\otimes h^{(2)}$ for the coproduct. Thus we can define a functor

$$\mathrm{Com\,Alg}_k\rightsquigarrow\ \mathrm{Groups},\quad A\mapsto\mathrm{Hom}_{\mathrm{Alg}}(H,\,A)$$

from the category of commutative k-algebras to the category of groups. This functor is clearly representable, as it is represented by our commutative Hopf algebra H.

Conversely, let $F\colon \operatorname{Com} \operatorname{Alg}_k \rightsquigarrow$ Groups be a *representable functor* represented by a commutative algebra H. Then $H \otimes H$ represents $F \times F$, and applying Yoneda's lemma we obtain maps $\Delta\colon H \to H \otimes H$, $\varepsilon\colon H \to \mathbb{C}$ and $S\colon H \to H$ satisfying axioms (1.20)–(1.22). This shows that H is in fact a Hopf algebra. Thus the category of affine group schemes is equivalent to the category of representable functors $\operatorname{Com} \operatorname{Alg}_k \to$ Groups.

Here is a concrete example. Consider the functor $\mu_n\colon \operatorname{Com} \operatorname{Alg}_k \to$ Groups which sends a commutative algebra A to the group of its n-th roots of unity. This functor is representable by the Hopf algebra $H = k[X]/(X^n - 1)$, the quotient of the polynomial algebra by the relation $X^n = 1$. Its coproduct, antipode, and counit are given by

$$\Delta(X) = X \otimes X, \quad S(X) = X^{n-1}, \quad \varepsilon(X) = 1.$$

In general, an algebraic group, such as GL_n or SL_n, is an affine group scheme, represented by its coordinate ring. See [176] for a good introduction to affine group schemes.

Example 1.7.2 (Hopf duality)**.** Let H be a finite dimensional Hopf algebra and let $H^* = \operatorname{Hom}(H, \mathbb{C})$ denote its linear dual. By dualizing the algebra, coalgebra, and antipode maps of H, we obtain the maps

$$m^*\colon H^* \to H^* \otimes H^*, \quad \eta^*\colon H^* \to \mathbb{C}, \quad \Delta^*\colon H^* \to H^* \otimes H^*,$$

$$\varepsilon^*\colon \mathbb{C} \to H^*, \quad S^*\colon H^* \to H^*.$$

It can be checked that these operations turn H^* into a Hopf algebra, called the *dual* of H. Notice that H is commutative (resp. cocommutative) if and only if H^* is cocommutative (resp. commutative). We also note that $H^{**} = H$ as Hopf algebras. For example, when G is a finite group we have $(\mathbb{C}G)^* = C(G)$. The isomorphism is induced by the map $\sum a_g g \mapsto \sum a_g \delta_g$.

We note that the linear dual of an infinite dimensional Hopf algebra is not a Hopf algebra. The main problem is that when we dualize the product we only obtain a map $m^*\colon H^* \to (H \otimes H)^*$, and when H is infinite dimensional $H^* \otimes H^*$ is only a proper subspace of $(H \otimes H)^*$. Notice that the dual of a coalgebra is always an algebra. One way to get around this problem is to consider the smaller *restricted dual* of Hopf algebras which are always a Hopf algebra [65], [167]. The main idea is to consider, instead of all linear functionals on H, only the *continuous* ones (with respect to the *linearly compact topology* on H). The restricted dual H° may be too small though.

A better way to think about Hopf duality which covers the infinite dimensional case as well is via a *Hopf pairing.* A Hopf pairing between Hopf algebras K and H is a bilinear map

$$\langle\, ,\, \rangle\colon H \otimes K \to \mathbb{C}$$

satisfying the following relations for all h, h_1, h_2 in H and g, g_1, g_2 in K:

$$\begin{aligned}\langle h_1 h_2,\, g\rangle &= \sum \langle h_1,\, g^{(1)}\rangle\langle h_2,\, g^{(2)}\rangle,\\ \langle h,\, g_1 g_2\rangle &= \sum \langle h^{(1)},\, g_1\rangle\langle h^{(2)},\, g_2\rangle,\\ \langle h,\, 1\rangle &= \varepsilon(h),\\ \langle 1,\, g\rangle &= \varepsilon(g).\end{aligned}$$

For example, let $H = U(\mathfrak{g})$ be the enveloping algebra of the Lie algebra $\mathfrak{g}$ of a Lie group G and let $K = \mathrm{Rep}(G)$ be the Hopf algebra of representable functions on G. There is a canonical non-degenerate pairing $H \otimes K \to \mathbb{C}$ defined by

$$\langle X_1 \otimes \cdots \otimes X_n,\, f\rangle = X_1(X_2(\ldots X_1(f))\ldots), \tag{1.23}$$

where

$$X(f) := \frac{d}{dt} f(e^{tX})|_{t=0}$$

(cf. [85] for a thorough discussion).

We shall see that there is an analogous pairing between compact quantum groups of classical Lie groups and their associated quantized enveloping algebras (cf. (1.24)).

Example 1.7.3 (Structure of cocommutative Hopf algebras)**.** Is every cocommutative Hopf algebra a universal enveloping algebra? The answer is negative since, for example, as we saw above group algebras are also cocommutative. We mention two major structure theorems which completely settle this question over an algebraically closed field of characteristic zero.

1. By a theorem of Kostant and, independently, Cartier [167], [32], any cocommutative Hopf algebra H over an algebraically closed field k of characteristic zero is isomorphic (as a Hopf algebra) to a crossed product algebra $H = U(P(H)) \rtimes G(H)$, where $P(H)$ is the Lie algebra of primitive elements of H and $G(H)$ is the group of all grouplike elements of H and $G(H)$ acts on $P(H)$ by inner automorphisms, $(g, h) \mapsto ghg^{-1}$, for $g \in G(H)$ and $h \in P(H)$. The coalgebra structure of $H = U(P(H)) \rtimes G(H)$ is simply the tensor product of the two coalgebras $U(P(H))$ and $kG(H)$.

2. (Cartier–Milnor–Moore). Let H be a Hopf algebra over a field k of characteristic zero (k need not be algebraically closed), let $\bar{H}$ denote the kernel of the counit map ε, and let $\bar{\Delta}\colon \bar{H} \to \bar{H} \otimes \bar{H}$ denote the *reduced coproduct*. By definition

$$\bar{\Delta}(h) = \Delta(h) - 1 \otimes h - h \otimes 1.$$

Let $\bar{H}_n \subset \bar{H}$ denote the kernel of the iterated coproduct $\bar{\Delta}_{n+1}\colon \bar{H} \to \bar{H}^{\otimes(n+1)}$. The increasing sequence of subspaces

$$\bar{H}_0 \subset \bar{H}_1 \subset \bar{H}_2 \subset \cdots$$

is called the *coradical filtration* of H. It is a *Hopf algebra filtration* in the sense that

$$\bar{H}_i \cdot \bar{H}_j \subset \bar{H}_{i+j} \quad \text{and} \quad \bar{\Delta}(\bar{H}_n) \subset \sum_{i+j=n} \bar{H}_i \otimes \bar{H}_j.$$

A Hopf algebra is called *connected* or *co-nilpotent* if its coradical filtration is exhaustive, that is, if $\bigcup_i \bar{H}_i = \bar{H}$, or, equivalently, for any $h \in \bar{H}$, there is an n such that $\bar{\Delta}^n(h) = 0$. Now we can state the Cartier–Milnor–Moore theorem:

Proposition 1.7.1. *A cocommutative Hopf algebra over a field of characteristic* 0 *is isomorphic, as a Hopf algebra, to the enveloping algebra of a Lie algebra if and only if it is connected.*

The Lie algebra in question is the Lie algebra of primitive elements of H, namely those elements h such that $\Delta(h) = h \otimes 1 + 1 \otimes h$. A typical application of the proposition is as follows. Let $H = \bigoplus_{i \geq 0} H_i$ be a graded cocommutative Hopf algebra. It is easy to see that H is connected if and only if $H_0 = k$. The theorem then implies that, if H is connected, we have $H = U(\mathfrak{g})$, an enveloping algebra.

Example 1.7.4 (Compact quantum groups)**.** A prototypical example of a compact quantum group is Woronowicz' $\mathrm{SU}_q(2)$, for $0 < q \leq 1$. As a C^*-algebra it is the unital C^*-algebra, denoted $C(\mathrm{SU}_q(2))$, generated by α and β subject to the relations

$$\beta\beta^* = \beta^*\beta, \quad \alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \alpha\alpha^* + q^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I.$$

Notice that these relations amount to saying that

$$U = \begin{pmatrix} \alpha & q\beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

is unitary, i.e., $UU^* = U^*U = I$. This C^*-algebra is not a Hopf algebra in the strict algebraic sense of Definition 1.7.1, but, as with compact topological groups, it has a dense subalgebra which is a Hopf algebra. Let $\mathcal{O}(\mathrm{SU}_q(2))$ denote the dense $*$-subalgebra of $C(\mathrm{SU}_q(2))$ generated by elements α and β. This is the analogue of the algebra of representative functions $(\mathrm{Rep}(\mathrm{SU}(2)))$ in Example 1.7.1 iv). We can turn $\mathcal{O}(\mathrm{SU}_q(2))$ into a Hopf algebra as follows. Its coproduct and antipode are defined by

$$\Delta \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

$$S(\alpha) = \alpha^*, \quad S(\beta) = -q^{-1}\beta, \quad S(\beta^*) = -q\beta^*, \quad S(\alpha^*) = \alpha.$$

Notice that the coproduct is only defined on the algebra $\mathcal{O}(\mathrm{SU}_q(2))$ of matrix elements on the quantum group, and its extension to $C(\mathrm{SU}_q(2))$ lands in the completed tensor product

$$\Delta \colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \hat{\otimes} C(\mathrm{SU}_q(2)).$$

At $q = 1$ we obtain the algebra of continuous functions on $\mathrm{SU}(2)$. We refer to [117] for a survey of compact and locally compact quantum groups.

Example 1.7.5 (The quantum enveloping algebra $U_q(su(2))$)**.** As an algebra over $\mathbb{C}$, $U_q(su(2))$ is generated by elements E, F and K, subject to the relations $KK^{-1} = K^{-1}K = 1$ and (see [111])

$$KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad [F, E] = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

One can then check that the following relations uniquely define the coproduct, the antipode, and the counit of $U_q(su(2))$:

$$\Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E,$$

$$S(K) = K^{-1}, \quad S(E) = -qE, \; S(F) = -q^{-1}F, \quad \varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$

There is a Hopf pairing

$$U_q(su(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{C} \tag{1.24}$$

given on generators by

$$\langle K, \alpha \rangle = q^{-1/2}, \quad \langle K^{-1}, \alpha \rangle = q^{1/2}, \quad \langle K, \alpha^* \rangle = q^{1/2}, \quad \langle K^{-1}, \alpha^* \rangle = q^{-1/2},$$
$$\langle E, \beta^* \rangle = -1, \quad \langle F, \beta \rangle = q^{-1},$$

and the pairing between all other couples of generators is 0. This pairing should be compared with its classical counterpart (1.23).

Example 1.7.6 (Symmetry in noncommutative geometry; Hopf algebra actions)**.** The idea of *symmetry* in classical geometry is encoded via the action of a group or Lie algebra on a space. This can be extended to noncommutative geometry by considering an *action* or *coaction* of a Hopf algebra on a noncommutative algebra (or coalgebra) representing a noncommutative space. We look at Hopf algebra actions first.

Let H be a Hopf algebra. An algebra A is called a left H-*module algebra* if A is a left H-module via a map $\rho \colon H \otimes A \to A$ and the multiplication and unit maps of A are morphisms of H-modules, that is,

$$h(ab) = \sum h^{(1)}(a)h^{(2)}(b) \quad \text{and} \quad h(1) = \varepsilon(h)1$$

for all $h \in H$ and $a, b \in A$.

Using the relations $\Delta h = h \otimes h$ and $\Delta h = 1 \otimes h + h \otimes 1$, for grouplike and primitive elements, it is easily seen that, in an H-module algebra, group-like elements act as unit preserving algebra automorphisms while primitive elements act as derivations. In particular, for $H = \mathbb{C}G$ the group algebra of a discrete group, an H-module algebra structure on A is simply an action of G by unit preserving algebra automorphisms of A. Similarly, we have a one-to-one correspondence between $U(\mathfrak{g})$-module algebra structures on A and Lie actions of the Lie algebra $\mathfrak{g}$ by derivations on A.

Really novel examples occur when one quantizes actions of Lie algebras on classical space. Here is an example. Recall from Example 1.2.2 that the Podleś quantum sphere S_q^2 is the $*$-algebra generated by elements a, a^* and b subject to the relations

$$aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1, \quad ab = q^{-2}ba, \quad a^*b = q^2ba^*.$$

By a direct computation one can show that the following formulas define a $U_q(su(2))$-module algebra structure on S_q^2:

$$K \cdot a = qa, \;\; K \cdot a^* = q^{-1}a^*, \;\; K \cdot b = b,$$
$$E \cdot b = q^{\frac{5}{2}}a, \quad E \cdot a^* = -q^{\frac{3}{2}}(1 + q^{-2})b, \quad E \cdot a = 0,$$
$$F \cdot a = q^{-\frac{7}{2}}(1 + q^2)b, \quad F \cdot b = -q^{-\frac{1}{2}}a^*, \quad F \cdot a^* = 0.$$

Recall from Example 1.2.2 that the quantum analogue of the Dirac (or Hopf) monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$:

$$e_q = \frac{1}{2}\begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

We can show that this noncommutative line bundle is *equivariant* in the following sense. Consider the 2-dimensional standard representation of $U_q(su(2))$ on $V = \mathbb{C}^2$ defined by

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{bmatrix}.$$

We obtain an action of $U_q(su(2))$ on $M_2(S_q^2) = M_2(\mathbb{C}) \otimes S_q^2$ as the tensor product of these two modules by the formula

$$h(m \otimes a) = \sum h^{(1)}(m)h^{(2)}(a) \quad \text{for all } h \in U_q(su(2)),\; m \in M_2(\mathbb{C}),\; a \in S_q^2.$$

The claim is that

$$h(e_q) = \varepsilon(h)e_q \tag{1.25}$$

for all $h \in U_q(su(2))$ (Exercise 1.7.4).

Example 1.7.7 (Symmetry in noncommutative geometry; Hopf algebra coactions)**.** Let H be a Hopf algebra. A (left) *corepresentation* or *comodule* for H is a vector space M together with a map $\rho\colon M \to H \otimes M$ such that

$$(\Delta \otimes I_M)\,\rho = (I_H \otimes \rho)\,\rho \quad \text{and} \quad (\varepsilon \otimes I_M)\,\rho = I_M.$$

These conditions are duals of axioms for a module over an algebra. Now an algebra A is called a left H-*comodule algebra* if A is a left H-comodule via a map $\rho\colon A \to H \otimes A$, and if ρ is a morphism of algebras.

For example, the coproduct $\Delta\colon H \to H \otimes H$ gives H the structure of a left H-comodule algebra. This is the analogue of the left action of a group on itself by translations.

For compact quantum groups like $\mathrm{SU}_q(2)$ and their algebraic analogues such as $\mathrm{SL}_q(2)$ the coaction is more natural. Formally they are obtained by 'dualizing and quantizing' group actions $G \times X \to X$ for classical groups. Here is an example.

Let q be a nonzero complex number and let $A = \mathbb{C}_q[x, y]$ be the algebra of coordinates on the quantum q-plane. It is defined as the quotient algebra

$$\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y\rangle/(yx - qxy),$$

where $\mathbb{C}\langle x, y\rangle$ is the free algebra with two generators and $(yx-qxy)$ is the two-sided ideal generated by $yx - qxy$. For $q \neq 1$, the algebra $\mathbb{C}_q[x, y]$ is noncommutative.

By direct computation one can show that there is a unique $\mathrm{SL}_q(2)$-comodule algebra structure $\rho\colon A \to \mathrm{SL}_q(2) \otimes A$ on the quantum plane $\mathbb{C}_q[x, y]$ with

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

The proof boils down to checking that the elements $\rho(x) = a \otimes x + b \otimes y$ and $\rho(y) = c \otimes x + d \otimes y$ satisfy the defining relation $\rho(y)\rho(x) = q\rho(x)\rho(y)$ for the quantum plane, which is straightforward.

We can also define the coaction of a Hopf algebra on a coalgebra, and the concept of a *comodule coalgebra*. This plays an important role in the next example.

Example 1.7.8 (Bicrossed products)**.** The examples of Hopf algebras that are really difficult to construct are the noncommutative and the non-cocommutative ones. Of course, one can always start with a noncommutative Hopf algebra U, which exist in abundance, say as universal enveloping algebras of Lie algebras, and a non-cocommutative Hopf algebra F, say the algebra of representative functions on a compact group, and form the *tensor product Hopf algebra*

$$F \otimes U \tag{1.26}$$

which is clearly neither commutative nor cocommutative. But this is not that significant. A variation of this method, however, provides really interesting examples as we shall explain. We should mention that another source of interesting examples of noncommutative and non-cocommutative Hopf algebras is the theory of quantum groups.

The idea is to deform the algebra and coalgebra structures in (1.26) via an action of U on F and a coaction of F on U, through crossed products. First we describe these crossed product constructions in general, which are of independent interest as well.

Let A be a left H-module algebra. The underlying vector space of the *crossed product algebra* $A \rtimes H$ is $A \otimes H$ and its product is determined by

$$(a \otimes g)(b \otimes h) = \sum a(g^{(1)}b) \otimes g^{(2)}h.$$

One checks that endowed with $1 \otimes 1$ as its unit, $A \rtimes H$ is an associative unital algebra. For example, for $H = \mathbb{C}G$, the group algebra of a discrete group G acting by automorphisms on an algebra A, the algebra $A \rtimes H$ is isomorphic to the crossed product algebra $A \rtimes G$.

For a second simple example, let a Lie algebra $\mathfrak{g}$ act by derivations on a commutative algebra A. Then the crossed product algebra $A \rtimes U(\mathfrak{g})$ is a subalgebra of the algebra of differential operators on A generated by derivations from $\mathfrak{g}$ and multiplication operators by elements of A. The simples example is when $A = \mathbb{C}[x]$ and $\mathfrak{g} = \mathbb{C}$ acting via the differential operator $\frac{d}{dx}$ on A. Then $A \rtimes U(\mathfrak{g})$ is the *Weyl algebra* of differential operators on the line with polynomial coefficients.

Let D be a right H-comodule coalgebra with coaction $d \in D \mapsto \sum d^{(0)} \otimes d^{(1)} \in D \otimes H$. The underlying linear space of the *crossed product coalgebra* $H \rtimes D$ is $H \otimes D$. It is a coalgebra under the coproduct and counit defined by

$$\Delta(h \otimes d) = \sum h^{(1)} \otimes (d^{(1)})^{(0)} \otimes h^{(2)}(d^{(1)})^{(1)} \otimes d^{(2)}, \quad \varepsilon(h \otimes d) = \varepsilon(d)\varepsilon(h).$$

The above two constructions deform multiplication or comultiplication of algebras or coalgebras, respectively. Thus to obtain a simultaneous deformation of multiplication and comultiplication of a Hopf algebra it stands to reason to try to apply both constructions simultaneously. This idea, going back to G. I. Kac in the 1960s in the context of Kac–von Neumann Hopf algebras, has now found its complete generalization in the notion of *bicrossed product* of *matched pairs* of Hopf algebras due to Shahn Majid. See [128] for extensive discussions and references. There are many variations of this construction, of which the most relevant for the structure of the Connes–Moscovici Hopf algebra is the following. Another special case is the *Drinfeld double* $D(H)$ of a finite dimensional Hopf algebra [128], [105].

Let U and F be two Hopf algebras. We assume that F is a left U-module algebra and U is a right F-comodule coalgebra via $\rho\colon U \to U \otimes F$. We say that (U, F) is a *matched pair* if the action and coaction satisfy the compatibility condition:

$$\varepsilon(u(f)) = \varepsilon(u)\varepsilon(f), \quad \Delta(u(f)) = (u^{(1)})^{(0)}(f^{(1)}) \otimes (u^{(1)})^{(1)}(u^{(2)}(f^{(2)})),$$
$$\rho(1) = 1 \otimes 1, \quad \rho(uv) = (u^{(1)})^{(0)} v^{(0)} \otimes (u^{(1)})^{(1)}(u^{(2)}(v^{(1)})),$$
$$(u^{(2)})^{(0)} \otimes (u^{(1)}(f))(u^{(2)})^{(1)} = (u^{(1)})^{(0)} \otimes (u^{(1)})^{(1)}(u^{(2)}(f)).$$

Given a matched pair as above, we define its bicrossed product Hopf algebra $F \rtimes U$ to be $F \otimes U$ with crossed product algebra structure and crossed coproduct coalgebra structure. Its antipode S is defined by

$$S(f \otimes u) = (1 \otimes S(u^{(0)}))(S(fu^{(1)}) \otimes 1).$$

It is a remarkable fact that, thanks to the above compatibility conditions, all the axioms of a Hopf algebra are satisfied for $F \rtimes U$.

The simplest and first example of this bicrossed product construction is as follows. Let $G = G_1 G_2$ be a *factorization* of a finite group G. This means that G_1 and G_2 are subgroups of G, $G_1 \cap G_2 = \{e\}$, and $G_1 G_2 = G$. We denote the factorization of g by $g = g_1 g_2$. The relation $g \cdot h := (gh)_2$ for $g \in G_1$ and $h \in G_2$

defines a left action of G_1 on G_2. Similarly $g \bullet h := (gh)_1$ defines a right action of G_2 on G_1. The first action turns $F = F(G_2)$ into a left $U = kG_1$-module algebra. The second action turns U into a right F-comodule coalgebra. The latter coaction is simply the dual of the map $F(G_1) \otimes kG_2 \to F(G_1)$ induced by the right action of G_2 on G_1. Details of this example can be found in [128] and [55].

Example 1.7.9 (Connes–Moscovici Hopf algebras)**.** A very important example for noncommutative geometry and its applications to transverse geometry and number theory is the family of *Connes–Moscovici Hopf algebras* $\mathcal{H}_n$ for $n \geq 1$ [55], [56], [57]. They are deformations of the group $G = \mathrm{Diff}(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$ and can also be thought of as deformations of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$. These algebras appeared for the first time as quantum symmetries of transverse frame bundles of codimension n foliations. We briefly treat the case $n = 1$ here. The main features of $\mathcal{H}_1$ stem from the fact that the group $G = \mathrm{Diff}(\mathbb{R}^n)$ has a *factorization* of the form

$$G = G_1 G_2,$$

where G_1 is the subgroup of diffeomorphisms φ that satisfy

$$\varphi(0) = 0, \quad \varphi'(0) = 1,$$

and G_2 is the $ax + b$ group of affine diffeomorphisms. We introduce two Hopf algebras corresponding to G_1 and G_2 respectively. Let F denote the Hopf algebra of polynomial functions on the pro-unipotent group G_1. It can also be defined as the *continuous dual* of the enveloping algebra of the Lie algebra of G_1. It is a commutative Hopf algebra generated by the Connes–Moscovici coordinate functions δ_n, $n = 1, 2, \ldots$, defined by

$$\delta_n(\varphi) = \frac{d^n}{dt^n}(\log(\varphi'(t))|_{t=0}.$$

The second Hopf algebra, U, is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_2$ of the $ax + b$ group. It has generators X and Y and one relation $[X, Y] = X$.

As we explained in the previous example, the factorization $G = G_1 G_2$ defines a *matched pair* of Hopf algebras consisting of F and U. More precisely, the Hopf algebra F has a right U-module algebra structure defined by

$$\delta_n(X) = -\delta_{n+1} \quad \text{and} \quad \delta_n(Y) = -n\delta_n.$$

The Hopf algebra U, on the other hand, has a left F-comodule coalgebra structure via

$$X \mapsto 1 \otimes X + \delta_1 \otimes X \quad \text{and} \quad Y \mapsto 1 \otimes Y.$$

One can check that they are a matched pair of Hopf algebras and the resulting bicrossed product Hopf algebra

$$\mathcal{H}_1 = F \bowtie U$$

is the Connes–Moscovici Hopf algebra $\mathcal{H}_1$. (See [55] for a slightly different approach and fine points of the proof.)

Thus $\mathcal{H}_1$ is the universal Hopf algebra generated by $\{X, Y, \delta_n;\ n = 1, 2, \dots\}$ with relations

$$[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_l] = 0,$$

$$\Delta Y = Y \otimes 1 + 1 \otimes Y, \quad \Delta\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$

$$\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,$$

$$S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1,$$

for $n, k, l = 1, 2, \dots$.

Another recent point of interaction between Hopf algebras and noncommutative geometry is the work of Connes and Kreimer in renormalization schemes of quantum field theory. We refer to [47], [48], [49], [51], [52] and references therein for this fascinating and still developing subject.

An important feature of $\mathcal{H}_1$, and in fact its *raison d'être*, is that it acts as quantum symmetries of various objects of interest in noncommutative geometry, like the noncommutative 'space' of leaves of codimension one foliations or the noncommutative 'space' of modular forms modulo the action of Hecke correspondences. Let M be a 1-dimensional manifold and $A = C_0^\infty(F^+M)$ denote the algebra of smooth functions with compact support on the bundle of positively oriented frames on M. Given a discrete group $\Gamma \subset \mathrm{Diff}^+(M)$ of orientation preserving diffeomorphisms of M, one has a natural prolongation of the action of Γ to $F^+(M)$ by

$$\varphi(y, y_1) = (\varphi(y), \varphi'(y)(y_1)).$$

Let $A_\Gamma = C_0^\infty(F^+M) \rtimes \Gamma$ denote the corresponding crossed product algebra. Thus the elements of A_Γ consist of finite linear combinations (over $\mathbb{C}$) of terms fU_φ^* with $f \in C_0^\infty(F^+M)$ and $\varphi \in \Gamma$. Its product is defined by

$$fU_\varphi^* \cdot gU_\psi^* = (f \cdot \varphi(g))U_{\psi\varphi}^*.$$

There is an action of $\mathcal{H}_1$ on A_Γ given by [55]:

$$Y(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y_1} U_\varphi^*, \quad X(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y} U_\varphi^*,$$

$$\delta_n(fU_\varphi^*) = y_1^n \frac{d^n}{dy^n}\left(\log \frac{d\varphi}{dy}\right) fU_\varphi^*.$$

Once these formulas are given, it can be checked, by a long computation, that A_Γ is indeed an $\mathcal{H}_1$-module algebra. In the original application, M is a transversal for a codimension one foliation and thus $\mathcal{H}_1$ acts via transverse differential operators [55].

Remark 1. The theory of Hopf algebras and *Hopf spaces* has its roots in algebraic topology and was born in the paper of H. Hopf in his celebrated computation of the rational cohomology of compact connected Lie groups [101]. The cohomology ring of such a Lie group is a Hopf algebra and this puts strong restrictions on its structure as an algebra where it was shown that it is isomorphic to an exterior algebra with odd generators. This line of investigation eventually led to the Cartier–Milnor–Moore theorem [32], [137] characterizing connected cocommutative Hopf algebras as enveloping algebras of Lie algebras.

A purely algebraic theory, with motivations independent from algebraic topology, was created by Sweedler in the 1960s who wrote the first book on Hopf algebras [167]. This line of investigation took a big leap forward with the work of Faddeev–Reshetikhin–Takhtajan, of Drinfeld, who coined the term quantum group [71], and of Jimbo, resulting in quantizing all classical Lie groups and Lie algebras. This latter line of investigations were directly motivated by the theory of quantum integrable systems and quantum inverse scattering methods developed by the Leningrad and Japanese school in the early 1980s.

In a different direction, immediately after *Pontryagin's duality* theorem for locally compact abelian groups, attempts were made to extend it to noncommutative groups. The *Tannaka–Krein duality* theorem was an important first step. This result was later sharpened by Grothendieck, Deligne, and, independently, Doplicher and Roberts. Note that the dual, in any sense of the word, of a noncommutative group cannot be a group and one is naturally interested in extending the category of groups to a larger category which is closed under duality and hopefully is even equivalent to its second dual, much as is the case for locally compact abelian groups. The Hopf–von Neumann algebras of G. I. Kac and Vainerman achieve this in the measure theoretic world of von Neumann algebras [75]. The theory of *locally compact quantum groups* of Kustermans and Vaes [117] which was developed much later achieves this goal in the category of C^*-algebras. An important step towards this program was the theory of compact quantum groups of S. L. Woronowicz (cf. [180] for a survey). We refer to [32], [105], [111], [128], [130], [131], [167], [169] for the general theory of Hopf algebras and quantum groups.

The first serious interaction between Hopf algebras and noncommutative geometry started in earnest in the paper of Connes and Moscovici on transverse index theory [54] (cf. also [55], [56], [57] for further developments). In that paper a noncommutative and non-cocommutative Hopf algebra appears as the quantum symmetries of the noncommutative space of codimension one foliations. The same Hopf algebra was later shown to act on the noncommutative space of modular Hecke algebras [58]. For a survey of Hopf algebras in noncommutative geometry the reader can consult [85], [173].

Exercise 1.7.1. The dual of a coalgebra is an algebra and in fact we have a functor $\mathrm{Coalg}_k \rightsquigarrow \mathrm{Alg}_k^{\mathrm{op}}$ which is an equivalence of categories if we restrict to finite dimensional algebras and coalgebras (k is a field).

Exercise 1.7.2. Give an example of a functor from $\mathrm{Com\,Alg}_k \to \mathrm{Groups}$ which is *not* representable.

Exercise 1.7.3. Show that, for a commutative Hopf algebra H and a commutative algebra A, $G := \mathrm{Hom}_{\mathrm{Alg}}(H, A)$ is a group under the convolution product.

Exercise 1.7.4. Verify the formula (1.25).

Chapter 2

Noncommutative quotients

In this chapter we recall the method of *noncommutative quotients* as advanced by Connes in [41]. This is a technique that allows one to replace the so called “bad quotients” by nice noncommutative spaces, represented by noncommutative algebras. The good news is that most of the noncommutative spaces which are currently in use in noncommutative geometry are constructed by this method. In general these noncommutative algebras are defined as groupoid algebras. In some cases, like quotients by group actions, the noncommutative quotient can be defined as a crossed product algebra as well, without using groupoids. In general however, like quotients by equivalence relations, the use of groupoids seems to be unavoidable. Thus *groupoids* and *groupoid algebras* provide a unified approach when dealing with bad quotients.

In Section 2.1 we recall the definition of a groupoid together with its various refinements like topological, smooth and étale groupoids. In Section 2.2 we define the groupoid algebra of a groupoid and give several examples. An important concept is Morita equivalence of algebras. When the classical quotient is a nice space, e.g. when a group acts freely and properly, the algebra of functions on the classical quotient is (strongly) Morita equivalent to the noncommutative quotient. As we shall see, this fully justifies the use of noncommutative quotients. We treat both the purely algebraic Morita theory as well as the notion of strong Morita equivalence for C^*-algebras in Sections 2.3 and 2.4. Finally, noncommutative quotients are defined in Section 2.5.

2.1 Groupoids

Definition 2.1.1. A *groupoid* is a small category in which every morphism is an isomorphism.

We recall that a category is called *small* if its objects form a set, and not just a class. As a consequence, the collection of all morphisms of a small category form a set as well, and we can safely talk about functions on the set of morphisms of a groupoid, as we shall often need to do in this chapter. Let $\mathcal{G}$ be a groupoid. We denote the set of objects of $\mathcal{G}$ by $\mathcal{G}^{(0)}$ and, by a small abuse of notation, the set of morphisms of $\mathcal{G}$ by $\mathcal{G}$ itself. Every morphism has a *source*, has a *target* and has an *inverse*. They define maps, denoted by s, t, and i, respectively:

$$s\colon \mathcal{G} \to \mathcal{G}^{(0)}, \quad t\colon \mathcal{G} \to \mathcal{G}^{(0)}, \quad i\colon \mathcal{G} \to \mathcal{G}.$$

There is also a canonical map

$$u\colon \mathcal{G}^{(0)} \to \mathcal{G}$$

which sends an object to the unit morphism from that object to itself. The composition $\gamma_1 \circ \gamma_2$ of morphisms γ_1 and γ_2 is only defined if $s(\gamma_1) = t(\gamma_2)$. Composition defines a map

$$\circ\colon \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2);\ s(\gamma_1) = t(\gamma_2)\} \to \mathcal{G}$$

which is associative in an obvious sense.

Example 2.1.1 (Groupoids from groups)**.** Every group G defines at least two groupoids in a natural way.

(i) Define a category $\mathcal{G}$ with one object $*$ and with

$$\mathrm{Hom}_{\mathcal{G}}(*, *) = G,$$

where the composition of morphisms is simply defined by the group multiplication. This is obviously a groupoid. In fact any groupoid with one object is defined as above.

(ii) Define a category $\mathcal{G}$ with

$$\mathrm{obj}\,\mathcal{G} = G \quad \text{and} \quad \mathrm{Hom}_{\mathcal{G}}(s, t) = \{g \in G;\ gsg^{-1} = t\}.$$

Again, with composition defined by group multiplication, $\mathcal{G}$ is a groupoid.

Examples (i) and (ii) are special cases of the following general construction. Let

$$G \times X \to X, \quad (g, x) \mapsto gx,$$

denote the action of a group G on a set X. We define a groupoid

$$\mathcal{G} = X \rtimes G$$

called the *transformation groupoid*, or *action groupoid*, as follows. Let

$$\mathrm{obj}\,\mathcal{G} = X \quad \text{and} \quad \mathrm{Hom}_{\mathcal{G}}(x, y) = \{g \in G;\ gx = y\}.$$

Composition of morphisms is defined via group multiplication. It is easily checked that $\mathcal{G}$ is a groupoid. Its set of morphisms can be identified as

$$\mathcal{G} \simeq X \times G,$$

where the composition of morphisms is given by

$$(gx, h) \circ (x, g) = (x, hg).$$

Note that (i) corresponds to the action of a group on a point and (ii) corresponds to the action of a group on itself via conjugation.

Example 2.1.2 (Groupoids from equivalence relations)**.** Let $\sim$ denote an equivalence relation on a set X. We define a groupoid $\mathcal{G}$, called the *graph* of $\sim$, as follows. Let

$$\mathrm{obj}\,\mathcal{G} = X,$$

and let

$$\mathrm{Hom}_{\mathcal{G}}(x,y) = \begin{cases} (x,y) & \text{if } x \sim y, \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that the set of morphisms of $\mathcal{G}$ is identified with the graph of the relation $\sim$:

$$\mathcal{G} = \{(x,y);\ x \sim y\} \subset X \times X.$$

Two extreme cases of this graph groupoid construction are particularly important. When the equivalence relation reduces to equality, i.e., $x \sim y$ if and only if $x = y$, we have

$$\mathcal{G} = \Delta(X) = \{(x,x);\ x \in X\}.$$

On the other extreme when $x \sim y$ for all x and y, we obtain the *groupoid of pairs* where

$$\mathcal{G} = X \times X.$$

As we shall see, one cannot get very far with just discrete groupoids. To get really interesting examples like the groupoids associated to continuous actions of topological groups and to foliations, one needs to consider topological as well as smooth groupoids, much in the same way as one studies topological and Lie groups.

A *topological groupoid* is a groupoid such that its set of morphisms $\mathcal{G}$ and set of objects $\mathcal{G}^{(0)}$ are topological spaces, and its composition, source, target and inversion maps are continuous.

A special class of topological groupoids, called étale groupoids, are particularly convenient to work with. An *étale groupoid* is a topological groupoid such that its set of objects $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space and its source map s (and hence its target map t) is an étale map, i.e., is a local homeomorphism.

A *Lie groupoid* is a groupoid such that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, the inclusion $\mathcal{G}^{(0)} \to \mathcal{G}$ as well as the maps s, t, i and the composition map $\circ$ are smooth, and s and t are submersions. This last condition will guarantee that the domain $\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2);\, s(\gamma_1) = t(\gamma_2)\}$ of the composition map is a smooth manifold.

Example 2.1.3. Let G be a discrete group acting by homeomorphisms on a locally compact Hausdorff space X. Then the transformation groupoid $\mathcal{G} = X \rtimes G$ is naturally an étale groupoid. In fact since $\mathcal{G} = X \times G$, we can endow $\mathcal{G}$ with the product topology. The composition, source and target maps

$$s(x,g) = x, \quad t(x,g) = gx$$

are clearly continuous and the t-fibers $\mathcal{G}^x$ are discrete subsets of $X \times G$.

Here is a concrete example of paramount importance for noncommutative geometry. Let $\mathbb{Z}$ act by rotation through a fixed angle $2\pi\theta$ on the circle $\mathbb{T}$. The corresponding transformation groupoid is the étale groupoid

$$\mathbb{T} \rtimes \mathbb{Z}.$$

As we shall see in the next section its groupoid algebra is the noncommutative torus.

If G is a locally compact group acting continuously on X, then clearly $X \rtimes G$ is a locally compact groupoid. Similarly, if G is a Lie group acting smoothly on a smooth manifold X, then the transformation groupoid $X \rtimes G$ is a Lie groupoid.

Example 2.1.4 (The fundamental groupoid of a space)**.** Let X be a topological space. We define an étale groupoid $\pi_1(X)$ as follows. The set of objects of $\pi_1(X)$ is X itself and for all $x, y \in X$, morphisms from x to y are homotopy classes of continuous paths from x to y. Under composition of paths this defines an abstract groupoid. This groupoid can be topologized as follows. Using the compact-open topology we first topologize the set of all continuous maps from $[0, 1] \to X$. The topology of $\pi_1(X)$ is defined by quotienting with respect to the homotopy equivalence relation. If X is locally path connected and locally simply connected, then $\pi_1(X)$ is an étale groupoid.

Example 2.1.5 (The holonomy groupoid of a foliation)**.** Let V be a smooth manifold and let TV denote its tangent bundle. A smooth subbundle $F \subset TV$ is called *integrable* if for any two vector fields X and Y on V with values in F their Lie bracket $[X, Y]$ takes its values in F. We call the pair (V, F) a *foliated manifold.* The *leaves* of the foliation (V, F) are the maximal connected submanifolds L of V with $T_x(L) = F_x$ for all $x \in L$. The *Frobenius integrability theorem* guarantees that there is a unique leaf passing through each point of V and this decomposes V into p-dimensional submanifolds. Here p is the rank of the integrable bundle F. We denote the leaf passing through x by L_x. A *foliation chart* for (V, F) is a coordinate system $\varphi\colon U \subset V \to \mathbb{R}^n$ for V such that for all $x \in U$,

$$\varphi^{-1}(\mathbb{R}^p \times y) = L_x \cap U,$$

where y is defined by $\varphi(x) = (y_0, y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$. A smooth *transversal* for (V, F) is a submanifold $N \subset V$ such that for all $x \in N$, the tangent space T_xN is transversal to F_x, i.e., $T_xV = T_xN \oplus F_x$. For example, fixing a foliation chart around $x \in V$ as above, $N_x := \varphi^{-1}(y_0 \times \mathbb{R}^{n-p})$ is a transversal through x.

Let L be a leaf and a and b be points in L. A *leafwise* path between a and b is a continuous path *in* L connecting a to b. We define an equivalence relation on the set of all continuous leafwise paths connecting a to b, called *holonomy equivalence* as follows. Given such a leafwise path $\alpha\colon [0, 1] \to L$, we can choose an integer k such that for all $0 \leq i < k$, $\alpha[\frac{i}{k}, \frac{i+1}{k}] \subset U_i$ is inside the domain of a foliation chart (U_i, φ_i). Let $N_0, N_1, \ldots, N_{k-1}$ denote the corresponding local transversals at $\alpha(\frac{i}{k})$, $i = 0, 1, \ldots, k-1$. We then have local diffeomorphisms $f_i \colon N_i \to N_{i+1}$ for all i, and by composing them we obtain a local diffeomorphism

$$H^{\alpha}_{ab} = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0 \colon N_a \to N_b.$$

It can be shown that the *germ* of this map is independent of the choice of foliation charts and local transversals N_i. Two leafwise paths α and β from a to b are called holonomy equivalent if their corresponding holonomy maps are equal:

$$H^{\alpha}_{ab} = H^{\beta}_{ab} \colon N_a \to N_b.$$

Leafwise homotopy equivalent paths are holonomy equivalent, though the converse need not be true. In particular if the leaf L is simply connected then all paths in L with the same endpoints are holonomy equivalent.

In general, holonomy equivalence defines an equivalence relation on the set of paths in L from a to b and we denote the quotient of the space of leafewise paths from a to b under this holonomy equivalence relation by $\mathrm{Hol}(a, b)$. Composition of paths and inversion define associative operations

$$\mathrm{Hol}(a, b) \times \mathrm{Hol}(b, c) \to \mathrm{Hol}(a, c),$$

$$\mathrm{Hol}(a, b) \to \mathrm{Hol}(b, a).$$

We can now define the *holonomy groupoid* of a foliation (V, F), denoted by $G(V, F)$.

Definition 2.1.2. Let (V, F) be a foliated space. The set of objects of $G(V, F)$ is V itself and its morphisms are defined by

$$\mathrm{Hom}(a, b) = \begin{cases} \varnothing & \text{if } a \text{ and } b \text{ are not in the same leaf,} \\ \mathrm{Hol}(a, b) & \text{if } a \text{ and } b \text{ are in the same leaf.} \end{cases}$$

It turns out that $G(V, F)$ is more than just an abstract groupoid and one can in fact define a smooth, but not necessarily Hausdorff, manifold structure on it to turn it into a smooth groupoid [41].

In practice it is a lot easier to work with an étale groupoid which is *Morita equivalent* to $G(V, F)$ in an appropriate sense. For Morita equivalence of groupoids see [98], [146]. Let us fix a complete transversal N in V. This means that N is transversal to the leaves of the foliation and each leaf has at least one intersection with N. The smooth étale groupoid $G^N(V, F)$ is defined as the full subcategory of the holonomy groupoid $G(V, F)$ whose set of objects is now equal to N. Changing the transversal N will change the groupoid $G^N(V, F)$, but its Morita equivalence class is independent of the choice of transversal. As we shall indicate later, this in particular implies that their corresponding groupoid algebras will be Morita equivalent as well.

Here is a concrete example. Let $\theta \in \mathbb{R}$. The *Kronecker foliation* of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is defined by the integral curves of the differential equation $dy = \theta dx$.

When θ is a rational number each leaf is compact and in fact is a circle, while for irrational θ all leaves are homeomorphic to $\mathbb{R}$ and are dense in $\mathbb{T}^2$. We assume that θ is irrational. Then leaves are simply connected and there is no holonomy. To describe the étale holonomy groupoid, let us fix a complete transversal, say the image of the x-axis under the projection map $\mathbb{R}^2 \to \mathbb{T}^2$. Then it is easy to see that the étale holonomy groupoid is the transformation groupoid of the action of $\mathbb{Z}$ on the circle $\mathbb{T}$ by rotation through an angle $2\pi\theta$.

For an insightful treatment of foliations and their place in noncommutative geometry see [41]. For an introduction to foliation theory and groupoids see [141], [142].

Exercise 2.1.1. Describe the étale holonomy groupoid of the Kronecker foliation when θ is rational.

Exercise 2.1.2. An *action* of a groupoid $\mathcal{G}$ on a small category $\mathcal{C}$ is by definition a functor $F\colon \mathcal{G} \to \mathcal{C}$. This idea extends the concept of action of a group on a set or a topological space. Define the transformation groupoid $\mathcal{C} \rtimes \mathcal{G}$ of an action. A particular case is the *action* of a groupoid $\mathcal{G}$ on a set X and the corresponding action groupoid $X \times \mathcal{G}$. Extend the definitions to topological and smooth groupoids.

2.2 Groupoid algebras

The *groupoid algebra* of a groupoid is a generalization of the notion of *group algebra* (or *convolution algebra*) of a group and it reduces to group algebras for groupoids with one object. To define the groupoid algebra of a locally compact topological groupoid in general one needs the analogue of a Haar measure for groupoids. For discrete groupoids as well as étale and Lie groupoids, however, the convolution product can be easily defined. We start by recalling the definition of the groupoid algebra of a discrete groupoid. As we shall see, in the discrete case the groupoid algebra can be easily described in terms of matrix algebras and group algebras. We also look at C^*-algebra completions of groupoid algebras.

Let $\mathcal{G}$ be a discrete groupoid and let

$$\mathbb{C}\,\mathcal{G} = \bigoplus_{\gamma \in \mathcal{G}} \mathbb{C}\gamma$$

denote the vector space generated by the set of morphisms of $\mathcal{G}$ as its basis. Thus an element of $\mathbb{C}\,\mathcal{G}$ is a *finite* sum $\sum a_\gamma \gamma$, where γ is a morphism of $\mathcal{G}$ and $a_\gamma = 0$ for all but a finite number of γ's. The formulas

$$\gamma_1\gamma_2 = \begin{cases} \gamma_1 \circ \gamma_2 & \text{if } \gamma_1 \circ \gamma_2 \text{ is defined,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and

$$\Big(\sum_\gamma a_\gamma \gamma\Big)^* = \sum_\gamma \overline{a_\gamma}\gamma^{-1} \tag{2.2}$$

define an associative $*$-algebra structure on $\mathbb{C}\,\mathcal{G}$. The resulting algebra is called the *groupoid algebra* of the groupoid $\mathcal{G}$. Note that $\mathbb{C}\,\mathcal{G}$ is unital if and only if the set $\mathcal{G}^{(0)}$ of objects of $\mathcal{G}$ is finite. The unit then is given by

$$1 = \sum_{x \in \mathcal{G}^{(0)}} \mathrm{id}_x.$$

An alternative description of the groupoid algebra $\mathbb{C}\,\mathcal{G}$ which is more appropriate for generalization to topological groupoids is as follows. We have an isomorphism

$$\mathbb{C}\,\mathcal{G} \simeq \{f\colon \mathcal{G} \to \mathbb{C};\ f \text{ has finite support}\},$$

under which the groupoid product formula (2.1) and the $*$-operation (2.2) are transformed to the *convolution product*

$$(f * g)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) = \sum_{\gamma_1 \in t^{-1}(\gamma)} f(\gamma_1) g(\gamma_1^{-1}\gamma),$$

and

$$(f^*)(\gamma) = \overline{f(\gamma^{-1})}.$$

Remark 2. Given a small category $\mathcal{C}$ and any ground ring k, let $k\mathcal{C}$ denote the k-module generated by the morphisms of $\mathcal{C}$. Then formula (2.1) defines an algebra structure on $k\mathcal{C}$. In practice we do not even need to start with a category. Instead we can start with a *quiver* Q (which is nothing but an *oriented graph*) and consider the set $\mathcal{P}Q$ of *paths* in Q. It is a category and the corresponding algebra is called the *quiver algebra* of Q.

The advantage of working with groupoids and over $\mathbb{C}$ is that in this case, thanks to (2.2), the groupoid algebra is a $*$-algebra and we can then complete it to a C^*-algebra. This will then enables us to bring in tools of functional analysis and operator algebras to probe the resulting noncommutative space.

Example 2.2.1. If the groupoid has only one object with automorphism group G, then the groupoid algebra is just the group algebra of G:

$$\mathbb{C}\,\mathcal{G} = \mathbb{C}G.$$

At the other extreme, if the groupoid is the *groupoid of pairs* on a finite set X, that is, the groupoid of the indiscrete equivalence relation, then

$$\mathbb{C}\,\mathcal{G} \simeq \mathrm{End}(V),$$

where V is the vector space generated by X. To see this assume that $X = \{1, 2, \ldots, n\}$ is a finite set of n elements. The morphisms of $\mathcal{G}$ can be indexed as

$$\mathcal{G} = \{(i, j);\ i, j = 1, \ldots, n\}$$

with composition given by

$$(l, k) \circ (j, i) = (l, i) \quad \text{if } k = j.$$

(Composition is not defined otherwise.)

We claim:

$$\boxed{\mathbb{C}\,\mathcal{G} \simeq M_n(\mathbb{C})}$$

Indeed, it is easily checked that the map

$$\sum a_{i,j}(i,j) \longmapsto \sum a_{i,j} E_{i,j},$$

where $E_{i,j}$ is the matrix with 1 in the $(i,\, j)$ entry and 0 elsewhere, defines an algebra isomorphism between our groupoid algebra and the algebra of $n \times n$ matrices.

Remark 3. As emphasized in the opening section of [41], the way matrices appear as a groupoid algebra is in fact closely related to the way Heisenberg discovered matrices in the context of his *matrix quantum mechanics* [94]. Roughly speaking, classical states are labeled by $i = 1, \ldots, n$, and $a_{i,j}$ are the *transition amplitudes* in moving from state i to state j. Thus, noncommutative algebras appeared first in quantum mechanics as a groupoid algebra! We recommend the reader to carefully examine the arguments of [94] and [41].

Example 2.2.2. We saw in the first example that matrix algebras and group algebras are examples of groupoid algebras. We now proceed to show that the groupoid algebra of any discrete groupoid can be expressed in terms of these two basic examples. Any groupoid $\mathcal{G}$ can be canonically decomposed as a disjoint union of *transitive groupoids*:

$$\mathcal{G} = \bigcup_i \mathcal{G}_i. \tag{2.3}$$

By definition a groupoid is called transitive if, for any two of its objects x and y, there is a morphism from x to y. From (2.3) we obtain a direct sum decomposition of the groupoid algebra $\mathbb{C}\,\mathcal{G}$:

$$\mathbb{C}\,\mathcal{G} \simeq \bigoplus_i \mathbb{C}\,\mathcal{G}_i.$$

Now, let $\mathcal{T}$ be a transitive groupoid and choose a point $x_0 \in \operatorname{obj}\mathcal{T}$ and let

$$G = \operatorname{Hom}_{\mathcal{T}}(x_0,\, x_0)$$

be the *isotropy group* of x_0. The isomorphism class of G is independent of the choice of the base point x_0. Then we have a (non-canonical) isomorphism of groupoids

$$\mathcal{T} \simeq \mathcal{T}_1 \times \mathcal{T}_2 \tag{2.4}$$

where $\mathcal{T}_1$ is a groupoid with one object with automorphism group G and $\mathcal{T}_2$ is the groupoid of pairs on the set of objects of $\mathcal{T}$. Assuming the set of objects of $\mathcal{T}$ is finite, from (2.4) we obtain an isomorphism

$$\mathbb{C}\mathcal{T} \simeq \mathbb{C}\mathcal{T}_1 \otimes \mathbb{C}\mathcal{T}_2 \simeq \mathbb{C}G \otimes M_n(\mathbb{C}),$$

where n is the number of elements of the set of objects of $\mathcal{T}$.

Putting everything together, we have shown that for any groupoid $\mathcal{G}$, at least when each connected component of $\mathcal{G}$ is finite, we have an algebra isomorphism

$$\boxed{\mathbb{C}\,\mathcal{G} \simeq \bigoplus_i \mathbb{C}G_i \otimes M_{n_i}(\mathbb{C})}$$

where the summation is over connected components of $\mathcal{G}$, n_i is the cardinality of the given connected component, and G_i is the corresponding isotropy group.

Example 2.2.3. The groupoid algebra of any discrete groupoid can be easily completed to a C^*-algebra. Although this is a special case of the C^*-algebra of a topological groupoid to be defined later in this section, we shall nevertheless discuss this case first. Let $\mathcal{G}$ be a discrete groupoid. For each object $x \in \mathcal{G}^{(0)}$ we define a $*$-representation of the groupoid algebra, $\pi_x \colon \mathbb{C}\,\mathcal{G} \to \mathcal{L}(\ell^2(\mathcal{G}_x))$, by

$$(\pi_x \gamma)(\gamma') = \gamma\gamma',$$

if the composition $\gamma\gamma'$ is defined, and is 0 otherwise. Here $\mathcal{G}_x = s^{-1}(x)$ is the fiber of the source map $s \colon \mathcal{G} \to \mathcal{G}^0$ at x. Then the formula

$$\|f\| := \sup\{\|\pi_x(f)\|;\ x \in \mathcal{G}^0\},$$

defines a pre-C^*-norm on $\mathbb{C}\,\mathcal{G}$. The reduced groupoid C^*-algebra of $\mathcal{G}$ is by definition the completion of $\mathbb{C}\,\mathcal{G}$ under this norm. For example, if $\mathcal{G}$ is the groupoid of pairs on a set X, the corresponding groupoid C^*-algebra can be shown to be isomorphic to the algebra of compact operators on the Hilbert space $H = \ell^2(X)$ with an orthonormal basis indexed by the set X:

$$C^*\mathcal{G} \simeq \mathcal{K}(H).$$

To define the convolution algebra of a topological groupoid and its C^*-completion in general, we need an analogue of Haar measure for groupoids. A *Haar measure* on a locally compact groupoid $\mathcal{G}$ is a family of measures μ^x on each t-fiber $\mathcal{G}^x = t^{-1}(x)$. The family is assumed to be continuous and left-invariant in an obvious sense (cf. [154] for details; notice that, unlike locally compact topological groups, locally compact groupoids need not have an invariant Haar measure in general). Let

$$C_c(\mathcal{G}) = \{f \colon \mathcal{G} \to \mathbb{C};\ f \text{ is continuous and has compact support}\}$$

denote the space of continuous, complex-valued functions on $\mathcal{G}$ with compact support. Given a Haar measure, we can then define, for functions with compact support
$f, g \in C_c(\mathcal{G})$, a convolution product by

$$(f * g)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) := \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma)\, d\mu^{t(\gamma)}. \tag{2.5}$$

This turns $C_c(\mathcal{G})$ into a $*$-algebra, where the involution is defined by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

To put a pre-C^*-norm on $C_c(\mathcal{G})$, note that for each t-fiber $\mathcal{G}^x = t^{-1}(x)$ we have a $*$-representation π_x of $C_c(\mathcal{G})$ on the Hilbert space $L^2(\mathcal{G}^x, \mu^x)$ defined by

$$(\pi_x f)(\xi)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2) := \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1)\xi(\gamma_1^{-1}\gamma)\, d\mu^{t(\gamma)}.$$

We can then define a pre-C^*-norm on $C_c(\mathcal{G})$ by

$$\|f\| := \sup\{\|\pi_x(f)\|;\ x \in \mathcal{G}^0\}.$$

The completion of $C_c(\mathcal{G})$ under this norm is the *reduced C^*-algebra* of the groupoid $\mathcal{G}$ and will be denoted by $C_r^*(\mathcal{G})$.

There are two special cases that are particularly important and convenient to work with: étale and smooth groupoids. Notice that for an étale groupoid each fiber is a discrete set and with the counting measure on each fiber we obtain a Haar measure. The convolution product is then given by

$$(f * g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) = \sum_{\mathcal{G}^{t(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma).$$

Notice that for each γ this is a finite sum since the support of f is compact and hence contains only finitely many points of each fiber.

We look at groupoid algebras of certain étale groupoids.

Example 2.2.4. We start with an example from [41]: an étale groupoid defined by an equivalence relation. Let

$$X = [0,1] \times \{1\} \cup [0,1] \times \{2\}$$

denote the disjoint union of two copies of the interval $[0,1]$. Let $\sim$ denote the equivalence relation that identifies $(x,1)$ in the first copy with $(x,2)$ in the second copy for $0 < x < 1$. Let $\mathcal{G}$ denote the corresponding groupoid with its topology inherited from $X \times X$. It is clear that $\mathcal{G}$ is an étale groupoid. The elements of the groupoid algebra $C_c(\mathcal{G})$ can be identified as continuous matrix-valued functions on $[0,1]$ satisfying a boundary condition:

$$C_c(\mathcal{G}) = \{f\colon [0,1] \to M_2(\mathbb{C});\ f(0) \text{ and } f(1) \text{ are diagonal}\}.$$

Example 2.2.5 (Non-Hausdorff manifolds)**.** Let

$$X = S^1 \times \{0,\, 1\}$$

be the disjoint union of two copies of the circle. We identify $(x,0) \sim (x,1)$ for all $x \neq 1$ in S^1. The quotient space $X/\sim$ is a non-Hausdorff manifold, as the points $(1,0)$ and $(1,1)$ cannot be separated. The groupoid of the equivalence relation $\sim$,

$$\mathcal{G} \,=\, \{(x,y) \,\subset X \times X;\, x \sim y\,\},$$

is a smooth étale groupoid. Its smooth groupoid algebra is given by

$$C^\infty(\mathcal{G}) = \{f \in C^\infty(S^1,\, M_2(\mathbb{C}));\ f(1) \text{ is diagonal}\}.$$

A second interesting case where one can do away with Haar measures is smooth groupoids. For this we need the notion of a *density* on an smooth manifold. Each complex number $\lambda \in \mathbb{C}$ defines a 1-dimensional representation of the group $\mathrm{GL}_n(\mathbb{R})$ by the formula $g \mapsto |\det(g)|^\lambda$. Let $|\Omega|^\lambda(M)$ denote the corresponding associated line bundle of the frame bundle of an n-dimensional manifold M. Using Jacobi's change of variable formula, one checks that for a smooth 1-density with compact support $\omega \in C_c^\infty(M, |\Omega|^1)$ the integral $\int_M \omega$ is well defined. Also note that $|\Omega|^\lambda(M) \otimes |\Omega|^\mu(M) \simeq |\Omega|^{\lambda+\mu}$ (M). Since the product of two half-densities is a 1-density and hence has a well-defined integral, we obtain an inner product $\langle \omega, \eta\rangle = \int_M \omega\bar{\eta}$ on the space of half-densities with compact support on any manifold M. Completing the pre-Hilbert space of smooth half-densities with compact support on M, we then obtain a Hilbert space canonically attached to M. In particular the first integral in (2.5) for $f,\, g \in C_c^\infty(\mathcal{G},\, |\Omega|^{\frac{1}{2}})$ is well defined and we obtain the smooth convolution algebra $C_c^\infty(\mathcal{G})$ [41], [142].

It is clear that if $\mathcal{G}$ is a groupoid with one object with automorphism group G then $C_r^*(\mathcal{G})$ is isomorphic to the reduced group C^*-algebra C_r^*G. As for groups, we can define the *full* C^**-algebra* of a groupoid by completing $C_c(\mathcal{G})$ with respect to the maximal norm

$$\|f\|_{\max} := \sup \|\pi(f)\|,$$

where π now ranges over the set of all bounded $*$representations of $C_c(\mathcal{G})$ on Hilbert spaces. The resulting C^*-algebra is denoted by $C^*(\mathcal{G})$. Notice that there is an obvious surjection $C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$. A groupoid is called *amenable* if this map is injective.

Example 2.2.6. Let X be a locally compact space with a Borel probability measure μ and $\mathcal{G}$ be the groupoid of pairs on X. Then for $f,\, g \in C_c(X \times X)$ the convolution product (2.5) reduces to

$$(f * g)(x, z) = \int_X f(x, y)g(y, z)\, d\mu(y),$$

which is reminiscent of matrix multiplication or products of integral operators. In fact the map $T\colon C_c(X \times X) \to \mathcal{K}(L^2(X, \mu))$ sending f to the integral operator

$$(Tf)(g)(x) = \int_X f(x, y)g(y)\, d\mu(y)$$

is clearly an algebra map, is one-to-one and its image is dense in the algebra of compact operators $\mathcal{K}(L^2(X, \mu))$. From this we conclude that $C^*(\mathcal{G}) \simeq \mathcal{K}(L^2(X, \mu))$.

On the other extreme, if $\mathcal{G}$ is the groupoid of the discrete equivalence relation (i.e., $x \simeq y$ if and only if $x = y$) on a locally compact space X then clearly

$$C_c(\mathcal{G}) \simeq C_c(X) \quad \text{and} \quad C_r^*(\mathcal{G}) \simeq C_0(X).$$

These two examples are continuous C^*-analogues of Example 2.2.1 above.

Example 2.2.7 (Crossed products)**.** When the groupoid is the transformation groupoid of a group action, the groupoid algebra reduces to a *crossed product algebra*. To motivate its definition, we start with a purely algebraic case first. Let $\mathrm{Aut}(A)$ denote the group of automorphisms of an algebra A. An *action* of a group G on A is a group homomorphism

$$\alpha\colon G \to \mathrm{Aut}(A).$$

Sometimes one refers to the triple $(A,\, G,\, \alpha)$ as a *noncommutative dynamical system*, or a *covariant system*. We use the simplified notation $g(a) := \alpha_g(a)$ to denote the action. The (algebraic) *crossed product* or *semidirect product* algebra $A \rtimes_\alpha G$ is defined as follows. As a vector space,

$$A \rtimes_\alpha G = A \otimes \mathbb{C}G.$$

Its product is uniquely defined by the rule

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh$$

for all $a, b \in A$ and $g, h \in G$. It is easily checked that, endowed with the above product, $A \rtimes_\alpha G$ is an associative algebra. It is unital if and only if A is unital and G acts by unital automorphisms.

One checks that $A \rtimes_\alpha G$ is the universal algebra generated by the subalgebras A and $\mathbb{C}G$ subject to the relation

$$gag^{-1} = g(a) \tag{2.6}$$

for all g in G and a in A. Representations of $A \rtimes_\alpha G$ are easily described. Given a representations $A \rtimes_\alpha G \to \mathrm{End}(V)$, we obtain a pair of representations $\pi\colon A \to \mathrm{End}(V)$ and $\rho\colon G \to \mathrm{GL}(V)$ satisfying the *covariance condition*

$$\rho(g)\pi(a)\rho(g)^{-1} = \pi(\alpha_g(a)).$$

Conversely, given such a pair of covariant representations ρ and π on the same vector space V one obtains a representation $\rho \rtimes \pi$ of $A \times_\alpha G$ on V by setting

$$(\rho \rtimes \pi)(a \otimes g) = \pi(a)\rho(g).$$

This defines a one-to-one correspondence between representations of $A \rtimes_\alpha G$ and covariant representations of (A, G, α).

Next, let us describe the C^*-algebraic analogue of the above construction. So, let

$$\alpha\colon G \to \mathrm{Aut}(A)$$

denote the action of a locally compact group G by $*$-automorphisms of a C^*-algebra A. We assume that the action is *continuous* in the sense that for all $a \in A$, the map $g \mapsto \alpha_g(a)$ from $G \to A$ is continuous. The triple $(A,\, G,\, \alpha)$ is sometimes called a *covariant system* or a C^*-*dynamical system*. A classical example is when $A = C_0(X)$ and the action of G on A is induced from its action on X by $(\alpha_g f)(x) = f(g^{-1}(x))$. A *covariant representation* of (A, G, α) is a C^*-representation π of A and a unitary representation ρ of G on the same Hilbert space H, satisfying the covariance condition

$$\rho(g)\pi(a)\rho(g)^{-1} = \pi(\alpha_g(a))$$

for all a in A and g in G. Now the idea behind the definition of the C^*-crossed product algebra is to construct a C^*-algebra whose representations are in one-to-one correspondence with covariant representations of (A, G, α). This universal problem indeed has a solution as we shall explain next.

Let us fix a right Haar measure μ on G and let $\Delta\colon G \to \mathbb{R}^+$ denote the *modular character* of G. The following formulae define a product (the *convolution product*) and an involution on the space $C_c(G, A)$ of continuous compactly supported functions on G with values in A:

$$(f * g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t))\, d\mu(s),$$
$$f^*(t) = \Delta_G(t^{-1})\alpha_t(f(t^{-1})^*).$$

Endowed with the L^1-norm $\|f\|_1 = \int_G \|f(t)\|\, d\mu(t)$, $C_c(G, A)$ is an involutive normed algebra. We let $L^1(G, A)$ denote its completion which is a Banach algebra, but not a C^*-algebra. Let (π, ρ) be a covariant representation of (A, G, α) on a Hilbert space H. We can define a representation $\pi \rtimes \rho$ of $L^1(G, A)$ by

$$(\pi \rtimes \rho)(f) = \int_G \pi(f(t))\rho(t)\, d\mu(t).$$

Much as in the definition of the full group C^*-algebra, we now define

$$\|f\| = \sup \|(\pi \rtimes \rho)(f)\|,$$

where the supremum is over the set of all covariant representations (π, ρ) of (A, G, α), and the norm on the right-hand side is the operator norm. This is clearly a finite number since $\|(\pi \rtimes \rho)(f)\| \leq \|f\|_1$ and is in fact a pre C^*-norm on $L^1(G, A)$. The *full crossed product* C^*-algebra $A \rtimes_\alpha G$ is defined as the completion of $L^1(G, A)$ under this norm. By its very definition it is almost obvious that $A \rtimes_\alpha G$ satisfies the required universal property with respect to covariant representations.

There is also a *reduced crossed product* C^*-algebra. To define it, let $\pi\colon A \to \mathcal{L}(H)$ be a faithful representation of A on a Hilbert space H, and let $L^2(G, H) = L^2(G) \hat{\otimes} H$ denote the Hilbert space tensor product of $L^2(G)$ and H. Using π we can define a covariant representation (π, ρ) for (A, G, α) on $L^2(G, H)$ by

$$(\pi(a)\xi)(t) = \alpha_t(a)(\xi(t)),$$
$$(\rho(t)\xi)(s) = \xi(t^{-1}s).$$

This then defines a (faithful) representation of $L^1(G, A)$ and a pre-C^*-norm, known as the reduced norm,

$$\|f\|_r = \|(\pi \rtimes \rho)(f)\|.$$

It can be shown that this norm is independent of the choice of the faithful representation π. The completion of $L^1(G, A)$ under this norm is called the reduced crossed product C^*-algebra. There is always a surjection $A \rtimes_\alpha G \to A \rtimes_\alpha^r G$, which is injective if G acts *amenably* on A. Amenable groups always act amenably [3]. Finally we mention that if G is a finite group, then the algebraic crossed product coincides with its C^*-algebraic version.

Fast forwarding to the next section for a moment we mention that one of the key ideas of noncommutative geometry is the following principle:

crossed product algebra = noncommutative quotient space

Example 2.2.8. Let $G = \mathbb{Z}_n$ be the cyclic group of order n and $X = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. The map $i \mapsto i+1 \pmod{n}$ defines an action of G on X and, by duality, on $A = C(\mathbb{Z}_n)$. The group algebra $\mathbb{C}\,\mathbb{Z}_n$ is generated by a unitary U and a single relation $U^n = 1$. The isomorphism $A \simeq \mathbb{C}\,\hat{\mathbb{Z}}_n$ shows that A is generated by a unitary V and a single relation $V^n = 1$. Using (2.6), the crossed product algebra $C(\mathbb{Z}_n) \rtimes \mathbb{Z}_n$ is then seen to be generated by elements U and V with relations

$$U^n = 1, \quad V^n = 1, \quad UVU^{-1} = \lambda^{-1} V,$$

where $\lambda = e^{\frac{2\pi i}{n}}$. We claim that

$$C(\mathbb{Z}_n) \rtimes \mathbb{Z}_n \simeq M_n(\mathbb{C}). \tag{2.7}$$

To verify this, consider the unitary $n \times n$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & \cdots & 0 & \lambda^{n-1} \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

They satisfy the relations

$$u^n = v^n = 1, \quad vu = \lambda uv. \tag{2.8}$$

It follows that there is a unique (C^*-)algebra map

$$C(\mathbb{Z}_n) \rtimes \mathbb{Z}_n \to M_n(\mathbb{C})$$

for which $U \mapsto u$ and $V \mapsto v$. Now it is easy to see that the subalgebra generated by u and v in $M_n(\mathbb{C})$ is all of $M_n(\mathbb{C})$. Since both algebras have dimension n^2 the map is an isomorphism.

This example is really fundamental in that it contains several key ideas that can be extended in various directions. For example, if G is a locally compact *abelian* group, then, as any other group, it acts on itself by left translation and, by duality, on $C_0(G)$. Now there is a generalization of (2.7) in the form

$$C_0(G) \rtimes G \simeq \mathcal{K}(L^2(G)),$$

where $\mathcal{K}$ denotes the algebra of compact operators.

As a further extension of (2.7), we recall the *Takai duality theorem* ([168], [15]). Let (A, G, α) be a C^* dynamical system where G is abelian. There is a *dual* action, denoted $\hat{\alpha}$, of the dual group $\hat{G}$ on $A \rtimes_\alpha G$ defined by

$$\hat{\alpha}_\chi(f)(g) = \chi(g) f(g).$$

One can thus form the *double crossed product* $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$. The Takai duality theorem states that there is a natural isomorphism of C^*-algebras

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \simeq A \otimes \mathcal{K}(L^2(G)). \tag{2.9}$$

Example 2.2.9 (Noncommutative torus)**.** As another example of a crossed product algebra we show that the noncommutative torus A_θ is a crossed product algebra. Consider the covariant system $(C(S^1),\, \alpha_\theta,\, \mathbb{Z})$, where $\mathbb{Z}$ acts on $S^1 = \mathbb{R}/\mathbb{Z}$, and hence dually on $C(S^1)$, by rotation through the angle $2\pi\theta$. We define a covariant representation (π, ρ) for this system on $H = L^2(S^1)$, where $\pi(f)$ is the multiplication operator by f and $\rho(n)$ is the unitary operator

$$(\rho(n)f)(x) = f(x + n\theta).$$

Invoking the universal property of crossed products, we obtain a C^*-algebra map $\pi \rtimes \rho\colon C(S^1) \rtimes_\theta \mathbb{Z} \to \mathcal{L}(H)$. Comparing with our original definition of the noncommutative torus in Example 1.1.7 and relations (1.8), we see that we have defined a surjection

$$\pi \rtimes \rho\colon C(S^1) \rtimes_\theta \mathbb{Z} \to A_\theta.$$

Using the universal property of A_θ we can construct an inverse to this map by sending U to $e^{2\pi i x}$ and V to 1, the generator of $\mathbb{Z}$.

Example 2.2.10. Let G be a locally compact topological group acting continuously on a locally compact Hausdorff space X, and let $\mathcal{G} = X \rtimes G$ denote the transformation groupoid associated to this action. Then we have an isomorphism of C^*-algebras

$$C_r^* \mathcal{G} \simeq C_0(X) \rtimes^r G,$$

where the action of G on $C_0(X)$ is defined by $(gf)(x) = f(g^{-1}x)$. There is also an isomorphism between the full groupoid C^*-algebra and the full crossed product algebra.

For $X = S^1$ and $G = \mathbb{Z}$ acting through rotation by an angle $2\pi\theta$, we recover the noncommutative torus as a groupoid algebra, which is one among many incarnations of A_θ.

Example 2.2.11 (Twisted group algebras)**.** A curious feature of the noncommutative torus A_θ is that although it is a deformation of $C(\mathbb{T}^2)$, it lacks one of the basic properties of the torus in that it is not a group or Hopf algebra. This is related to the fact that A_θ is *not* a group algebra. It is however a *twisted group algebra*, as we shall show in this example. First a general definition.

Let G be a locally compact group. A 2-*cocycle* (or *multiplier*) on G is a measurable function $c\colon G \times G \to \mathbb{T}$ satisfying the *cocycle condition*

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_1, g_2g_3)c(g_2, g_3) \tag{2.10}$$

for all g_1, g_2, g_3 in G. Using this 2-cocycle c, the convolution product on $C_c(G)$ can be twisted (or deformed) to

$$(f *_c g)(t) = \int_G f(s)g(s^{-1}t)c(s, s^{-1})\, d\mu(s).$$

Thanks to the cocycle property of c, this new product is associative and the C^*-completion of $C_c(G)$ is the twisted group C^*-algebra $C^*(G, c)$.

For example, let us define a 2-cocycle c_θ on $\mathbb{Z}^2$ by

$$c_\theta((m, n), (m', n')) = \exp(2\pi i\theta(mn' - nm')). \tag{2.11}$$

Then it is easy to see that the resulting twisted group algebra is isomorphic to the noncommutative 2-torus: $C^*(\mathbb{Z}^2, c_\theta) = A_\theta$.

We indicate two common sources of group 2-cocycles: from projective representations and from magnetic perturbations of quantum mechanical Hamiltonians (cf.[41] and [134]). Let $\rho\colon G \to \mathrm{PU}(H)$ be a *projective unitary representation* of a group G on a Hilbert space. Here $\mathrm{PU}(H)$ is the quotient of the group of unitary operators on a Hilbert space H by its center $\mathbb{T}$. For each $g \in G$, let $\tilde{\rho}(g) \in U(H)$ be a lift of $\rho(g)$. Let

$$c(g_1, g_2) := \tilde{\rho}(g_1g_2)^{-1}\tilde{\rho}(g_1)\tilde{\rho}(g_2) \in \mathbb{T}1.$$

It is easy to see that $c(g_1, g_2)$ is indeed a scalar multiple of the identity operator and it satisfies the cocycle condition (2.10). One may not be able to choose a continuous lifting, but a measurable lifting is enough. Notice that if G is discrete this problem will not arise and any lifting can be used.

For another series of examples, let $G = \pi_1(M)$ be the fundamental group of a smooth manifold M. Let $\omega \in \Omega^2(\tilde{M})$ be a *closed invariant* 2-form, representing a *magnetic field strength* on the universal cover of M. By invariance we mean $g^*\omega = \omega$ for all $g \in G$ under the natural action of G on the simply connected manifold $\tilde{M}$. We have $\omega = d\theta$ for a 1-form θ on $\tilde{M}$, where the 1-form θ is known as the *vector potential*. By invariance of ω we have $d(\theta - g^*\theta) = 0$ and hence $\theta - g^*\theta = d\varphi_g$, for a smooth function φ_g. It is given by the line integral $\varphi_g(x) = \int_{x_0}^x (\theta - g^*\theta)$. Let

$$c(g_1, g_2) := \exp(i\varphi_{g_1}(g_2x_0)). \tag{2.12}$$

It can be checked that c is indeed a group 2-cocycle on G. (Notice that $\varphi_{g_1}(x) - \varphi_{g_2}(g_1 x) - \varphi_{g_1 g_2}(x)$ is independent of x.) As is shown in [41], this is exactly the way noncommutative tori appeared in the noncommutative geometry treatment of the quantum Hall effect (cf. [12] for a full treatment, and also [134] for recent developments).

Exercise 2.2.1. We saw that the algebra of $n \times n$ matrices $M_n(\mathbb{C})$ is a groupoid algebra. Show that it is not a group algebra for any group ($n \geq 2$).

Exercise 2.2.2. Show that the algebra of upper triangular $n \times n$ matrices is a quiver algebra. (It is however not a groupoid algebra.)

Exercise 2.2.3 (Lack of functoriality)**.** Show that the association $\mathcal{G} \mapsto \mathbb{C}\,\mathcal{G}$ from the category of groupoids to the category of algebras is *not* a functor. Notice that $G \mapsto \mathbb{C}G$ is a functor from the category of groups to the category of algebras.

Exercise 2.2.4. Define the disjoint union $\mathcal{G}_1 \cup \mathcal{G}_2$ and cartesian products $\mathcal{G}_1 \times \mathcal{G}_2$ of two groupoids and show that

$$\mathbb{C}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathbb{C}(\mathcal{G}_1) \oplus \mathbb{C}(\mathcal{G}_2) \quad \text{and} \quad \mathbb{C}(\mathcal{G}_1 \times \mathcal{G}_2) = \mathbb{C}(\mathcal{G}_1) \otimes \mathbb{C}(\mathcal{G}_2).$$

(For the latter assume that the sets of objects of $\mathcal{G}_1$ and $\mathcal{G}_2$ are finite.)

Exercise 2.2.5. Let $p\colon E \to M$ be a smooth vector bundle on a smooth manifold M. Fiberwise addition of vectors turns E into a groupoid with M as its set of objects and its source and target maps given by $s = t = p\colon E \to M$. Show that $C_r^* E \simeq C_0(M, E)$.

Exercise 2.2.6. Let $\mathcal{G} = \pi_1(S^1)$ be the fundamental groupoid of the circle. It is an étale smooth groupoid. Describe the groupoid algebras $C_c^\infty(\mathcal{G})$ and $C_r^*(\mathcal{G})$.

Exercise 2.2.7. Let $\Theta = (\theta_{ij})$ be a skew-symmetric real $n \times n$ matrix. Show that the map $c\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ defined by

$$c(u, v) = \exp(2\pi i \langle \Theta u, v \rangle)$$

is a 2-cocycle on $\mathbb{Z}^n$. The corresponding twisted group C^*-algebra $A_\Theta = C^*(\mathbb{Z}^n, \Theta)$ is called the n-dimensional noncommutative torus and can be alternatively defined as the universal C^*-algebra generated by unitaries $U_1, \ldots, U_n$ subject to the relations

$$U_i U_j = \exp(2\pi i \theta_{ij}) U_j U_i, \quad i, j = 1, \ldots, n.$$

Exercise 2.2.8. Let a group G act by automorphisms on an algebra A and let $\tau_0\colon A \to \mathbb{C}$ be a trace. Define a linear map $\tau\colon A \rtimes_\alpha G \to \mathbb{C}$ by

$$\tau(a \otimes g) = \begin{cases} \tau_0(a) & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Show that τ is a trace if and only if τ_0 is G-*invariant* in the sense that

$$\tau_0(g(a)) = \tau_0(a) \quad \text{for all } g \in G,\ a \in A.$$

Extend this to topological crossed products. Show that the canonical trace on the noncommutative torus is obtained this way.

Exercise 2.2.9. Give a proof of (2.9) for $G = \mathbb{Z}_n$ along the lines of the proof of (2.7) given in the text.

Exercise 2.2.10. Let the cyclic group $\mathbb{Z}_n$ act on $C(S^1)$ by rotation through an angle $\frac{2\pi}{n}$. Show that $C(S^1) \rtimes \mathbb{Z}_n \simeq M_n(C(S^1))$.

Exercise 2.2.11. Let G be a finite group acting on itself by left translation. Show that $C(G) \rtimes G \simeq M_n(\mathbb{C})$ where n is the order of G.

Exercise 2.2.12. Show that the 2-cocycle (2.11) is an example of (2.12). Show that it can also be defined via a projective representation. (Hint: For the first part let $M = \mathbb{T}^2$ be the 2-torus and let ω be the volume form of $\mathbb{R}^2$.)

2.3 Morita equivalence

Morita theory, which has its origins in the representation theory of algebras [143] (cf. also [4], [8] for a textbook treatment), turns out to be a powerful tool for noncommutative geometry as well. Morita equivalent algebras share many common features, in particular they have isomorphic K-theory, Hochschild and cyclic cohomology. Moreover, for good quotients, the commutative algebra of functions on the classical quotient is Morita equivalent to the noncommutative quotient algebra. To formulate this latter result properly one needs to extend the Morita theory to the context of C^*-algebras. This will be done in the next section. We start by sketching the purely algebraic theory in this section.

In this section *algebra* means an associative, not necessarily commutative, unital algebra over a commutative ground ring k. In particular k need not be a field. All modules are assumed to be unitary in the sense that the unit of the algebra acts as the identity operator on the module. Let A and B be algebras. We denote by $\mathcal{M}_A$, ${}_A\mathcal{M}$, and ${}_A\mathcal{M}_B$ the categories of right A-modules, left A-modules and A–B-bimodules, respectively. They are abelian categories. In the following definition the equivalence of categories is assumed to be implemented by an additive functor.

Definition 2.3.1. The algebras A and B are called *Morita equivalent* if there is an equivalence of categories

$$\mathcal{M}_A \simeq \mathcal{M}_B.$$

In general there are many ways to define an additive functor $F\colon \mathcal{M}_A \to \mathcal{M}_B$. By a result of Eilenberg [72] and Watts [178], however, if F is right exact and commutes with arbitrary direct sums then there exists an essentially unique A–B-bimodule X $(= F(A))$ such that

$$F(M) = M \otimes_A X \quad \text{for all } M \in \mathcal{M}_A. \tag{2.13}$$

Notice that the converse statement is obvious. Now an equivalence of module categories is certainly right exact and commutes with direct sums and therefore is of the form (2.13). Composition of functors obtained in this way simply corresponds to the balanced tensor product of the defining bimodules.

It is therefore clear that algebras A and B are Morita equivalent if and only if there exists an A–B-bimodule X and a B–A-bimodule Y such that we have isomorphisms of bimodules

$$X \otimes_B Y \simeq A, \quad Y \otimes_A X \simeq B, \tag{2.14}$$

where the A-bimodule structure on A is defined by $a(b)c = abc$, and similarly for B. Such bimodules are called *equivalence* (*or invertible*) *bimodules*. It also follows from this result that A and B are Morita equivalent if and only if we have an equivalence of categories of left modules

$${}_A\mathcal{M} \simeq {}_B\mathcal{M}.$$

Similarly it can be shown that A and B are Morita equivalent if and only if we have an equivalence of categories of bimodules

$${}_A\mathcal{M}_A \simeq {}_B\mathcal{M}_B.$$

Example 2.3.1. Any unital algebra A is Morita equivalent to the algebra $M_n(A)$ of $n \times n$ matrices over A. The A–$M_n(A)$ equivalence bimodules are defined by spaces of *row* and *column vectors*. That is, $X = A^n$ considered as row vectors with obvious left A-action and right $M_n(A)$-action and $Y = A^n$ considered as column vectors with its obvious left $M_n(A)$ and right A-modules structures. The maps

$$\begin{aligned}(a_1, \ldots, a_n) \otimes (b_1, \ldots, b_n) &\mapsto \sum a_i b_i, \\ (a_1, \ldots, a_n) \otimes (b_1, \ldots, b_n) &\mapsto (a_i b_j)\end{aligned}$$

induce the isomorphisms (2.14). Thus an algebra cannot be recovered from its module category. Also note that even when A is commutative $M_n(A)$ is not commutative $(n \geq 2)$. This example will be generalized below.

In general it is rather hard to characterize equivalence bimodules satisfying (2.14). Given an A–B-bimodule X, we define algebra homomorphisms

$$\begin{aligned}A &\to \mathrm{End}_B(X), & B^{\mathrm{op}} &\to \mathrm{End}_A(X), \\ a &\mapsto L_a, & b &\mapsto R_b,\end{aligned}$$

where L_a is the operator of left multiplication by a and R_b is the operator of right multiplication by b. The following theorem is one of the main results of Morita (cf. [8] for a proof):

Theorem 2.3.1. *An A–B-bimodule X is an equivalence bimodule if and only if X is finitely generated and projective both as a left A-module and as a right B-module, and the natural maps*

$$A \to \mathrm{End}_B(X), \quad B^{\mathrm{op}} \to \mathrm{End}_A(X),$$

are algebra isomorphisms.

Example 2.3.2. Let P be a finitely generated projective left A-module and let

$$B = \mathrm{End}_A(P)^{\mathrm{op}}.$$

The algebras A and B are Morita equivalent. An equivalence A–B-bimodule is given by $X = P$ with obvious A–B-bimodule structure. As a special case we obtain the following geometric example.

Example 2.3.3. Let X be a compact Hausdorff space and E be a complex vector bundle on X. The algebras $A = C(X)$ of continuous functions on X and $B = \Gamma(\mathrm{End}(E))$ of continuous global sections of the endomorphism bundle of E are Morita equivalent. In fact, in view of Swan's theorem, this is a special case of the last example with $P = \Gamma(E)$ the module of global sections of E. There are analogous results for real as well as quaternionic vector bundles. If X happens to be a smooth manifold and E a smooth vector bundle, we can let A be the algebra of smooth functions on X and B be the algebra of smooth sections of $\mathrm{End}(E)$. The next example is a special case where it is shown that the noncommutative tori A_θ for rational values of θ are Morita equivalent to a commutative algebra.

Example 2.3.4 (Rational noncommutative tori)**.** We showed in Proposition 1.1.1 that when $\theta = \frac{p}{q}$ is a rational number there is a (flat) vector bundle E of rank q over $\mathbb{T}^2$ such that the noncommutative torus $A_{\frac{p}{q}}$ is isomorphic to the algebra of continuous sections of the endomorphism bundle of E,

$$A_{\frac{p}{q}} \simeq \Gamma(\mathrm{End}(E)).$$

In view of the above example this shows that $A_{\frac{p}{q}}$ is Morita equivalent to $C(\mathbb{T}^2)$. Every continuous vector bundle on a smooth manifold has a natural smooth structure. This shows that the smooth noncommutative torus $\mathcal{A}_{\frac{p}{q}}$ is Morita equivalent to $C^\infty(\mathbb{T}^2)$.

Given a category $\mathcal{C}$, we can consider its *functor category* $\mathrm{Fun}(\mathcal{C})$ whose objects are functors from $\mathcal{C} \to \mathcal{C}$ and whose morphisms are natural transformations between functors. The *center* of a category $\mathcal{C}$ is by definition the set of natural transformations from the identity functor to itself:

$$\mathcal{Z}(\mathcal{C}) := \mathrm{Hom}_{\mathrm{Fun}\,(\mathcal{C})}(\mathrm{id}, \mathrm{id}).$$

Equivalent categories obviously have isomorphic centers.

Let $Z(A) = \{a \in A;\ ab = ba \text{ for all } b \in A\}$ denote the center of an algebra A. It is easily seen that for $\mathcal{C} = \mathcal{M}_A$, the category of right A-modules, the natural map

$$Z(A) \to \mathcal{Z}(\mathcal{C}), \quad a \mapsto R_a,$$

where $R_a(m) = ma$ for any module M and any $m \in M$, is one-to-one and onto and identifies the center of $\mathcal{M}_A$ with the center of A. It follows that Morita equivalent algebras have isomorphic centers:

$$A \overset{\text{M.E.}}{\sim} B \ \Rightarrow\ Z(A) \simeq Z(B).$$

In particular two commutative algebras are Morita equivalent if and only if they are isomorphic.

We notice that commutativity is *not* a *Morita invariant property* in that a commutative algebra can be Morita equivalent to a noncommutative algebra (as with A and $M_n(A)$). In general, a property P of algebras is called a Morita invariant property if for any two Morita equivalent algebras, either both satisfy the property P or does not satisfy it. Similarly one can speak of a Morita invariant cohomology theory for algebras. In a sense, Morita invariant properties of algebras are those properties that can be expressed completely in terms of module categories, even if they were originally defined in terms of the algebra itself.

For example, as we saw above, the center of an algebra can be completely described in terms of its module category and hence is Morita invariant. It can be shown that being simple (i.e., having no non-trivial two-sided ideal), or being semisimple is a Morita invariant property for algebras. In fact an algebra A is semisimple if and only if any short exact sequence in ${}_A\mathcal{M}$ splits, which clearly shows that semisimplicity is a Morita invariant property. It can also be shown that there is a one-to-one correspondence between the lattices of two-sided ideals of Morita equivalent algebras (cf. [4], [8], [152] for a complete discussion).

We shall see in Chapter 3 that Morita equivalent algebras have isomorphic Hochschild and cyclic cohomology groups. See Exercise 2.3.4 below for a warmup. They have isomorphic algebraic K-theory as well (cf. Exercise 2.3.6 below).

Example 2.3.5 (Azumaya algebra)**.** An *Azumaya algebra* over a smooth manifold M is an algebra $A = \Gamma(\mathcal{A})$ of global sections of a smooth bundle of algebras with fibers isomorphic to $M_n(\mathbb{C})$. A simple example is obtained with $A = \Gamma(\text{End}(P))$, the algebra of sections of the endomorphism bundle of a vector bundle. This algebra is of course Morita equivalent to $C^\infty(M)$. But not all Azumaya algebras are Morita equivalent to the algebra of functions on the base. A less trivial example is obtained with $\mathcal{A} = \text{Cliff}(TM)$ the complex Clifford algebra bundle of the tangent bundle of an even-dimensional Riemannian manifold. It can be shown that this Azumaya algebra is Morita equivalent to $C^\infty(M)$ if and only if M has a Spin^c structure. In fact the existence of a Spin^c structure on M is equivalent to the existence of a complex vector bundle S on M with a pointwise *irreducible* action of $\text{Cliff}(TM)$. This in turn implies that $\text{Cliff}(TM) \simeq \Gamma(\text{End}(S))$. See [150], [85], [174] and references therein for full details.

Example 2.3.6 (The Morita category of noncommutative spaces)**.** We can think of the category Alg_k of unital k-algebras and unital algebra homomorphisms as the opposite of a category of noncommutative spaces. One problem with this category is that it does not have enough morphisms. For example, if A is a simple algebra, then, unless $A = k$, there are no algebra maps $A \to k$. So, it is better to embed this category into a 'larger' category $\widetilde{\mathrm{Alg}_k}$ which has the same objects as Alg_k but with more morphisms. A (generalized) morphism from A to B is now an *isomorphism class* of an A–B-bimodule. Composition of morphisms is defined as tensor products of bimodules

$$Y \circ X := Y \otimes_B X$$

which is clearly associative. Our first characterization of Morita equivalence in terms of equivalence bimodules as in (2.14) shows that algebras A and B are Morita equivalent if and only if they are isomorphic as objects of the category $\widetilde{\mathrm{Alg}_k}$. Let $\mathrm{Mor}(A, B)$ denote the set of isomorphism classes of A–B equivalence bimodules, or, equivalently, the set of isomorphism from A to B in $\widetilde{\mathrm{Alg}_k}$. Clearly, A is Morita equivalent to B if and only if $\mathrm{Mor}(A, B)$ is non-empty and in that case $\mathrm{Mor}(A, B)$ is a *torsor*, i.e., a principal homogeneous space, for the group $\mathrm{Mor}(A, A)$ acting on the left, or the isomorphic group $\mathrm{Mor}(B, B)$ acting on the right.

Given an algebra map $f\colon A \to B$, we can turn B into an A–B-bimodule by defining $a \cdot b \cdot b' = f(a)bb'$. This in fact defines a functor $\mathrm{Alg}_k \to \widetilde{\mathrm{Alg}_k}$. The group of *self Morita equivalences* of A, sometimes called the *noncommutative Picard group* of A ([26]), is the group of invertible A–A-bimodules

$$G = \mathrm{Aut}_{\widetilde{\mathrm{Alg}_k}}(A).$$

Notice that there is a natural map

$$\mathrm{Aut}(A) \to \mathrm{Aut}_{\widetilde{\mathrm{Alg}_k}}(A).$$

Remark 4. There is a similar problem with the category of C^*-algebras and C^*-morphisms between them. There are not that many morphisms available in the noncommutative case. For example, if A is a *simple* C^*-algebra, any naive attempt at defining the fundamental group of A as the set of homotopy classes of C^*-maps $A \to C(S^1)$ is bound to fail as there are no such maps in this case. There is now a proposal [136] for a *homotopy category of noncommutative spaces* based on C^*-algebras called the KK-category. Objects of the KK-category are C^*-algebras and morphisms are Kasparov's KK-groups. It is a triangulated category with many features that resembles the category of spectra in algebraic topology.

There is a very useful method of constructing Morita equivalence bimodules, in particular finite projective modules, for *involutive algebras* that we recall now. This is the purely algebraic counterpart of some of the concepts introduced in the next section for C^*-algebras. By an involutive algebra, also known as an $*$-algebra, we mean an algebra B over the field of complex numbers endowed with a conjugate linear map $*: B \to B$, $b \mapsto b^*$ such that for all $a, b \in B$ we have $(ab)^* = b^* a^*$ and $(a^*)^* = a$. Let B be a unital $*$-algebra and X be a unitary right B-module. A *B-valued inner product* on X is a sesquilinear form, conjugate linear in the first variable and linear in the second variable,

$$\langle \cdot , \cdot \rangle_B : X \times X \to B$$

such that for all x, y in X and b in B we have

i) $\langle x, y\rangle_B = \langle y, x\rangle_B^*$,

ii) $\langle x, yb\rangle_B = \langle x, y\rangle_B \, b$.

The inner product is called *full* if in addition we have

$$\langle X, X\rangle_B = B,$$

that is for any $b \in B$, there are elements x_i, y_i, $i = 1, \dots, n$, such that

$$b = \sum_i \langle x_i, y_i\rangle_B$$

Similarly one can define an A-valued inner product for a left module over a $*$-algebra A. In this case the inner product

$${}_A\langle \cdot , \cdot \rangle : X \times X \to A$$

is assumed to be linear in the first variable and conjugate linear in the second variable and it satisfies

i)′ ${}_A\langle x , y\rangle = {}_A\langle y , x\rangle^*$,

ii)′ ${}_A\langle ax , y\rangle = a {}_A\langle y , x\rangle^*$.

Now let A and B be unital $*$-algebras over $\mathbb{C}$. let X be an A–B-bimodule endowed with full A-valued and B-valued inner products ${}_A\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle_B$, respectively. We further assume that these two inner products satisfy the *associativity condition*: that is, for all x, y, z in X, we have

$${}_A\langle x , y\rangle z = x\langle y , z\rangle_B.$$

In this case we say X is an *imprimitivity bimodule*. Here is a simple example.

Example 2.3.7. Let A be a unital $*$-algebra and $B = M_n(A)$ the algebra of n by n matrices over A which is an $*$-algebra in a natural way. Let $X = A^n$. Considered as the space of column matrices, X is an $M_n(A)$–A-bimodule in a natural way. Then the inner products

$$\langle x , y\rangle_A = x^* y = \sum_i x_i^* y_i,$$

$${}_{M_n(A)}\langle x , y\rangle = xy^* = (x_i y_j^*)_{ij}$$

turn X into an imprimitivity $M_n(A)$–A-bimodule.

Now let X be an imprimitivity A–B-bimodule. We claim that X is finite projective both as a left A-module and as a right B-module. To see this, let 1_B be the unit of B. By fullness of $\langle \cdot , \cdot \rangle_B$, we can find $x_i, y_i, i = 1, \ldots, k$, in X such that

$$1_B = \sum_{i=1}^{k} \langle x_i , y_i \rangle_B .$$

Let $e_i, i = 1, \ldots, k$, be a basis for A^k. Define the map

$$P \colon A^k \to X, \quad P(e_i) = y_i .$$

We claim that P splits as an A-module map and hence X is a finite projective left A-module. To this end consider the A-linear map

$$I \colon X \to A^k, \quad I(x) = \sum_i {}_A\langle x , x_i \rangle e_i .$$

We have

$$\begin{aligned} (PI)(x) &= \sum_i {}_A\langle x , x_i \rangle y_i = \sum_i x \langle x_i , y_i \rangle_B \quad \text{(by associativity)} \\ &= x. \end{aligned}$$

A similar proof shows that X is finite and projective as a right B-module. But in fact more is true. We can show that X is an equivalence bimodule. To this end let X^* denote the *complex conjugate* of the complex vector space X whose elements we denote by $\bar{x}$, $x \in X$. It is a B–A-bimodule by

$$b\bar{x}a := \overline{a^* x b^*} .$$

Although we do not need it here, we can also endow X^* with algebra valued inner products by setting

$${}_B\langle \bar{x}, \bar{y} \rangle = \langle x, y \rangle_B \quad \text{and} \quad \langle \bar{x}, \bar{y} \rangle_A = {}_A\langle x, y \rangle .$$

Consider the maps

$$f \colon X \otimes_B X^* \to A, \quad x \otimes \bar{y} \mapsto {}_A\langle x , y \rangle, \quad g \colon X^* \otimes_A X \to B, \quad \bar{x} \otimes y \mapsto \langle x , y \rangle_B .$$

Both maps are clearly bimodule maps. They are also surjective thanks to the fullness of inner products. It follows from Proposition 4.4 in [8] that f and g are isomorphisms. This of course shows that X is a Morita equivalence bimodule and the algebras A and B are Morita equivalent. We give a couple of examples illustrating this method.

Example 2.3.8. Let M be a smooth compact manifold and let E be a smooth complex vector bundle on M. Let $X = C^\infty(E)$ denote the set of smooth sections of E. It is an A–B-bimodule in an obvious way, with $A = C^\infty(\operatorname{End} E))$ acting on the left and $B = C^\infty(M)$ acting on the right. A Hermitian inner product $\langle\, ,\rangle$ on E, which is assumed to be conjugate linear in the first variable, will enable us to define algebra valued inner products ${}_A\langle\cdot\,,\cdot\rangle$ and $\langle\cdot\,,\cdot\rangle_B$ on X by

$$\langle\xi,\,\eta\rangle_B(m) = \langle\xi(m),\,\eta(m)\rangle, \quad {}_A\langle\xi\,,\,\eta\rangle(\zeta)(m) = \langle\eta(m),\,\zeta(m)\rangle\,\xi(m),$$

where ξ,η, and ζ are sections of E and $m \in M$. It is easy to see that ${}_AX_B$ satisfies both the fullness and associativity conditions and hence is an equivalence bimodule. This gives another proof of the Morita equivalence of $C^\infty(M)$ and $C^\infty(\mathrm{End}(E))$.

Example 2.3.9. Let us show that the smooth noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\frac{1}{\theta}}$ are Morita equivalent by exhibiting an equivalence bimodule between them. Let $X = \mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}$. It is easily checked that the following formulae define a *right* $\mathcal{A}_\theta$-module structure on X:

$$(\xi\cdot U)(x) = \xi(x+\theta), \quad (\xi\cdot V)(x) = e^{2\pi i x}\xi(x)$$

for all $\xi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. Let us denote the generators of $\mathcal{A}_{\frac{1}{\theta}}$ by $\bar{U}$ and $\bar{V}$. We can define a *left* action of $\mathcal{A}_{\frac{1}{\theta}}$ on X by

$$(\bar{U}\xi)(x) = \xi(x+1), \quad (\bar{V}\xi)(x) = e^{\frac{-2\pi i x}{\theta}}\xi(x).$$

Since the two actions commute, X is an $\mathcal{A}_{\frac{1}{\theta}}$–$\mathcal{A}_\theta$-bimodule. To show that it is an equivalence bimodule we endow X with algebra valued inner products:

$${}_{\frac{1}{\theta}}\langle\cdot\,,\cdot\rangle\colon X\otimes X \to \mathcal{A}_{\frac{1}{\theta}}, \quad \langle\cdot\,,\cdot\rangle_\theta\colon X\otimes X \to \mathcal{A}_\theta,$$

given by

$${}_{\frac{1}{\theta}}\langle\xi,\,\eta\rangle = \sum_{m,n}\sum_k \xi(n-k)\bar{\eta}(n-k-m\theta)\bar{U}^m\bar{V}^n$$

and

$$\langle\xi,\,\eta\rangle_\theta = \sum_{m,n}\sum_k \bar{\xi}(n-k\theta)\eta(n-m-k\theta)U^mV^n.$$

We leave it to the reader to check the axioms of inner products and the associativity condition (see Exercise 2.3.9 below).

Exercise 2.3.1. Let X be a Morita equivalence A–B-bimodule. Show that X is finitely generated projective both as an A-module and as a B-module.

Exercise 2.3.2. Show that if two discrete groupoids, with finite sets of objects, are equivalent (as categories), then their groupoid algebras are Morita equivalent. This can be extended to topological groupoids and their corresponding C^*-algebras and it forms an important principle in noncommutative geometry. For example, it allows one to replace the holonomy groupoid of a foliation by a Morita equivalent étale holonomy groupoid which is much easier to work with.

Exercise 2.3.3 (Duality)**.** Let X be a Morita equivalence A–B-bimodule and let $X^* = \mathrm{Hom}_A(X, A)$. Show that X^* is a B–A-bimodule and we have isomorphism of bimodules:

$$X \otimes_B X^* \simeq A, \quad X^* \otimes_A X \simeq B.$$

Thus in (2.14) we can take $Y = X^*$.

Exercise 2.3.4. For an algebra A, let $[A, A]$ denote the linear subspace of A spanned by commutators $ab - ba$ for $a, b \in A$ (the *commutator subspace* of A). Show that the 'trace map'

$$\mathrm{Tr}\colon M_n(A) \to A, \quad \mathrm{Tr}(a_{ij}) = \sum a_{ii}$$

induces a k-linear isomorphism between the commutator quotient spaces of A and $M_n(A)$:

$$\frac{M_n(A)}{[M_n(A),\, M_n(A)]} \simeq \frac{A}{[A,\, A]}.$$

In particular the space of traces of A and of $M_n(A)$ are isomorphic. Extend this fact to arbitrary Morita equivalent algebras. This is a special case of Morita invariance of Hochschild and cyclic homology, another important principle in noncommutative geometry, to be discussed in the next chapter.

Exercise 2.3.5. Let $e \in A$ be an idempotent and let $f = 1 - e$. Show that the algebras eAe and fAf are Morita equivalent. Example 2.3.1 is a special case of this exercise.

Exercise 2.3.6 (Morita invariance of K-theory)**.** Show that an additive equivalence of categories $F\colon \mathcal{M}_A \rightsquigarrow \mathcal{M}_B$ sends a finitely generated projective module over A to a module of the same type over B and induces an equivalence of the corresponding categories of finitely generated projective modules. Since the definition of the algebraic K-theory of an algebra depends solely on its category of finitely generated projective modules (cf. e.g. [160]), it follows that Morita equivalent algebras have isomorphic algebraic K-theories.

Exercise 2.3.7. Let G be a finite group acting on a finite set X. Show that the algebras $C(X/G)$ and $C(X) \rtimes G$ are Morita equivalent if and only if the action of G is *free*. When the action is free, define an equivalence $C(X) \rtimes G$–$C(X/G)$-bimodule structure on $C(X)$ with algebra valued inner products.

Exercise 2.3.8. Let G be a *finite* group and H and K be subgroups of G. Then H acts by right multiplication on the right coset space $K\backslash G$ and K acts by left multiplication on the left coset space G/H. Show that the algebras

$$A = C(K\backslash G) \rtimes_\alpha H \quad \text{and} \quad B = C(G/H) \rtimes_\beta K$$

are Morita equivalent. (Hint: Let $X = C(G)$ and turn X into an equivalence A–B-bimodule by defining on X covariant representations for $(C(K\backslash G),\, H,\, \alpha)$ and $(C(G/H),\, K,\, \beta)$ and algebra valued inner products $X \times X \to A$ and $X \times X \to B$. See also Example 2.4.5.)

Exercise 2.3.9. Verify the claim at the end of Example 2.3.9.

2.4 Morita equivalence for C^*-algebras

To extend the Morita theory to non-unital and to topological algebras needs more work and new ideas. For C^*-algebras we have Rieffel's notion of Morita equivalence [155], originally called *strong Morita equivalence*, that we recall in this section. For a complete account see [15], [41], [152] and references therein. In this section C^*-algebras are not assumed to be unital.

For C^*-algebras one is mostly interested in their $*$-representations by bounded operators on a Hilbert space as opposed to general representations. Let A and B be C^*-algebras. To compare their corresponding categories of representations, we need A–B-bimodules X such that if H is a Hilbert space and a right A-module, then $H \otimes_A X$ is also a Hilbert space in natural way. This observation leads one to the concepts of *Hilbert module* and *equivalence bimodule* recalled below. Hilbert modules can also be thought of as simultaneous generalizations of notions of Hilbert space and C^*-algebra.

Definition 2.4.1. Let B be a C^*-algebra. A right *Hilbert B-module* is a right B-module X endowed with a B-valued inner product

$$\langle \cdot , \cdot \rangle \colon X \times X \to B$$

such that X is complete with respect to its natural norm. More precisely, $\langle \cdot , \cdot \rangle$ is a sesquilinear form, conjugate linear in the first variable and linear in the second variable, such that for all x, y in X and b in B we have

i) $\langle x, y\rangle \,=\, \langle y, x\rangle^*$,

ii) $\langle x, yb\rangle \,=\, \langle x, y\rangle\, b$,

iii) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ implies that $x = 0$,

iv) X is complete with respect to the norm

$$\|x\| := \|\langle x, x\rangle\|^{1/2}.$$

When only conditions i)–iii) are satisfied, X is called a right *pre-Hilbert B-module*. It can be easily shown that any B-valued inner product satisfying conditions i)–iii) above satisfies the *generalized Cauchy–Schwarz inequality*

$$\langle x, y\rangle^* \langle y, x\rangle \;\leq\; \|\langle x, x\rangle\|\, \langle y, y\rangle \quad \text{as elements of } B.$$

From this it follows that $\| \cdot \| := \langle x, x\rangle^{1/2}$ defines a norm on X. Any pre-Hilbert module can be completed to a Hilbert module. A Hilbert B-module X is called *full* if the ideal

$$I = \operatorname{span}\{\langle x, y\rangle;\; x, y \in X\}$$

is dense in B. Notice that if B is unital and its unit acts as the identity operator on X, then X is automatically full. The notion of a left Hilbert B-module is defined similarly.

Example 2.4.1. 1. For $B = \mathbb{C}$, a right Hilbert B-module is just a Hilbert space where the inner product is antilinear in the first factor (physicists' convention).

2. For any C^*-algebra B, $X = B$ with its natural right B-module structure and the inner product

$$\langle a,\, b\rangle = a^* b$$

is clearly a right Hilbert B-module. Notice that the norm of B as a Hilbert B-module is the same as its norm as a C^*-algebra because of the C^*-axiom $\|a^*a\| = \|a\|^2$.

3. For any C^*-algebra B let $X = \ell^2(B)$ denote the space of square summable sequences $(b_1, b_2, \dots)$ from B with the natural right B-action and the inner product

$$\langle (a_1, a_2, \dots), (b_1, b_2, \dots)\rangle = \sum_{i=1}^{\infty} a_i^* b_i.$$

It is clearly a Hilbert B-module.

4. A nice geometric example to keep in mind is the following. Let M be a locally compact Hausdorff space, let E be a complex vector bundle on M endowed with a Hermitian inner product, and let $X = \Gamma_0(E)$ be the space of continuous sections of E vanishing at infinity. One defines a Hilbert $C_0(M)$-module structure on X by letting

$$\langle s,\, t\rangle(m) = \langle s(m),\, t(m)\rangle$$

for continuous sections s and t of E. It is however not true that all Hilbert modules over $C_0(M)$ are of this type. It can be shown that an arbitrary Hilbert module over $C_0(M)$ is isomorphic to the space of continuous sections of a *continuous field of Hilbert spaces* over X (cf. [69], [41] for definitions).

Every bounded linear operator on a Hilbert space has an *adjoint*. The analogous statement for Hilbert modules, however, is not true. This is simply because, for the existence of the adjoint operator, the existence of complementary submodules is crucial, but, even purely algebraically, a submodule of a module need not have a complementary submodule.

Definition 2.4.2. Let X be a Hilbert B-module. A B-linear map $T\colon X \to X$ is called *adjointable* if there is a B-linear map $T^*\colon X \to X$ such that for all x and y in X

$$\langle Tx,\, y\rangle = \langle x,\, T^*y\rangle.$$

It is easy to show that an adjointable operator is bounded in the operator norm and the algebra $\mathcal{L}_B(X)$ of adjointable operators on X is a C^*-algebra. For $B = \mathbb{C}$ we recover the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space.

Another important C^*-algebra attached to a Hilbert module is its algebra of *compact operators*. For any $x, y \in X$ the operator $T = |x\rangle\langle y| \in \mathcal{L}_B(X)$ is defined, using Dirac's 'bra-ket' notation, by

$$T(z) = |x\rangle\langle y|\,(z) := x\,\langle y,\, z\rangle \quad \text{for all } z \in X.$$

The operator T is clearly B-linear and adjointable with adjoint given by $T^* = |y\rangle\langle x|$. The linear span of such *'rank one operators'* is a two-sided $*$-ideal of $\mathcal{L}_B(X)$, called the ideal of *finite rank operators*. The C^*-algebra $\mathcal{K}_B(X)$ of compact endomorphism of X is defined as the norm closure of the ideal of finite rank operators in $\mathcal{L}_B(X)$. Again, for $B = \mathbb{C}$, we recover the algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space.

Finite projective modules over C^*-algebras have a very useful characterization in terms of Hilbert modules [155]:

Proposition 2.4.1. *Let B be a unital C^*-algebra. A right B-module E is finite and projective if and only if there is a right Hilbert B-module structure on E such that*

$$1_E \in \mathcal{K}_B(E).$$

Definition 2.4.3. Let A and B be C^*-algebras. An A–B equivalence bimodule is an A–B-bimodule X such that

(a) X is a full left Hilbert A-module and a full right Hilbert B-module,

(b) for all $x, y, z \in X$, the "associativity formula"

$$_A\langle x, y\rangle\, z = x\, \langle y, z\rangle_B \tag{2.15}$$

holds.

Equivalence bimodules are also known as *imprimitivity bimodules*.

Definition 2.4.4. Two C^*-algebras A and B are called (strongly) Morita equivalent if there exists an equivalence A–B-bimodule.

Thanks to associativity condition (2.15) and fullness, the operators of left and right multiplications $L_a(x) = ax$ and $R_b(x) = xb$ are easily seen to be adjointable. In fact, for all $a \in A$, $b \in B$ and $x, y, z \in X$ we have

$$z\langle ax, y\rangle_B = {}_A\langle z, ax\rangle\, y = {}_A\langle z, x\rangle\, a^* x = z\langle x, a^* y\rangle_B,$$

and hence $\langle ax, y\rangle_B = \langle x, a^* y\rangle_B$, since X is a full right Hilbert B-module. It follows that L_a is adjointable with $L_a^* = L_{a^*}$. Similarly R_a is adjointable with $R_a^* = R_{a^*}$. We obtain C^*-algebra representations

$$L\colon A \to \mathcal{L}_B(X) \quad \text{and} \quad R\colon B^{\mathrm{op}} \to \mathcal{L}_A(X).$$

Example 2.4.2. 1. Any C^*-algebra A is Morita equivalent to itself where the equivalence A–A-bimodule is $X = A$ with inner products $_A\langle x, y\rangle = xy^*$ and $\langle x, y\rangle_A = x^* y$.

2. Any Hilbert space H is an equivalence $\mathcal{K}(H)$–$\mathbb{C}$-bimodule. This shows that the algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space is Morita equivalent to $\mathbb{C}$ and hence for $H = \mathbb{C}^n$ one recovers the basic Morita equivalence between $M_n(\mathbb{C})$ and $\mathbb{C}$. More generally, any full right Hilbert B-module X is an equivalence $\mathcal{K}_B(X)$–B-bimodule.

3. It can be shown that two *unital* C^*-algebras are Morita equivalent as C^*-algebras if and only if they are Morita equivalent as abstract algebras as defined in Section 2.3 (see [11]). It is also known that σ-unital C^*-algebras A and B are (strongly) Morita equivalent if and only if their stabilizations are isomorphic: $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$. (A C^*-algebra is called σ-unital if it has a countable approximate identity.) In particular this implies that Morita equivalent σ-unital C^*-algebras have isomorphic K-theories (cf. [23]). It is shown in [77] that Morita equivalent C^*-algebras have the same K-theory, whether they are σ-unital or not.

Example 2.4.3 (Correspondences, tensor products, and duality)**.** Let A and B be C^*-algebras. A *correspondence* (or *generalized homomorphism*) from A to be B consists of a right Hilbert B-module X and a homomorphism $\phi\colon A \to \mathcal{L}_B(X)$. The most obvious correspondences are defined by homomorphisms $f\colon A \to B$ since we can then take $X = B$ with its right Hilbert B-module structure as in Example 2.4.1. 2) and with left A-action induced by f. A less trivial example is an equivalence bimodule.

Correspondences can be composed and the composition is via *tensor products* of Hilbert modules. In general, let X be a Hilbert right A-module, Y a Hilbert right B-module and let $\phi\colon A \to \mathcal{L}_B(Y)$ be a $*$-homomorphism. Then Y is an A–B-bimodule and we can form the algebraic tensor product $X \otimes_A Y$ and define a B-valued inner product on it by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1,\, \phi(\langle x_1, x_2\rangle_A) y_2 \rangle.$$

The completion of $X \otimes_A Y$ under the above inner product is a Hilbert B-module, denoted by $X \otimes_\phi Y$. This operation of tensor product is associative up to natural isomorphism. We can thus conceive of a category whose objects are C^*-algebras and whose morphisms are isomorphism classes of correspondences between C^*-algebras. Since correspondences can be added it is an additive category. This should be compared with the purely algebraic *Morita category of noncommutative spaces* introduced in Example 2.3.6. The category of correspondences is closely related to Kasparov's KK-theory. For example the KK-group $KK(A, B)$ is the set of homotopy classes of correspondences with a generalized Fredholm operator.

There is an alternative description of Morita equivalence via tensor products of correspondences [41]. Let 1_A denote the equivalence A–A-bimodule defined in Example 2.4.2. 1). It can be shown that C^*-algebras A and B are Morita equivalent if and only if there is an A-B correspondence X and a B-A correspondence Y with

$$X \otimes_B Y \simeq 1_A, \quad Y \otimes_A X \simeq 1_B.$$

In fact, if X is an equivalence A–B-bimodule we can take $Y = X^*$, the *complex conjugate* of X. It is a B–A-bimodule by $b\bar{x}a := \overline{a^* x b^*}$. With inner products ${}_B\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle_B$ and $\langle \bar{x}, \bar{y}\rangle_A = {}_A\langle x, y\rangle$ it is an equivalence B–A-bimodule, and one can show that

$$X \otimes_B X^* = 1_A, \quad X^* \otimes_A X = 1_B.$$

Example 2.4.4 (Morita equivalence and group actions)**.** Let G be a locally compact topological group acting continuously on a locally compact Hausdorff space X. The action is called *proper* if the map $G \times X \to X \times X$ sending (g, x) to (gx, x) is proper. For example the action of a compact group is always proper. One checks that the orbit space X/G of a proper action is locally compact and Hausdorff. To gain a better feeling for proper actions, let G be a discrete group. Then an action of G on a topological space X is continuous if and only if each element of G acts as a homeomorphism of X. In this case the action is proper if and only if for any compact subset $\Delta \subset X$ the set

$$\{g \in G;\ g\Delta \cap \Delta \neq \varnothing\}$$

is a *finite* subset of G. Roughly speaking, each compact subset of X is 'eventually pushed to infinity'. The action is called *free* if for all $g \in G, x \in X$, $gx = x$ iff $g = e$, the identity of the group.

We show, following [156], that when the action is *free and proper* the commutative C^*-algebra $B = C_0(X/G)$ of continuous functions on the quotient space vanishing at infinity, is Morita equivalent to the crossed product algebra $A = C_0(X) \rtimes G$. To this end, we turn

$$E = C_c(X)$$

into a pre-Hilbert right B-module on which B acts by pointwise multiplication (elements of B are regarded as G-invariant functions on X). The B-valued inner product on E is defined by

$$\langle f\,,\, g\rangle_B(x) = \int_G \bar{f}(t^{-1}x) g(x^{-1}t)\, dt.$$

To define a left action of the crossed product algebra $A = C_0(X) \rtimes G$ on E we need a covariant action of G and $C_0(X)$ on X. Let $C_0(X)$ act on $C_c(X)$ by pointwise multiplication and G act on $C_c(X)$ by

$$(t \cdot f)(x) = \Delta(t)^{-\frac{1}{2}} f(t^{-1}(x)) \quad \text{for all } t \in G,$$

where Δ is the modular character of G. Then the covariance condition is easily checked. The associativity condition

$${}_A\langle f,\, g\rangle h = f\langle g\,,\, h\rangle_B$$

of course fixes the A-valued inner product. This defines a pre-equivalence bimodule which one can then complete to obtain an equivalence bimodule.

In particular, for $X = G$ and G acting by right translations, which is obviously a free and proper action, we obtain the known Morita equivalence

$$\mathbb{C} \overset{\mathrm{M}}{\sim} C^*(G, G) = \mathcal{K}(L^2(G)).$$

Example 2.4.5 (Morita equivalence and noncommutative tori)**.** We saw in Example 2.3.9 that smooth noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{\frac{1}{\theta}}$ are Morita equivalent by defining an explicit equivalence bimodule for these two algebras. A similar technique works to show that the corresponding C^*-algebras A_θ and $A_{\frac{1}{\theta}}$ are Morita equivalent. Let us recall a much more general result from [156] that contains this as a special case. Let G be a locally compact topological group and let H and K be closed subgroups of G. Then H acts by right multiplication on the right coset space $K\backslash G$ and K acts by left multiplication on the left coset space G/H. Let α and β denote the corresponding dual actions of H and K on $C_0(K\backslash G)$ and $C_0(G/H)$, respectively. Then the result is that the crossed product algebras

$$A = C_0(K\backslash G) \rtimes_\alpha H, \quad B = C_0(G/H) \rtimes_\beta K \tag{2.16}$$

are Morita equivalent. We refer to [156] for a description of the equivalence bimodule in this case (cf. also [85] for an exposition).

As a special case we can take $G = \mathbb{R}$, $H = \mathbb{Z}$, and $K = \theta\mathbb{Z}$. It is easy to see that the corresponding crossed product algebras are A_θ and $A_{\frac{1}{\theta}}$. Now the group $\mathrm{GL}_2(\mathbb{Z})$ acts on the space of parameters $\theta \in \mathbb{R}$ of noncommutative tori by linear fractional transformations $\theta \mapsto \frac{a\theta+b}{c\theta+d}$, for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$. We have seen that for $\theta' = \frac{1}{\theta}$ and $\theta' = \theta + 1$ the noncommutative tori $A_{\theta'}$ and A_θ are Morita equivalent; in fact $A_{\theta+1} = A_\theta$. Since $\mathrm{GL}_2(\mathbb{Z})$ is generated by these transformations, we see that for any $g \in \mathrm{GL}_2(\mathbb{Z})$, the noncommutative tori A_θ and $A_{g\theta}$ are Morita equivalent:

$$A_\theta \overset{\mathrm{M}}{\sim} A_{g\theta}.$$

The converse is also true: if A_θ and $A_{\theta'}$ are Morita equivalent then there is a matrix $g \in \mathrm{GL}_2(\mathbb{Z})$ such that $\theta' = g\theta$. These results have been extended to higher dimensional noncommutative tori in [126], [159].

Exercise 2.4.1. Consider the antipodal action of $\mathbb{Z}_2$ on the two-sphere S^2. Show that the algebras $M_2(C(S^2))$ and $C(S^2) \rtimes \mathbb{Z}_2$ are *not* Morita equivalent. (Hint: By Example 2.4.4, $C(S^2) \rtimes \mathbb{Z}_2$ is Morita equivalent to $C(S^2/\mathbb{Z}_2)$, the algebra of continuous functions on the real projective plane.)

2.5 Noncommutative quotients

From a purely set theoretic point of view, all that one needs in order to form a *quotient space* $X/\sim$ is an equivalence relation $\sim$ on a set X. If X is a topological space, then there is of course a canonical topology, the quotient topology, on the quotient space. If X has some extra features like being a Hausdorff space, a manifold, or a smooth algebraic variety, etc., we may want these features to be shared by the quotient space as well. If this can be done, then we say that we have a *good quotient*, and otherwise the quotient is called a *bad quotient*. Now the quotient of a Hausdorff space may easily fail to be Hausdorff. Similarly the quotient of a manifold by an equivalence relation can easily fail to be a manifold again. So we have a problem: *how to deal with bad quotients*. In the context of algebraic geometry, bad quotients are dealt with by enlarging the category of schemes to the category of *stacks*, about which we shall say nothing, but check [139] for an introduction to this circle of ideas.

The solution of this problem, as pioneered by Connes, in noncommutative geometry involves extending the category of classical spaces to noncommutative spaces. It hinges on the fact that an equivalence relation is usually obtained from a much richer structure by forgetting part of this structure. For example, an equivalence relation $\sim$ may arise from an action of a group G on a set X where $x \sim y$ if and only if $gx = y$ for some g in G (*orbit equivalence*). Note that there may be, in general, many g's with this property. That is, x may be identifiable with y in more than one way. In particular an element $x \in X$ may have a non-trivial group of automorphisms, or self-equivalences. When we form the equivalence relation this extra information is completely lost. The key idea in dealing with bad quotients in noncommutative geometry is to keep track of this extra information and organize it into a (discrete, topological, or smooth) groupoid. We call, rather vaguely, this extra structure the *quotient data*.

Now Connes' dictum in forming noncommutative quotients can be summarized as follows:

$$\boxed{\text{quotient data} \rightsquigarrow \text{groupoid} \rightsquigarrow \text{groupoid algebra}}$$

where the noncommutative quotient is defined to be the groupoid algebra itself:

$$\boxed{\text{noncommutative quotient space} = \text{groupoid algebra}} \tag{2.17}$$

Depending on what we want to do, we may want to consider a purely algebraic, smooth, continuous, or measure theoretic groupoid algebra, represented by an abstract algebra, a topological Fréchet algebra, a C^*-algebra, or a von Neumann algebra, respectively. The type of algebra is usually dictated by the nature of the problem at hand.

Why is this a reasonable approach? The answer is that first of all, by Theorem 2.5.1 below, when the classical quotient is a reasonable space, the algebra of continuous functions on the classical quotient is Morita equivalent to the groupoid algebra. Now it is known that Morita equivalent algebras have isomorphic K-theory, Hochschild, and cyclic cohomology groups. Thus the topological invariants defined via noncommutative geometry are the same for classical and noncommutative quotients and no information is lost.

For bad quotients there is no reasonable classical space to deal with but we think of the noncommutative algebra defined as a groupoid algebra as representing a noncommutative quotient space. Thanks to noncommutative geometry, tools like K-theory, K-homology, cyclic cohomology, the local index formula, etc., can be applied to great advantage in the study of these noncommutative spaces.

Example 2.5.1. We start with a simple example from [41]. Let $X = \{a, b\}$ be a set with two elements and define an equivalence relation on X that identifies a

and b, $a \sim b$:

$$\overset{a}{\bullet} \leftrightsquigarrow \overset{b}{\bullet}$$

The corresponding groupoid $\mathcal{G}$ here is the *groupoid of pairs* on the set X whose morphisms (arrows) are indexed by pairs (a,a), (a,b), (b,a), (b,b). By Example 2.2.1 the groupoid algebra $\mathbb{C}\mathcal{G}$ is isomorphic to the algebra of 2 by 2 matrices $M_2(\mathbb{C})$. The algebra isomorphism is given by the map

$$f_{aa}(a,a) + f_{ab}(a,b) + f_{ba}(b,a) + f_{bb}(b,b) \mapsto \begin{pmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{pmatrix}.$$

The algebra of functions on the classical quotient, on the other hand, is given by

$$\{f \colon X \to \mathbb{C};\ f(a) = f(b)\} \simeq \mathbb{C}.$$

Thus the classical quotient and the noncommutative quotient are Morita equivalent.

$$M_2(\mathbb{C}) \xleftarrow{\text{noncommutative quotient}} \overset{a}{\bullet} \leftrightsquigarrow \overset{b}{\bullet} \xrightarrow{\text{classical quotient}} \mathbb{C}$$

Example 2.5.2. The above example can be generalized. For example let X be a finite set with n elements with the equivalence relation $i \sim j$ for all i, j in X.

$$\overset{1}{\bullet} \leftrightsquigarrow \overset{2}{\bullet} \cdots\cdots\cdots \overset{n-1}{\bullet} \leftrightsquigarrow \overset{n}{\bullet}$$

The corresponding groupoid $\mathcal{G}$ is the groupoid of pairs on X and, again by Example 2.2.1, its groupoid algebra, representing the noncommutative quotient, is

$$\mathbb{C}\mathcal{G} \simeq M_n(\mathbb{C}).$$

The algebra of functions on the classical quotient is given by

$$\{f \colon X \to \mathbb{C};\ f(a) = f(b) \text{ for all } a, b \in X\} \simeq \mathbb{C}.$$

Again the classical quotient is obviously Morita equivalent to the noncommutative quotient.

Example 2.5.3. Let G be a finite group acting on a finite set X. The algebra of functions on the classical quotient is

$$C(X/G) = \{f \colon X \to \mathbb{C};\ f(x) = f(gx) \text{ for all } g \in G,\ x \in X\} \simeq \bigoplus_{\mathcal{O}} \mathbb{C},$$

where $\mathcal{O}$ denotes the set of orbits of X under the action of G.

The noncommutative quotient, on the other hand, is defined to be the groupoid algebra of the transformation groupoid $\mathcal{G} = X \rtimes G$. Note that as we saw before this algebra is isomorphic to the crossed product algebra $C(X) \rtimes G$. From Section 2.2 we have

$$\mathbb{C}\,\mathcal{G} \simeq C(X) \rtimes G \simeq \bigoplus_{i \in \mathcal{O}} \mathbb{C}G_i \otimes M_{n_i}(\mathbb{C}),$$

where G_i is the isotropy group of the i-th orbit, and n_i is the size of the i-th orbit. Comparing the classical quotient with the noncommutative quotient we see the following:

i) The two algebras are Morita equivalent if and only if the action of G is free, that is $G_i = \{1\}$ for all orbits i (compare with Theorem 2.5.1). In this case we have

$$C(X/G) \simeq \bigoplus_{\mathcal{O}} \mathbb{C} \overset{M}{\sim} \bigoplus_{i \in \mathcal{O}} M_{n_i}(\mathbb{C}) \simeq \mathbb{C}\,\mathcal{G}.$$

It is still interesting to spell out the form of the equivalence bimodule when the action is free. Let $M = C(X)$ be the space of functions on X. In Exercise 2.3.7 we ask the reader to turn M into a $\mathbb{C}\,\mathcal{G}$-$C(X/G)$ equivalence bimodule.

ii) When the action is not free, the information about the isotropy groups is not lost in the noncommutative quotient construction, whereas the classical quotient totally neglects the isotropy groups.

Example 2.5.4. Let G be a finite group acting freely and continuously on a compact Hausdorff space X. The classical quotient X/G is a compact Hausdorff space and we consider this a good quotient. What about the noncommutative quotient? By (2.17) the noncommutative quotient should be the groupoid algebra of the transformation groupoid $X \rtimes G$. We saw in Example 2.2.10 that the groupoid algebra in this case is the crossed product algebra $C(X) \rtimes G$. We can describe the noncommutative quotient algebra $C(X) \rtimes G$ geometrically as follows. Consider the principal G-bundle

$$G \to X \to X/G.$$

There is an action of G, by algebra automorphisms, on the finite dimensional matrix algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}G)$ defined by

$$(gT)(h) = T(g^{-1}h) \quad \text{for all } T \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}G),\ g, h \in G.$$

The associated vector bundle is a flat vector bundle $\mathcal{E}$ of finite dimensional $n \times n$ matrix algebras over X/G, where n is the order of the finite group G. Let $\Gamma(\mathcal{E})$ denote the algebra of continuous sections of $\mathcal{E}$. We leave it to the reader to show that there is an algebra isomorphism

$$C(X) \rtimes G \simeq \Gamma(\mathcal{E}). \tag{2.18}$$

Notice that since $\mathcal{E}$ is in general a non-trivial bundle, the crossed product algebra $C(X) \rtimes G$ is not always isomorphic to $C(X/G) \otimes M_n(\mathbb{C})$, though we know that it is Morita equivalent to $C(X/G)$.

Here is a concrete example. Let $X = \mathbb{T}^2$ and $\lambda = \exp(2\pi i \frac{p}{q})$ where p and q are positive relatively prime integers with $q > 1$. The map $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$

defines an free action of $G = \mathbb{Z}_q$ on $\mathbb{T}^2$. The corresponding crossed product algebra is the rational noncommutative torus for $\theta = \frac{p}{q}$:

$$C(\mathbb{T}^2) \rtimes \mathbb{Z}_q \simeq \Gamma(\mathcal{E}) = A_{\frac{p}{q}}.$$

Example 2.5.5. Let G be a locally compact group acting continuously on a locally compact Hausdorff space X. Assume the action is free and proper. Then the classical quotient space X/G is locally compact and Hausdorff. This is a good quotient. The corresponding noncommutative quotient is defined as the groupoid algebra of the transformation groupoid $X \rtimes G$, which is isomorphic to the crossed product algebra $C_0(X) \rtimes G$. As we saw in Example 2.4.4, in this case the commutative C^*-algebra of functions on the classical quotient, $C_0(X/G)$, is Morita equivalent to the crossed product algebra $C_0(X) \rtimes G$. That is, when the action is free and proper, the classical quotient and the noncommutative quotient are Morita equivalent. Due to its fundamental importance in justifying the whole philosophy of noncommutative quotients we record this result in a theorem.

Theorem 2.5.1. *Let a locally compact group G act freely and properly on a locally compact Hausdorff space X. Then the C^*-algebras $C_0(X/G)$ and $C_0(X) \rtimes G$ are Morita equivalent.*

Example 2.5.6. Let X be a locally compact Hausdorff space and consider the equivalence relation $\sim$ where $x \sim y$ for all x and y in X. The corresponding groupoid is the groupoid of pairs, identifying all points with each other. The classical quotient consists of a single point. To find out about the noncommutative quotient, notice that the groupoid is a locally compact topological groupoid and its groupoid C^*-algebra as we saw in Example 2.2.6 is the algebra of compact operators $\mathcal{K}(L^2(X, \mu))$. This algebra is obviously Morita equivalent to the classical quotient algebra $\mathbb{C}$.

Example 2.5.7 (Noncommutative torus)**.** Let $\theta \in \mathbb{R}$ be a fixed real number. Consider the action of $\mathbb{Z}$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ via rotation through an angle $2\pi\theta$:

$$(n, z) \mapsto e^{2\pi i n\theta} z.$$

For $\theta = \frac{p}{q}$ a rational number, the quotient space $\mathbb{T}/\mathbb{Z}$ is a circle and hence the classical quotient algebra is given by

$$C(\mathbb{T}/\mathbb{Z}) = \{f \in C(\mathbb{T});\ f(e^{2\pi i n\theta} z) = f(z) \text{ for all } n,\ z\} \simeq C(\mathbb{T}).$$

Notice that even though the action is proper in this case, it is not free, and the classical quotient algebra completely forgets any information about the isotropy groups of fixed points.

If θ is irrational, the action is not proper though it is certainly free. In fact the action is *ergodic* in this case; in particular each orbit is dense and the quotient space $\mathbb{T}/\mathbb{Z}$ has only two open sets. Thus the classical quotient is an uncountable set with a trivial topology. In particular it is not Hausdorff. Obviously, a continuous function on the circle which is constant on each orbit is necessarily constant since orbits are dense. Therefore the classical quotient algebra, for irrational θ, is given by

$$C(\mathbb{T}/\mathbb{Z}) \simeq \mathbb{C}.$$

The noncommutative quotient algebra, on the other hand, for *any* value of θ, is the crossed product algebra

$$C(\mathbb{T}) \rtimes_\theta \mathbb{Z},$$

which is the same as the groupoid algebra of the transformation groupoid $\mathbb{T} \rtimes \mathbb{Z}$ of the action. As we saw in Example 2.2.9 this algebra is isomorphic to the unital C^*-algebra generated by two unitaries U and V subject to the relation $VU = e^{2\pi i\theta}UV$, i.e., to the noncommutative torus A_θ.

Example 2.5.8 (Orbifolds)**.** The map $x \mapsto -x$ defines an action of the group $\mathbb{Z}_2$ on $\mathbb{R}$. The resulting quotient space is a simple example of an *orbifold*, a concept that generalizes the notion of a smooth manifold. A more elaborate example is the quotient of $\mathbb{R}^2$ by the action of the cyclic group $\mathbb{Z}_n$ through rotation by an angle $\frac{2\pi}{n}$. In general, an orbifold is a topological space that locally looks like a quotient of a Euclidean space $\mathbb{R}^n$ by an action of a finite group of diffeomorphisms. It is further assumed that the transition functions are compatible with group actions (see [141] for precise definitions and more general examples). There is a way to define an étale groupoid for any orbifold [141]. One can then define the noncommutative quotient as the groupoid algebra of this groupoid. In the simplest case when the orbifold is a quotient X/G the corresponding groupoid is the transformation groupoid $X \rtimes G$. In general it is an interesting problem to study orbifold invariants like the *orbifold Euler characteristic* through invariants of this noncommutative quotient space (cf. [134]).

Example 2.5.9 (The noncommutative space of leaves of a foliation)**.** Recall that in Example 2.1.5 we defined the notion of a foliation and its holonomy groupoid. The *space of leaves* of a foliated manifold (V, F) is, by definition, the quotient space $V/\sim$, where points x and y in V are considered equivalent if they belong to the same leaf. For a generic foliation the resulting quotient space is poorly behaved under its natural topology. For example, when there is an everywhere dense leaf, the quotient space has only two open sets and is highly singular. An example of this is the Kronecker foliation of the 2-torus by lines of constant *irrational* slope θ as we saw in Example 2.1.5. Now to form the corresponding noncommutative quotient space of leaves of a foliation we need a groupoid. The right groupoid turns out to be the *holonomy groupoid* of the foliation. As we saw in Example 2.1.5, there are two options here: the holonomy groupoid and the étale holonomy groupoid. These two groupoids are in a certain sense Morita equivalent and as a result their corresponding groupoid algebras are Morita equivalent. We can summarize the situation as follows:

foliation $\rightsquigarrow$ holonomy groupoid $\rightsquigarrow$ groupoid algebra

For a specific example, we look at the Kronecker foliation of the 2-torus. We assume that the slope parameter θ is irrational. Then, since each leaf is homeomorphic to $\mathbb{R}$ and is simply connected, there is no holonomy and the holonomy groupoid is reduced to the transformation groupoid

$$\mathcal{G} = \mathbb{T}^2 \rtimes_\theta \mathbb{R},$$

where the $\mathbb{R}$-action implementing the flow lines of the differential equation $dy = \theta dx$ on the torus is given by

$$(t,\, (z_1, z_2)) \mapsto (z_1, e^{2\pi i\theta} z_2). \tag{2.19}$$

The corresponding groupoid algebra, called the *foliation algebra*, is thus a crossed product algebra and is given by

$$C^*(\mathbb{T}^2, F_\theta) = C(\mathbb{T}^2) \rtimes_\theta \mathbb{R}.$$

What is this algebra? Does it have any relation with the noncommutative torus? To identify this algebra we can use the Morita equivalence of algebras in (2.16), as explained in [85]. Let $G = \mathbb{T} \times \mathbb{R}$ with subgroups $H = \{1\} \times \mathbb{Z}$ and $K = \{(e^{2\pi it},\, t);\ t \in \mathbb{R}\}$. Then G/K is the circle $\mathbb{T}$ and the action of $H = \mathbb{Z}$ is by rotation through the angle $-\frac{2\pi}{\theta}$. Also, $G/H = \mathbb{T}^2$ where under this identification the action of $K = \mathbb{R}$ is given by (2.19). It follows that we have a Morita equivalence of C^*-algebras

$$C^*(\mathbb{T}^2,\, F_\theta) = C(\mathbb{T}^2) \rtimes_\theta \mathbb{R} \overset{M}{\sim} A_{-\frac{1}{\theta}}.$$

Now of course $A_{-\frac{1}{\theta}}$ is isomorphic to $A_{\frac{1}{\theta}}$ (why?), and we saw in Example 2.4.5 that the latter algebra is Morita equivalent to A_θ. We have therefore a Morita equivalence

$$C^*(\mathbb{T}^2,\, F_\theta) \overset{M}{\sim} A_\theta \tag{2.20}$$

which identifies, up to Morita equivalence, the foliation algebra of the Kronecker foliation F_θ with the noncommutative torus. What about the étale holonomy groupoid and its groupoid algebra? As we saw in Example 2.1.5, by choosing a circle on $\mathbb{T}^2$ which is transverse to the foliation, the étale holonomy groupoid can be identified with the transformation groupoid $\mathbb{T} \rtimes_\theta \mathbb{Z}$ where the action of $\mathbb{Z}$ is through rotation by $2\pi\theta$. The corresponding groupoid algebra is again a crossed product algebra and is of course given by $C^*(\mathbb{T} \rtimes_\theta \mathbb{Z}) = C(\mathbb{T}) \rtimes_\theta \mathbb{Z} = A_\theta$. Thus (2.20) is an illustration of the general fact that the holonomy groupoid algebra of a foliation is Morita equivalent to its étale holonomy groupoid algebra. For a geometric interpretation of these Morita equivalences and more examples of foliation algebras we refer to [41].

Example 2.5.10 (The unitary dual of a non-compact group)**.** Let $\hat{G}$ denote the set of isomorphism classes of (continuous) irreducible unitary representations of a locally compact group G. It is possible to put a topology on $\hat{G}$ (cf. [68]) but unless G is compact or abelian, or obtained from such groups by a finite extension, this space tends to be non-Hausdorff, and can be quite singular (cf. Section

II.4 of [41] for a dramatic example with a discrete group $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$). Connes' noncommutative replacement for the unitary dual $\hat{G}$ is the (full) group C^*-algebra $C^*(G)$, while the reduced group C^*-algebra $C_r^*(G)$ is the noncommutative space representing the closed subspace $\hat{G}_r$ of $\hat{G}$ corresponding to irreducible representations that are weakly contained in the left regular representation (the support of the Plancherel measure).

Why is all this reasonable? Part of the answer consists in looking at what happens when G is abelian, or compact. In fact, if G is abelian, by Gelfand–Naimark's theorem we know that then $C^*(G) \simeq C_0(\hat{G})$ is indeed the algebra of continuous functions (vanishing at infinity) on the unitary dual of G. Also, if G is compact then $\hat{G}$ is a discrete space, while $C^*(G) \subset \prod_{\lambda\in\hat{G}} \mathrm{End}(V_\lambda)$ consists of sequences of finite dimensional matrices whose norm converges to zero. Then of course the noncommutative dual $C^*(G)$ and the classical dual $C_0(\hat{G})$ are Morita equivalent. In general, for noncommutative G there are close relations between the noncommutative dual $C^*(G)$ and the *classifying space* (for proper actions) of G through the *Baum–Connes conjecture*. In fact this was one of the main motivations behind noncommutative geometry and its applications to topology. We refer to [41] for a first-hand account.

Exercise 2.5.1 (A noncommutative circle)**.** A classical example of a smooth but non-Hausdorff manifold is obtained by gluing two copies of a circle at all except one point:

$$X = (S^1 \dot{\cup} S^1)/\sim .$$

Identify the corresponding noncommutative quotient space. This example can be generalized; e.g. one can glue the two circles at all but n points.

Exercise 2.5.2. Prove the isomorphism (2.18). Give an example of a free action of a finite group G on a space X where $C(X) \rtimes G$ and $C(X/G) \otimes M_n(\mathbb{C})$ are not isomorphic (n is the order of G).

Exercise 2.5.3. Let $M = \bigcup_i U_i$ be a finite open *covering* of a smooth manifold M by coordinate charts $U_i \subset M$. Let $V = \dot{\bigcup}_i U_i$ be the disjoint union of open sets U_i and $p\colon V \to M$ the natural projection. Define an equivalence relation on V by $x \sim y$ if $p(x) = p(y)$. Show that the corresponding groupoid $\mathcal{G}$, the graph of $\sim$, is a smooth étale groupoid and there is a Morita equivalence

$$C^\infty(\mathcal{G}) \overset{M}{\sim} C^\infty(M).$$

Exercise 2.5.4. In Example 2.5.9 show that even when θ is *rational* the foliation algebra $C^*(\mathbb{T}^2, F_\theta)$ is Morita equivalent to A_θ.

2.6 Sources of noncommutative spaces

We finish this chapter by making some general remarks on the sources of noncommutative spaces. At present we can identify at least four methods by which noncommutative spaces are constructed in noncommutative geometry:

i) noncommutative quotients;

ii) algebraic and C^*-algebraic deformations;

iii) Hopf algebras and quantum groups;

iv) cohomological constructions.

We stress that these methods are not mutually exclusive; there are in fact intimate relations between these sources and sometimes a noncommutative space can be described simultaneously by several methods, as is the case with noncommutative tori. The majority of examples, by far, however, fall into the first category. We shall not discuss the last method here (cf. [44]). Very briefly, the idea is that if one writes the conditions for the Chern character of an idempotent in cyclic homology to be trivial on the level of chains, then one obtains interesting examples of algebras such as noncommutative spheres and spherical manifolds, and Grassmannians.

Chapter 3

Cyclic cohomology

Cyclic cohomology was discovered by Connes in 1981 and was announced in that year, with full details, in a conference in Oberwolfach [36]. One of his main motivations came from index theory on foliated spaces. The K-theoretic index of a transversally elliptic operator on a foliated manifold is an element of the K-theory group of a noncommutative algebra, called the foliation algebra of the given foliated manifold. Connes realized that to identify this class it would be desirable to have a noncommutative analogue of the Chern character with values in an, as yet unknown, cohomology theory for noncommutative algebras. This theory would then play the role of the de Rham homology of *currents* on smooth manifolds.

Now, to define a noncommutative de Rham theory for noncommutative algebras is a highly non-trivial matter. This is in sharp contrast with the situation in K-theory where extending the topological K-theory to noncommutative Banach algebras is straightforward. Note that the usual algebraic formulation of de Rham theory is based on the module of Kähler differentials and its exterior algebra, which has no analogue for noncommutative algebras.

Instead, the noncommutative analogue of de Rham homology was found by a careful analysis of the algebraic structures deeply hidden in (*super*)*traces of products of commutators*. These expressions are directly defined in terms of an elliptic operator and its parametrix and were shown, via an index formula, to give the index of the operator when paired with a K-theory class. This connection with elliptic theory, K-homology, and K-theory, is mainly explored in the next chapter.

Let us read what Connes wrote in the Oberwolfach conference notebook after his talk, summarizing his discovery and how he arrived at it [36]:

"The transverse elliptic theory for foliations requires as a preliminary step a purely algebraic work, of computing for a noncommutative algebra $\mathcal{A}$ the cohomology of the following complex: n-cochains are multilinear functions $\varphi(f^0, \ldots, f^n)$ of $f^0, \ldots, f^n \in \mathcal{A}$ where

$$\varphi(f^1, \ldots, f^0) = (-1)^n \varphi(f^0, \ldots, f^n)$$

and the boundary is

$$b\varphi(f^0,\dots,f^{n+1}) = \varphi(f^0f^1,\dots,f^{n+1}) - \varphi(f^0,f^1f^2,\dots,f^{n+1}) + \cdots$$
$$+ (-1)^{n+1}\varphi(f^{n+1}f^0,\dots,f^n).$$

The basic class associated to a transversally elliptic operator, for $\mathcal{A}=$ the algebra of the foliation, is given by:

$$\varphi(f^0,\dots,f^n)=\operatorname{Trace}(\varepsilon F[F,f^0][F,f^1]\dots[F,f^n]),\quad f^i\in\mathcal{A},$$

where

$$F=\begin{pmatrix}0 & Q\\ P & 0\end{pmatrix},\quad \varepsilon=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and Q is a parametrix of P. An operation

$$S\colon H^n(\mathcal{A})\to H^{n+2}(\mathcal{A})$$

is constructed as well as a pairing

$$K(\mathcal{A})\times H(\mathcal{A})\to\mathbb{C},$$

where $K(\mathcal{A})$ is the algebraic K-theory of A. It gives the index of the operator from its associated class φ. Moreover $\langle e,\varphi\rangle=\langle e,S\varphi\rangle$ so that the important group to determine is the inductive limit $H_p=\varinjlim H^n(\mathcal{A})$ for the map S. Using the tools of homological algebra the groups $H^n(\mathcal{A},\mathcal{A}^*)$ of Hochschild cohomology with coefficients in the bimodule $\mathcal{A}^*$ are easier to determine and the solution of the problem is obtained in two steps:

1. *The construction of a map*

   $$B\colon H^n(\mathcal{A},\mathcal{A}^*)\to H^{n-1}(\mathcal{A})$$

   and the proof of a long exact sequence

   $$\cdots\to H^n(\mathcal{A},\mathcal{A}^*)\xrightarrow{B}H^{n-1}(\mathcal{A})\xrightarrow{S}H^{n+1}(\mathcal{A})\xrightarrow{I}H^{n+1}(\mathcal{A},\mathcal{A}^*)\to\cdots,$$

   where I is the obvious map from the cohomology of the above complex to the Hochschild cohomology.

2. *The construction of a spectral sequence with E_2 term given by the cohomology of the degree -1 differential $I\circ B$ on the Hochschild groups $H^n(\mathcal{A},\mathcal{A}^*)$ and which converges strongly to a graded group associated to the inductive limit.*

This purely algebraic theory is then used. For $\mathcal{A}=C^\infty(V)$ one gets the de Rham homology of currents, and for the pseudo-torus, i.e., the algebra of the Kronecker foliation, one finds that the Hochschild cohomology depends on the Diophantine nature of the rotation number while the above theory gives H_p^0 of dimension 2 and H_p^1 of dimension 2, as expected, but from some remarkable cancelations."

Cyclic cohomology has strong connections with group cohomology and Lie algebra cohomology as well. For example a theorem of Burghelea [25], to be recalled later in this chapter, computes the cyclic cohomology of the group algebra of a discrete group in terms of group cohomology; or a theorem of Loday–Quillen–Tsygan [125], [171] states that cyclic cohomology of an algebra A is isomorphic to the primitive part (in the sense of Hopf algebras) of the Lie algebra cohomology of the Lie algebra $\mathfrak{gl}(A)$ of stable matrices over A. We shall not pursue this connection in this book (cf. [124] for a full account and extensions; see also [64], [135] for alternative approaches to cyclic cohomology).

In Sections 3.1–3.5 we recall basic notions of Hochschild (co)homology theory and give several computations. A theorem of Hochschild–Kostant–Rosenberg [100] which identifies the Hochschild homology of the algebra of regular functions on a smooth affine variety with differential forms on that variety is recalled. We also recall a result of Connes [39] which identifies the continuous Hochschild cohomology of the algebra of smooth functions on a smooth closed manifold with the space of de Rham currents on that manifold. Relations between Hochschild cohomology and deformation theory of algebras are also indicated.

In Sections 3.6–3.10 we study cyclic cohomology theory in some detail. First we define cyclic cohomology via Connes' *cyclic complex*, and establish *Connes' long exact sequence*, relating Hochschild and cyclic cohomology groups. This naturally leads to a definition of the operator B and the *periodicity operator* S. A second, and more powerful, definition of cyclic cohomology via Connes' (b, B)-*bicomplex* is our next topic. A key result here is the vanishing of the E^2 term of the associated spectral sequence of this bicomplex. This then gives us Connes' *Hochschild to cyclic spectral sequence* which is a very powerful tool for computations. Finally we recall Connes' computation of the cyclic cohomology of the algebra of smooth functions on a manifold as well as the noncommutative torus, and Burghelea's result for the cyclic homology of group algebras. For this chapter we assume that the reader is familiar with basic notions of homological algebra up to derived functors and spectral sequences. This material is covered, e.g. in [19], [30], [124], [179].

3.1 Hochschild cohomology

Hochschild cohomology of associative algebras was defined by Hochschild through an explicit complex in [99]. This complex is a generalization of the standard complex for group cohomology. Formulating a dual homology theory is straightforward. One of the original motivations was to give a cohomological criterion for separability of algebras as well as a classification of (simple types) of algebra extensions in terms of second Hochschild cohomology. Once it was realized, by Cartan and Eilenberg [30], that Hochschild cohomology is an example of their newly discovered theory of derived functors, tools of homological algebra like resolutions became available. Hochschild cohomology plays an important role in the *deformation theory* of associative algebras [83], [84] and the closely related theory of $*$-*products* in quantum mechanics [10]. More recent applications of the Hochschild homology are to Khovanov homology and to invariants of knots and links like the Jones invariant [110].

The Hochschild–Kostant–Rosenberg theorem [100] and its smooth version by Connes [39] identifies the Hochschild homology of the algebra of regular functions on a smooth affine variety, or the algebra of smooth functions on a manifold, with differential forms and is among the most important results of this theory. Because of this result one usually thinks of Hochschild homology of an algebra A with coefficients in A as a noncommutative analogue of differential forms on A.

As we shall see later in this chapter, Hochschild cohomology is related to cyclic cohomology through Connes' long exact sequence and, even better, through a spectral sequence, also due to Connes. For this reason computing the Hochschild cohomology is often the first step in computing the cyclic cohomology of a given algebra. In the following all tensor products $\otimes$ and Hom's are over $\mathbb{C}$ unless specified otherwise.

Let A be an algebra over $\mathbb{C}$ and M be an A-bimodule. Thus M is a left and a right A-module and the two actions are compatible in the sense that $a(mb) = (am)b$ for all a, b in A and m in M. The *Hochschild cochain complex of A with coefficients in M*,

$$\boxed{C^0(A, M) \xrightarrow{\delta} C^1(A, M) \xrightarrow{\delta} C^2(A, M) \xrightarrow{\delta} \cdots} \tag{3.1}$$

denoted $(C^*(A, M), \delta)$, is defined by

$$C^0(A, M) = M, \quad C^n(A, M) = \operatorname{Hom}(A^{\otimes n}, M), \quad n \geq 1,$$

where the differential $\delta \colon C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\begin{aligned}
(\delta m)(a) &= ma - am, \\
(\delta f)(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^{i+1} f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\quad + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}.
\end{aligned}$$

Here $m \in M = C^0(A, M)$, and $f \in C^n(A, M)$, $n \geq 1$.

One checks that

$$\delta^2 = 0.$$

The cohomology of the complex $(C^*(A, M), \delta)$ is by definition the *Hochschild cohomology* of the algebra A with coefficients in the A-bimodule M and will be denoted by $H^n(A, M)$, $n = 0, 1, 2, \ldots$.

Among all bimodules over an algebra A, the following two bimodules play an important role.

1) $M = A$, with bimodule structure $a(b)c = abc$ for all a, b, c in A. In this case the Hochschild complex $C^*(A, A)$ is also known as the *deformation* or *Gerstenhaber complex* of A. It plays an important role in deformation theory of associative

algebras pioneered by Gerstenhaber [83], [84]. For example, it can be shown that $H^2(A, A)$ is the space of equivalence classes of *infinitesimal deformations* of A and $H^3(A, A)$ is the *space of obstructions* for deformations of A (cf. Section 3.3).

2) $M = A^* := \mathrm{Hom}(A, \mathbb{C})$, the linear dual of A, with A-bimodule structure defined by

$$(afb)(c) = f(bca)$$

for all a, b, c in A and f in A^*. This bimodule is relevant to cyclic cohomology. Indeed, as we shall see later in this chapter, the Hochschild groups $H^n(A, A^*)$ and the cyclic cohomology groups $HC^n(A)$ enter into a long exact sequence. Using the identification

$$\mathrm{Hom}(A^{\otimes n}, A^*) \simeq \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{C}), \quad f \mapsto \varphi,$$

$$\varphi(a_0, a_1, \ldots, a_n) = f(a_1, \ldots, a_n)(a_0),$$

the Hochschild differential δ is transformed into a differential, denoted b, given by

$$\begin{aligned}(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} &(-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n).\end{aligned}$$

Thus for $n = 0, 1, 2$ we have the following formulas for b:

$$\begin{aligned}(b\varphi)(a_0, a_1) &= \varphi(a_0 a_1) - \varphi(a_1 a_0), \\ (b\varphi)(a_0, a_1, a_2) &= \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1), \\ (b\varphi)(a_0, a_1, a_2, a_3) &= \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) \\ &\quad + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2).\end{aligned}$$

Due to its frequent occurrence in this text, from now on the Hochschild complex $C^*(A, A^*)$ will be simply denoted by $C^*(A)$ and the Hochschild cohomology $H^*(A, A^*)$ by $HH^*(A)$.

Example 3.1.1. We give a few examples of Hochschild cohomology, starting in low dimensions.

1. $n = 0$. It is clear that

$$H^0(A, M) = \{m \in M;\ ma = am \text{ for all } a \in A\}.$$

In particular for $M = A^*$,

$$H^0(A, A^*) = \{f \colon A \to \mathbb{C};\ f(ab) = f(ba) \text{ for all } a, b \in A\}$$

is the *space of traces* on A.

2. $n = 1$. A Hochschild 1-*cocycle* $f \in C^1(A, M)$ is simply a *derivation*, i.e., a $\mathbb{C}$-linear map $f \colon A \to M$ such that

$$f(ab) = af(b) + f(a)b$$

for all a, b in A. A 1-cocycle is a *coboundary* if and only if the corresponding derivation is *inner*, that is there should exists an m in M such that $f(a) = ma - am$ for all a in A. Therefore

$$H^1(A,\, M) = \frac{\text{derivations}}{\text{inner derivations}}.$$

Sometimes this is called the space of *outer derivations* of A with values in the A-bimodule M. In view of Exercise 3.1.6, for an algebra A, commutative or not, we can think of $\mathrm{Der}(A,\, A)$ as the Lie algebra of noncommutative vector fields on the noncommutative space represented by A. Notice that, unless A is commutative, $\mathrm{Der}(A,\, A)$ need not be an A-module.

3. $n = 2$. We show, following Hochschild [99], that $H^2(A, M)$ classifies *abelian extensions* of A by M. Let A be a unital algebra and M be an A-bimodule. By definition, an abelian extension of A by M is an exact sequence of algebras

$$0 \to M \to B \to A \to 0$$

such that B is unital, M has trivial multiplication (i.e., $M^2 = 0$), and the induced A-bimodule structure on M coincides with the original bimodule structure. Two such extensions (M, B, A) and (M, B', A) are called isomorphic if there is a unital algebra map $f\colon B \to B'$ which induces identity maps on M and A. (Notice that if such an f exists then it is necessarily an isomorphism.) Let $E(A, M)$ denote the set of isomorphism classes of such extensions. We define a natural bijection

$$E(A, M) \simeq H^2(A, M)$$

as follows. Given an extension as above, let $s\colon A \to B$ be a linear splitting for the projection $B \to A$, and let $f\colon A \otimes A \to M$ be its *curvature*, defined by

$$f(a, b) = s(ab) - s(a)s(b)$$

for all a, b in A. One can easily check that f is a Hochschild 2-cocycle and its class is independent of the choice of the splitting s. In the other direction, given a 2-cochain $f\colon A \otimes A \to M$, we try to define a multiplication on $B = A \oplus M$ via

$$(a, m)(a', m') = (aa', am' + ma' + f(a, a')).$$

It can be checked that this defines an associative multiplication if and only if f is a 2-cocycle. The extension associated to a 2-cocycle f is the extension

$$0 \to M \to A \oplus M \to A \to 0.$$

It can be checked that these two maps are bijective and inverse to each other.

4. A simple computation shows that when $A = \mathbb{C}$ is the ground field we have

$$HH^0(\mathbb{C}) = \mathbb{C} \quad \text{and} \quad HH^n(\mathbb{C}) = 0 \quad \text{for } n \geq 1.$$

Example 3.1.2. Let M be a closed (i.e., compact without boundary), smooth, oriented, n-dimensional manifold and let $A = C^\infty(M)$ denote the algebra of complex valued smooth functions on M. For $f^0, \dots, f^n \in C^\infty(M)$, let

$$\varphi(f^0, \dots, f^n) = \int_M f^0\, df^1 \dots df^n.$$

The $(n+1)$-linear cochain $\varphi\colon A^{\otimes(n+1)} \to \mathbb{C}$ has three properties: it is *continuous* with respect to the natural Fréchet space topology of A (cf. Section 3.4 for more on this point); it is a *Hochschild cocycle*; and it is a *cyclic cochain* (cf. Section 3.6 for more on this). The Hochschild cocycle property that concerns us here, $b\varphi = 0$, can be checked as follows:

$$\begin{aligned}(b\varphi)(f^0, \dots, f^{n+1}) &:= \sum_{i=0}^{n} (-1)^i \varphi(f^0, \dots, f^i f^{i+1}, \dots, f^{n+1}) \\ &\qquad + (-1)^{n+1} \varphi(f^{n+1} f^0, \dots, \dots, f^n) \\ &= \sum_{i=0}^{n} (-1)^i \int_M f^0\, df^1 \dots d(f^i f^{i+1}) \dots df^{n+1} \\ &\qquad + (-1)^{n+1} \int_M f^{n+1} f^0\, df^1 \dots df^n \\ &= 0\end{aligned}$$

for all $f^0, \dots, f^{n+1} \in A$. Here we used the Leibniz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega^* M,\, d)$ of differential forms on M.

We have thus associated a Hochschild cocycle to the orientation cycle of the manifold. This construction admits a vast generalization, as we explain now. Let

$$\Omega_p M := \mathrm{Hom}_{\mathrm{cont}}(\Omega^p M,\, \mathbb{C}) \tag{3.2}$$

denote the *continuous linear dual* of the space of p-forms on M. Here, the (locally convex) topology of $\Omega^p M$ is defined by the sequence of seminorms

$$\|\omega\|_n = \sup |\partial^\alpha\, \omega_{i_1, \dots, i_p}|, \quad |\alpha| \le n,$$

where the supremum is over a fixed, finite, coordinate cover for M, and over all partial derivatives ∂^α of total degree at most n of all components $\omega_{i_1, \dots, i_p}$ of ω. Elements of $\Omega_p M$ are called *de Rham p-currents* on M. For $p = 0$ we recover the notion of a *distribution* on M. Since the de Rham differential $d\colon \Omega^k M \to \Omega^{k+1} M$, $k = 0, 1, \dots$, is continuous in the topology of differential forms, by dualizing it we obtain differentials $d^*\colon \Omega_k M \to \Omega_{k-1} M$, $k = 1, 2, \dots$ and the *de Rham complex of currents* on M:

$$\Omega_0 M \xleftarrow{d^*} \Omega_1 M \xleftarrow{d^*} \Omega_2 M \xleftarrow{d^*} \cdots.$$

The homology of this complex is called the *de Rham homology* of M and we shall denote it by $H_n^{\mathrm{dR}}(M)$, $n = 0, 1, \dots$.

It is easy to check that for *any* m-current C, closed or not, the cochain φ_C defined by

$$\varphi_C(f^0, f^1, \dots, f^m) := \langle C,\, f^0 df^1 \dots df^m \rangle$$

is a Hochschild cocycle on A. As we shall explain in Section 3.4, φ_C is continuous in the natural topology of $A^{\otimes(m+1)}$ and we obtain a canonical map

$$\boxed{\Omega_m M \to HH^m_{\mathrm{cont}}(C^\infty(M))}$$

from the space of m-currents on M to the continuous Hochschild cohomology of $C^\infty(M)$. By a theorem of Connes [39] this map is an isomorphism. We refer to Section 3.5 for more details and a dual statement relating differential forms with Hochschild *homology*. The corresponding statement for the algebra of regular functions on a smooth affine variety, the Hochschild–Kostant–Rosenberg theorem, will be discussed in that section as well.

Exercise 3.1.1. Let $A_1 = \mathbb{C}[x, \frac{d}{dx}]$ denote the *Weyl algebra* of differential operators with polynomial coefficients, where the product is defined as the composition of operators. Equivalently, A_1 is the unital universal algebra generated by elements x and $\frac{d}{dx}$ with relation $\frac{d}{dx}x - x\frac{d}{dx} = 1$. Show that $HH^0(A_1) = 0$; that is, A_1 carries no nonzero trace.

Exercise 3.1.2. Show that any derivation of the Weyl algebra $A_1 = \mathbb{C}\,[x, \frac{d}{dx}]$ is inner, i.e., $H^1(A_1, A_1) = 0$.

Exercise 3.1.3. Show that any derivation of the algebra $C(X)$ of continuous functions on a compact Hausdorff space X is zero. (Hint: If $f = g^2$ and $g(x) = 0$ for some $x \in X$ then, for any derivation δ, $(\delta f)(x) = 0$.)

Exercise 3.1.4. Show that any derivation of the matrix algebra $M_n(\mathbb{C})$ is inner. (This was proved by Dirac in his first paper on quantum mechanics [67], where derivations are called *quantum differentials*).

Exercise 3.1.5. Let $Z(A)$ denote the center of the algebra A. Show that the Hochschild groups $H^n(A, M)$ are $Z(A)$-modules.

Exercise 3.1.6 (Derivations and vector fields)**.** Let $U \subset \mathbb{R}^n$ be an open set and let

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$$

be a smooth vector field on U. Define a derivation $\delta_X \colon C^\infty(U) \to C^\infty(U)$ by

$$\delta_X(f) = \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}.$$

Show that the map $X \mapsto \delta_X$ defines a 1-1 correspondence between vector fields on U and derivations of $C^\infty(U)$ to itself. Under this isomorphism, the bracket of vector fields corresponds to commutators of derivations:

$$\delta_{[X,\,Y]} = [\delta_X,\, \delta_Y].$$

Fix a point $m \in U$ and define an A-module structure on $\mathbb{C}$ by the map $f \otimes 1 \mapsto f(m)$. Show that the set $\mathrm{Der}(C^\infty(U), \mathbb{C})$ of $\mathbb{C}$-valued derivations of $C^\infty(U)$ is canonically isomorphic to the (complexified) tangent space of U at m. Extend these correspondences to arbitrary smooth manifolds. (These considerations form the beginnings of a purely algebraic approach to some *'soft'* aspects of differential geometry including differential forms and tensor analysis, connection and curvature formalism and Chern–Weil theory [115], [148], and is part of 'differential geometry over commutative algebras'. It can also be adapted to algebraic geometry.)

3.2 Hochschild cohomology as a derived functor

The original complex (3.1) that we used to define the Hochschild cohomology is rarely useful for computations. Instead, it is the fact that Hochschild cohomology is a *derived functor* that will allows us, in specific cases, to replace the standard complex (3.1) by a much smaller complex and to compute the Hochschild cohomology. In this section we show that Hochschild cohomology is a derived functor; more precisely it is an Ext functor. References for the general theory of derived functors and homological algebra include [19], [30], [81], [124], [179].

Let A^{op} denote the *opposite algebra* of an algebra A. Thus, as a vector space $A^{\mathrm{op}} = A$ and the new multiplication is defined by $a \cdot b := ba$. There is a one-to-one correspondence between A-bimodules and left $A \otimes A^{\mathrm{op}}$-modules defined by

$$(a \otimes b^{\mathrm{op}})m = amb.$$

Define a functor from the category of left $A \otimes A^{\mathrm{op}}$-modules to the category of complex vector spaces by

$$M \mapsto \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A,\, M) = \{m \in M;\ ma = am \text{ for all } a \in A\} = H^0(A,\, M).$$

We show that Hochschild cohomology is the left derived functor of the functor $M \rightsquigarrow H^0(A,\, M)$. We assume that A is *unital*. Since A is naturally a left $A \otimes A^{\mathrm{op}}$-module, we can consider its *bar resolution*. It is defined by

$$0 \leftarrow A \xleftarrow{b'} B_1(A) \xleftarrow{b'} B_2(A) \xleftarrow{b'} \cdots, \tag{3.3}$$

where $B_n(A) = A \otimes A^{\mathrm{op}} \otimes A^{\otimes n}$ is the free left $A \otimes A^{\mathrm{op}}$-module generated by $A^{\otimes n}$.

The differential b' is defined by

$$\begin{aligned}b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) &= aa_1 \otimes b \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (a \otimes b \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n (a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1}).\end{aligned}$$

Define the operators $s\colon B_n(A) \to B_{n+1}(A)$, $n \geq 0$, by

$$s(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes b \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

One checks that

$$b's + sb' = \mathrm{id},$$

which shows that $(B(A), b')$ is acyclic and hence is a free resolution of A as a left $A \otimes A^{\mathrm{op}}$-module. Now, for any A-bimodule M we have an isomorphism of cochain complexes

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(B(A), M) \simeq (C^*(A, M), \delta),$$

which shows that Hochschild cohomology is the left derived functor of the Hom functor:

$$H^n(A, M) \simeq \mathrm{Ext}^n_{A\otimes A^{\mathrm{op}}}(A, M) \quad \text{for all } n \geq 0.$$

One can therefore use any projective resolution of A, or any injective resolution of M, as a left $A \otimes A^{\mathrm{op}}$-module to compute the Hochschild cohomology groups.

Before proceeding further let us recall the definition of the *Hochschild homology* of an algebra A with coefficients in a bimodule M. The *Hochschild homology complex of A with coefficients in M* is the complex

$$C_0(A, M) \xleftarrow{\delta} C_1(A, M) \xleftarrow{\delta} C_2(A, M) \xleftarrow{\delta} \cdots \tag{3.4}$$

denoted by $(C_*(A, M), \delta)$, where

$$C_0(A, M) = M \quad \text{and} \quad C_n(A, M) = M \otimes A^{\otimes n}, \quad n = 1, 2, \ldots,$$

and the *Hochschild boundary* $\delta\colon C_n(A, M) \to C_{n-1}(A, M)$ is defined by

$$\begin{aligned}\delta(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_n.\end{aligned}$$

The Hochschild homology of A with coefficients in M is, by definition, the homology of the complex $(C_*(A, M), \delta)$. We denote this homology by $H_n(A, M)$, $n = 0, 1, \ldots$. It is clear that

$$H_0(A, M) = M/[A, M],$$

where $[A,\, M]$ is the $\mathbb{C}$-linear subspace of M spanned by commutators $am - ma$ for a in A and m in M.

The following facts are easily established:

1) Hochschild homology, $H_*(A,\, M)$, is the right derived functor of the functor $M \rightsquigarrow A \otimes_{A\otimes A^{\mathrm{op}}} M = H_0(A, M)$ from the category of left $A \otimes A^{\mathrm{op}}$-modules to the category of complex vector spaces, i.e.,

$$H_n(A,\, M) \simeq \mathrm{Tor}_n^{A\otimes A^{\mathrm{op}}}(A, M).$$

For the proof one can simply use the bar resolution (3.3) as we did for cohomology.

2) (Duality) Let $M^* = \mathrm{Hom}(M, \mathbb{C})$. It is an A-bimodule via $(afb)(m) = f(bma)$. One checks that the natural isomorphism

$$\mathrm{Hom}(A^{\otimes n},\, M^*) \simeq \mathrm{Hom}(M \otimes A^{\otimes n}, \mathbb{C}), \quad n = 0, 1, \ldots$$

is compatible with differentials. Thus, since we are over a field of characteristic 0, we have natural isomorphisms

$$H^n(A,\, M^*) \simeq (H_n(A,\, M))^*, \quad n = 0, 1, \ldots$$

From now on the Hochschild homology groups $H_*(A, A)$ will be denoted by $HH_*(A)$. In view of the above duality, we have the isomorphisms

$$HH^n(A) \simeq HH_n(A)^*, \quad n \geq 0,$$

where by our earlier convention $HH^n(A)$ stands for $H^n(A, A^*)$.

Example 3.2.1. Let $A = \mathbb{C}[x]$ be the algebra of polynomials in one variable. It is easy to check that the following complex is a resolution of A as a left $A \otimes A$-module:

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{\varepsilon} \mathbb{C}[x] \otimes \mathbb{C}[x] \xleftarrow{d} \mathbb{C}[x] \otimes \mathbb{C}[x] \otimes \mathbb{C} \leftarrow 0, \tag{3.5}$$

where the differentials are the unique $A \otimes A$-linear extensions of the maps

$$\varepsilon(1 \otimes 1) = 1, \quad d(1 \otimes 1 \otimes 1) = x \otimes 1 - 1 \otimes x. \tag{3.6}$$

To check its acyclicity, notice that it is isomorphic to the complex

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{\varepsilon} \mathbb{C}[x, y] \xleftarrow{d} \mathbb{C}[x, y] \leftarrow 0,$$

where now

$$\varepsilon(f(x, y)) = f(x, x), \quad d(f(x, y)) = (x - y)f(x, y).$$

By tensoring this resolution with the right $A \otimes A$-module A, we obtain a complex with zero differentials

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{0} \mathbb{C}[x] \leftarrow 0$$

and hence

$$HH_i(\mathbb{C}[x]) \simeq \begin{cases} \mathbb{C}[x] & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

The complex (3.5) is a simple example of a *Koszul resolution*. In the next example we generalize it to polynomials in several variables. (cf. [19] for the general theory in the commutative case.)

Example 3.2.2. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ be the algebra of polynomials in n variables. Let V be an n-dimensional complex vector space over. The Koszul resolution of A, as a left $A \otimes A$-module, is defined by

$$0 \leftarrow A \xleftarrow{\varepsilon} A{\otimes}A \xleftarrow{d} A{\otimes}A{\otimes}\Omega^1 \leftarrow \cdots \leftarrow A{\otimes}A{\otimes}\Omega^i \leftarrow \cdots \leftarrow A{\otimes}A{\otimes}\Omega^n \leftarrow 0, \quad (3.7)$$

where $\Omega^i = \bigwedge^i V$ is the i-th exterior power of V. The differentials ε and d are defined in (3.6). d has a unique extension to a graded derivation of degree -1 on the graded commutative algebra $A \otimes A \otimes \bigwedge V$. Notice that $A \simeq S(V)$, the symmetric algebra of the vector space V.

Let $K(S(V))$ denote the Koszul resolution (3.7). To show that it is exact we notice that

$$K(S(V \oplus W)) \simeq K(S(V) \otimes K(S(W))).$$

Since the tensor product of two exact complexes is again exact (notice that we are over a field of characteristic zero), the exactness of $K(S(V))$ can be reduced to the case where V is 1-dimensional, which was treated in the last example. See Exercise 3.2.6 for an explicit description of the resolution (3.7).

As in the one dimensional case, the differentials in the complex $A \otimes_{A \otimes A} K(S(V))$ are all zero and we obtain

$$\begin{aligned} HH_i(S(V)) &= \mathrm{Tor}_i^{S(V)\otimes S(V)}(S(V), S(V) \\ &= S(V) \otimes \textstyle\bigwedge^i V. \end{aligned}$$

The right-hand side is isomorphic to the module of algebraic differential forms on $S(V)$. So we can write this result as

$$HH_i(S(V)) = \Omega^i(S(V)),$$

which is a special case of the Hochschild–Kostant–Rosenberg theorem mentioned before. More generally, if M is a *symmetric* A-bimodule, the differentials of $M \otimes_{A \otimes A} K(S(V))$ vanish and we obtain

$$H_i(S(V), M) \simeq M \otimes \textstyle\bigwedge^i V, \quad i = 0, 1, \ldots, n,$$

and 0 otherwise.

Example 3.2.3. In Section 3.4 we shall define the continuous analogues of Hochschild and cyclic (co)homology as well as Tor and Ext functors. Here is a simple example. The continuous analogue of the resolution (3.5) for the topological algebra $A = C^\infty(S^1)$ is the *topological Koszul resolution*

$$0 \leftarrow C^\infty(S^1) \overset{\varepsilon}{\longleftarrow} C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \overset{d}{\longleftarrow} C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \otimes \mathbb{C} \leftarrow 0, \qquad (3.8)$$

with differentials given by (3.6). Here $\hat{\otimes}$ denotes the *projective* tensor product of locally convex spaces (cf. Section 3.4 for definitions). To verify the exactness, the only non-trivial step is to check that $\ker \varepsilon \subset \operatorname{im} d$. To this end, notice that if we identify

$$C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \simeq C^\infty(S^1 \times S^1),$$

the differentials are given by

$$(\varepsilon f)(x) = f(x, x), \quad (d_1 f)(x, y) = (x - y) f(x, y).$$

Now the homotopy formula

$$f(x, y) = f(x, x) - (x - y) \int_0^1 \frac{\partial}{\partial y} f(x,\, y + t(x - y))\, dt$$

shows that $\ker \varepsilon \subset \operatorname{im} d$. Alternatively, one can use Fourier series to establish the exactness (cf. Exercise 3.2.5).

To compute the continuous Tor functor, we apply the functor $- \hat{\otimes}_{A \hat{\otimes} A}$ to the above complex. We obtain

$$0 \leftarrow C^\infty(S^1) \overset{0}{\longleftarrow} C^\infty(S^1) \leftarrow 0$$

and hence

$$HH_i^{\text{cont}}(C^\infty(S^1)) = \begin{cases} \Omega^i S^1 & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

where $\Omega^i S^1 \simeq C^\infty(S^1) dx^i$ is the space of differential forms of degree i on S^1.

A similar computation, using a continuous version of Ext by applying the functor $\operatorname{Hom}^{\text{cont}}_{A \hat{\otimes} A}(-, A)$ gives

$$HH^i_{\text{cont}}(C^\infty(S^1)) = \begin{cases} \Omega_i S^1 & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Here $\Omega_i S^1 = (\Omega^i S^1)^*$, the continuous dual of i-forms, is the space of i-currents on S^1.

Notice how the identification $C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \simeq C^\infty(S^1 \times S^1)$ played an important role in the above proof. The algebraic tensor product $C^\infty(S^1) \otimes C^\infty(S^1)$, on the other hand, is only *dense* in $C^\infty(S^1 \times S^1)$ and this makes it very difficult to write a resolution to compute the *algebraic* Hochschild groups of $C^\infty(S^1)$. In fact these groups are not known so far!

Example 3.2.4 (Cup product)**.** Let A and B be unital algebras. What is the relation between the Hochschild homology groups of $A \otimes B$ and those of A and B? One can construct (cf. [30], [124] for details) chain maps

$$C_*(A \otimes B) \to C_*(A) \otimes C_*(B),$$
$$C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$$

inducing inverse isomorphisms. We obtain

$$HH_n(A \otimes B) \simeq \bigoplus_{p+q=n} HH_p(A) \otimes HH_q(B) \quad \text{for all } n \geq 0.$$

Now, if A is commutative, the multiplication $m\colon A \otimes A \to A$ is an algebra map and, in combination with the above map, induces an associative and graded commutative product on $HH_*(A)$.

Exercise 3.2.1. Let A and B be unital algebras. Give a direct proof of the isomorphism

$$HH_0(A \otimes B) \simeq HH_0(A) \otimes HH_0(B).$$

Dually, show that there is a natural map

$$HH^0(A) \otimes HH^0(B) \to HH^0(A \otimes B),$$

but it need not be surjective in general.

Exercise 3.2.2. Let

$$A = T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots,$$

be the tensor algebra of a vector space V. Show that the complex

$$0 \leftarrow A \overset{\varepsilon}{\longleftarrow} A \otimes A^{\mathrm{op}} \overset{d}{\longleftarrow} A \otimes A^{\mathrm{op}} \otimes V \leftarrow 0,$$

with differentials induced by

$$\varepsilon(1 \otimes 1) = 1, \quad d(1 \otimes 1 \otimes v) = v \otimes 1 - 1 \otimes v, \quad v \in V,$$

is a free resolution of A as a left $A \otimes A^{\mathrm{op}}$-module. Conclude that A has Hochschild homological dimension 1 in the sense that $H_n(A, M) = 0$ for all A-bimodules M and all $n \geq 2$. Compute $H_0(A, M)$ and $H_1(A, M)$.

Exercise 3.2.3 (Normalization)**.** Let M be an A-bimodule. A cochain $f\colon A^{\otimes n} \to M$ is called *normalized* if $f(a_1, \ldots, a_n) = 0$ whenever $a_i = 1$ for some i. Show that normalized cochains $C^*_{\mathrm{norm}}(A, M)$ form a subcomplex of the Hochschild complex $C^*(A, M)$ and that the inclusion

$$C^*_{\mathrm{norm}}(A, M) \hookrightarrow C^*(A, M)$$

is a quasi-isomorphism. (Hint: Introduce a normalized version of the bar resolution.)

Exercise 3.2.4. Let $A = \mathbb{C}[x]/(x^2)$ denote the algebra of *dual numbers*. Use the normalized Hochschild complex to compute $HH_*(A)$.

Exercise 3.2.5. Use Fourier series to show that the sequence (3.8) is exact.

Exercise 3.2.6. Let V be an n-dimensional vector space. Show that the following complex is a free resolution of $S(V)$, the symmetric algebra of V, as a left $S(V) \otimes S(V)$-module

$$S(V) \xleftarrow{\varepsilon} S(V^2) \xleftarrow{i_X} S(V^2) \otimes E_1 \xleftarrow{i_X} S(V^2) \otimes E_2 \xleftarrow{i_X} \cdots \xleftarrow{i_X} S(V^2) \otimes E_n \leftarrow 0,$$

where $E_k = \bigwedge^k V$, and i_X is the interior multiplication (contraction) with respect to the vector field

$$X = \sum_{i=1}^{n} (x_i - y_i) \frac{\partial}{\partial y_i}$$

on $V^2 = V \times V$. (Hint: Use the Cartan homotopy formula $d i_X + i_X d = L_X$ to find a contracting homotopy for i_X.)

Exercise 3.2.7 (A resolution for the algebraic noncommutative torus [39])**.** Let

$$A = \mathbb{C}\langle U_1, U_2 \rangle / (U_1 U_2 - \lambda U_2 U_1)$$

be the universal unital algebra generated by invertible elements U_1 and U_2 with relation $U_1 U_2 = \lambda U_2 U_1$. We assume that $\lambda \in \mathbb{C}$ is not a root of unity. Let $\Omega^i = \bigwedge^i V$, where V is a 2-dimensional vector space with basis e_1 and e_2. Consider the complex of left $A \otimes A^{\mathrm{op}}$-modules

$$0 \leftarrow A \xleftarrow{\varepsilon} A \otimes A^{\mathrm{op}} \xleftarrow{d_0} A \otimes A^{\mathrm{op}} \otimes \Omega^1 \xleftarrow{d_1} A \otimes A^{\mathrm{op}} \otimes \Omega^2 \leftarrow 0, \tag{3.9}$$

where ε is the multiplication map and the other differentials are defined by

$$d_0(1 \otimes 1 \otimes e_j) = 1 \otimes U_j - U_j \otimes 1, \quad j = 1, 2,$$

$$d_1(1 \otimes 1 \otimes e_1 \wedge e_2) = (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2.$$

Show that (3.9) is a resolution of A as an $A \otimes A^{\mathrm{op}}$-module and use it to compute $HH_*(A)$.

Exercise 3.2.8. The Weyl algebra A_1 is defined in Exercise 3.1.1. By giving a 'small' resolution of length two for A_1 as a left $A_1 \otimes A_1^{\mathrm{op}}$-module show that

$$HH_i(A_1) \simeq \begin{cases} \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}$$

Show that $HH_2(A_1)$ is generated by the class of the 2-cycle

$$1 \otimes p \otimes q - 1 \otimes q \otimes p + 1 \otimes 1 \otimes 1,$$

where $q = x$ and $p = \frac{d}{dx}$. Extend this result to higher order Weyl algebras $A_n = A_1^{\otimes n}$ and show that

$$HH_i(A_n) \simeq \begin{cases} \mathbb{C} & \text{if } i = 2n, \\ 0 & \text{if } i \neq 2n. \end{cases}$$

Can you give an explicit formula for the generator of $HH_{2n}(A_n)$?

3.3 Deformation theory

Let A be a unital complex algebra. An increasing *filtration* on A is an increasing sequence of subspaces of A, $F^i(A) \subset F^{i+1}(A)$, $i = 0, 1, 2, \ldots$, with $1 \in F^0(A)$, $\bigcup_i F^i(A) = A$, and

$$F^i(A)F^j(A) \subset F^{i+j}(A) \quad \text{for all } i,\, j.$$

Let $F^{-1}(A) = 0$. The *associated graded algebra* of a filtered algebra is the graded algebra

$$\mathrm{Gr}(A) = \bigoplus_{i \geq 0} \tfrac{F^i(A)}{F^{i-1}(A)}.$$

Definition 3.3.1. An *almost commutative* algebra is a filtered algebra whose associated graded algebra $\mathrm{Gr}(A)$ is commutative.

Being almost commutative is equivalent to the commutator condition

$$[F^i(A),\, F^j(A)] \subset F^{i+j-1}(A) \tag{3.10}$$

for all i, j. As we shall see, Weyl algebras and, more generally, algebras of differential operators on a smooth manifold, and universal enveloping algebras are examples of almost commutative algebras.

Let A be an almost commutative algebra. The original Lie algebra bracket $[x, y] = xy - yx$ on A induces a Lie algebra bracket $\{\ \}$ on $\mathrm{Gr}(A)$ via the formula

$$\{x + F^i, y + F^j\} := [x, y] + F^{i+j-2}.$$

Notice that by the almost commutativity assumption (3.10), $[x, y]$ is in $F^{i+j-1}(A)$ and $\mathrm{Gr}(A)$, with its grading shifted by one, is indeed a *graded* Lie algebra. The induced Lie bracket on $\mathrm{Gr}(A)$ is compatible with its multiplication in the sense that for all $a \in \mathrm{Gr}(A)$, the map $b \mapsto \{a,\, b\}$ is a derivation. The algebra $\mathrm{Gr}(A)$ is called the *semiclassical limit* of the almost commutative algebra A. It is an example of a Poisson algebra as we recall later in this section.

Notice that as *vector spaces*, $\mathrm{Gr}(A)$ and A are linearly isomorphic, but their algebra structures are different as $\mathrm{Gr}(A)$ is always commutative but A need not be commutative. A linear isomorphism

$$q\colon\ \mathrm{Gr}(A) \to A$$

can be regarded as a *'naive quantization map'*. Of course, linear isomorphisms always exist but they are hardly interesting. One usually demands more. For example one wants q to be a Lie algebra map in the sense that

$$q\{a, b\} = [q(a), q(b)] \tag{3.11}$$

for all a, b in $\mathrm{Gr}(A)$. This is one form of *Dirac's quantization rule*, going back to Dirac's paper [67]. One normally thinks of A as the algebra of quantum observables of a system acting as operators on a Hilbert space, and of $\mathrm{Gr}(A)$ as the algebra of classical observables of functions on the phase space. *No-go theorems*, e.g. the celebrated Groenewold–Van Hove Theorem (cf. [1], [87] for discussions and precise statements; see also Exercise 3.3.2), states that, under reasonable irreducibility conditions, this is almost never possible. The remedy is to have q defined only for a special class of elements of $\mathrm{Gr}(A)$, or satisfy (3.11) only in an *asymptotic sense* as Planck's constant h goes to zero. As we shall discuss later in this section, this can be done in different ways, for example in the context of formal deformation quantization [10], [29], [120] or through strict C^*-algebraic deformation quantization [158], [112].

The notion of a Poisson algebra captures the structure of semiclassical limits.

Definition 3.3.2. Let P be a commutative algebra. A *Poisson structure* on P is a Lie algebra bracket $(a,b) \mapsto \{a,b\}$ on A such that for any $a \in A$, the map $b \mapsto \{a,b\}$: $A \to A$ is a derivation of A. That is, for all b, c in A we have

$$\{a, bc\} = \{a,b\}c + b\{a,c\}.$$

In geometric examples (see below) the vector field defined by the derivation $b \mapsto \{a,b\}$ is called the *Hamiltonian vector field* of the *Hamiltonian function* a.

Definition 3.3.3. A *Poisson algebra* is a pair $(P, \{\ ,\ \})$ where P is a commutative algebra and $\{\ ,\ \}$ is a Poisson structure on P.

We saw that the semiclassical limit $P = \mathrm{Gr}(A)$ of any almost commutative algebra A is a Poisson algebra. Conversely, given a Poisson algebra P one may ask if it is the semiclassical limit of an almost commutative algebra. This is one form of the problem of quantization of Poisson algebras, the answer to which for general Poisson algebras is negative. We give a few concrete examples of Poisson algebras (cf. also [29], [33]).

Example 3.3.1. A *Poisson manifold* is a manifold M whose algebra of smooth functions $A = C^\infty(M)$ is a Poisson algebra (we should also assume that the bracket $\{\ ,\ \}$ is continuous in the Fréchet topology of A, or, equivalently, is a bidifferential operator). It is not difficult to see that all Poisson structures on A are of the form

$$\{f, g\} := \langle df \wedge dg, \pi\rangle,$$

where $\pi \in C^\infty(\bigwedge^2(TM))$ is a smooth 2-vector field on M. This bracket clearly satisfies the Leibniz rule in each variable and one checks that it satisfies the Jacobi identity if and only if $[\pi, \pi] = 0$, where the *Schouten bracket* $[\pi, \pi] \in C^\infty(\bigwedge^3(TM))$ is defined in local coordinates by

$$[\pi, \pi]_{ijk} = \sum_{l=1}^{n} \left(\pi_{lj} \frac{\partial \pi_{ik}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{kj}}{\partial x_l} + \pi_{lk} \frac{\partial \pi_{ji}}{\partial x_l} \right).$$

The Poisson bracket in local coordinates is given by

$$\{f, g\} = \sum_{ij} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Symplectic manifolds are the simplest examples of Poisson manifolds. They correspond to non-degenerate Poisson structures. Recall that a *symplectic form* on a manifold is a non-degenerate closed 2-form on the manifold. Given a symplectic form ω, the associated Poisson bracket is given by

$$\{f, g\} = \omega(X_f, X_g),$$

where the vector field X_f is the *symplectic dual* of df and is defined by requiring that the equation $df(Y) = \omega(X_f, Y)$ holds for all smooth vector fields Y on M.

Let $C^\infty_{\text{poly}}(T^*M)$ be the algebra of smooth functions on T^*M which are polynomial in the cotangent direction. It is a Poisson algebra under the natural symplectic structure of T^*M. This Poisson algebra is the semiclassical limit of the algebra of differential operators on M, as we shall see in the next example.

Example 3.3.2 (Differential operators on commutative algebras)**.** Let A be a commutative unital algebra. We define an algebra $\mathcal{D}(A) \subset \operatorname{End}_{\mathbb{C}}(A)$ inductively as follows. Let

$$\mathcal{D}^0(A) = A = \operatorname{End}_A(A) \subset \operatorname{End}_{\mathbb{C}}(A)$$

denote the set of differential operators of order zero on A, i.e., A-linear maps from $A \to A$. Assuming $\mathcal{D}^k(A)$ has been defined for $0 \leq k < n$, we let $\mathcal{D}^n(A)$ be the set of all operators D in $\operatorname{End}_{\mathbb{C}}(A)$ such that for any $a \in A$, $[D, a] \in \mathcal{D}^{n-1}(A)$. The set

$$\mathcal{D}(A) = \bigcup_{n \geq 0} \mathcal{D}^n(A)$$

is a subalgebra of $\operatorname{End}_{\mathbb{C}}(A)$, called the *algebra of differential operators* on A. It is an almost commutative algebra under the filtration given by subspaces $\mathcal{D}^n(A)$, $n \geq 0$. Elements of $\mathcal{D}^n(A)$ are called differential operators of order n. For example, a linear map $D\colon A \to A$ is a differential operator of order one if and only if it is of the form $D = \delta + a$, where δ is a derivation on A and $a \in A$.

For general A, the semiclassical limit $\operatorname{Gr}(\mathcal{D}(A))$ and its Poisson structure are not easily identified except for coordinate rings of smooth affine varieties or algebras of smooth functions on a manifold. In this case a differential operator D of order k is locally given by

$$D = \sum_{|I| \leq k} a_I(x) \partial^I,$$

where $I = (i_1, \dots, i_n)$ is a multi-index, $\partial^I = \partial_{i_1}\partial_{i_2}\dots\partial_{i_n}$ is a mixed partial derivative, and n is the dimension of the manifold. This expression depends on the local coordinates but its leading terms of total degree n have an invariant meaning provided that we replace ∂_i with $\xi_i \in T^*M$. For $\xi \in T_x^*M$, let

$$\sigma_p(D)(x,\xi) := \sum_{|I|=k} a_I(x)\xi^I.$$

Then the function $\sigma_p(D)\colon T^*M \to \mathbb{C}$, called the *principal symbol* of D, is invariantly defined and belongs to $C^\infty_{\text{poly}}(T^*M)$. The algebra $C^\infty_{\text{poly}}(T^*M)$ inherits a canonical Poisson structure as a subalgebra of the Poisson algebra $C^\infty(T^*M)$ and we have the following

Proposition 3.3.1. *The principal symbol map induces an isomorphism of Poisson algebras*

$$\sigma_p\colon \operatorname{Gr}\mathcal{D}(C^\infty(M)) \xrightarrow{\ \simeq\ } C^\infty_{\text{poly}}(T^*M).$$

See [33] for a proof of this or, even better, try to prove it yourself by proving it for Weyl algebras first.

Example 3.3.3 (Weyl algebra)**.** Let $A_1 := \mathcal{D}\,\mathbb{C}[X]$ be the *Weyl algebra* of differential operators on the line. Alternatively, A_1 can be described as the unital complex algebra defined by generators x and p with

$$px - xp = 1.$$

The map $x \mapsto x$, $p \mapsto \frac{d}{dx}$ defines the isomorphism. Physicists prefer to write the defining relation as the *canonical commutation relation* $pq - qp = \frac{h}{2\pi i}1$, where h is Planck's constant and p and q represent momentum and position operators. This is not without merit because we can then let $h \to 0$ and obtain the commutative algebra of polynomials in p and q as the semiclassical limit. Also, i is necessary if we want to consider p and q as selfadjoint operators (why?). Then one can use the representation $q \mapsto x$, $p \mapsto \frac{h}{2\pi i}\frac{d}{dx}$.

Any element of A_1 has a unique expression as a differential operator with polynomial coefficients $\sum a_i(x)\frac{d^i}{dx^i}$ where the standard filtration is by degree of the differential operator. The principal symbol map

$$\sigma_p\Big(\sum_{i=0}^n a_i(x)\frac{d^i}{dx^i}\Big) = a_n(x)y^n.$$

defines an algebra isomorphism $\operatorname{Gr}(A_1) \simeq \mathbb{C}[x,y]$. The induced Poisson bracket on $\mathbb{C}[x,y]$ is the classical Poisson bracket

$$\{f,\, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

In principle, the Weyl algebra A_n is the algebra of differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. Alternatively, it can be defined as the universal algebra defined by $2n$ generators $x_1, \ldots, x_n, p_1, \ldots, p_n$ with

$$[p_i, x_i] = \delta_{ij} \quad \text{and} \quad [p_i, p_j] = [x_i, x_j] = 0$$

for all i, j. Notice that $A_n \simeq A_1 \otimes \cdots \otimes A_1$ (n factors). A lot is known about Weyl algebras and a lot remains to be known, including the Dixmier conjecture about the automorphisms of A_n. The Hochschild and cyclic cohomology of A_n are computed in [79] (cf. also [124]).

Example 3.3.4 (Universal enveloping algebras)**.** Let $U(\mathfrak{g})$ denote the *enveloping algebra* of a Lie algebra $\mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. For $p \geq 0$, let $F^p(U(\mathfrak{g}))$ be the subspace generated by tensors of degree at most p. This turns $U(\mathfrak{g})$ into a filtered algebra and the Poincaré–Birkhoff–Witt theorem asserts that its associated graded algebra is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. The algebra isomorphism is induced by the *symmetrization map* $s\colon S(\mathfrak{g}) \to \mathrm{Gr}(U(\mathfrak{g}))$, defined by

$$s(X_1 X_2 \ldots X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \ldots X_{\sigma(p)}.$$

Note that $S(\mathfrak{g})$ is the algebra of polynomial functions on the dual space $\mathfrak{g}^*$, which is a Poisson manifold under the bracket

$$\{f, g\}(X) = [Df(X), \ \ Dg(X)]$$

for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $X \in \mathfrak{g}^*$. Here we have used the canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{**}$, to regard the differential $Df(X) \in \mathfrak{g}^{**}$ as an element of $\mathfrak{g}$. The induced Poisson structure on polynomial functions coincides with the Poisson structure in $\mathrm{Gr}(U(\mathfrak{g}))$.

Example 3.3.5 (Algebra of formal pseudodifferential operators on the circle)**.** This algebra is obtained by formally inverting the differentiation operator $\partial := \frac{d}{dx}$ and then completing the resulting algebra. A formal pseudodifferential operator on the circle is an expression of the form $\sum_{-\infty}^{n} a_i(x)\partial^i$, where each $a_i(x)$ is a Laurent polynomial. The multiplication is uniquely defined by the rules $\partial x - x\partial = 1$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. We denote the resulting algebra by Ψ_1. The *Adler–Manin trace* on Ψ_1 [129], also called the *noncommutative residue*, is defined by

$$\mathrm{Tr}\Big(\sum_{-\infty}^{n} a_i(x)\partial^i\Big) = \mathrm{Res}(a_{-1}(x);\, 0) = \frac{1}{2\pi i}\int_{S^1} a_{-1}(x).$$

This is a trace on Ψ_1. In fact one can show that $\Psi_1/[\Psi_1,\, \Psi_1]$ is 1-dimensional which means that any trace on Ψ_1 is a multiple of Tr. Notice that for the Weyl algebra A_1 we have $[A_1,\, A_1] = A_1$.

Another interesting difference between Ψ_1 and A_1 is that Ψ_1 admits non-inner derivations (see exercise below). The algebra Ψ_1 has a nice generalization to algebras of pseudodifferential operators in higher dimensions. The appropriate extension of the above trace is the *noncommutative residue* of Wodzicki (cf. [177]; see also [41] for relations with the Dixmier trace and its role in noncommutative Riemannian geometry).

So far in this section we saw at least one way to formalize the idea of quantization through the notion of an almost commutative algebra and its semiclassical limit which is a Poisson algebra. A closely related notion is *formal deformation quantization*, or *star products*, going back to [10], [13], [145]. It is also closely related to the theory of deformations of associative algebras developed originally by Gerstenhaber, as we recall now.

Let A be an algebra, which may be noncommutative, over $\mathbb{C}$, and let $A[[h]]$ be the algebra of formal power series over A. A *formal deformation* of A is an associative $\mathbb{C}[[h]]$-linear multiplication

$$*_h \colon A[[h]] \otimes A[[h]] \to A[[h]]$$

such that $*_0$ is the original multiplication. Writing

$$a *_h b = B_0(a, b) + hB_1(a, b) + h^2 B_2(a, b) + \cdots ,$$

where $B_i \colon A \otimes A \to A$ are Hochschild 2-cochains on A with values in A, we see that the initial value condition on $*_h$ is equivalent to $B_0(a, b) = ab$ for all $a, b \in A$. Let us define a bracket $\{\ ,\ \}$ on A by

$$\{a,\, b\} = B_1(a, b) - B_1(b, a)$$

or, equivalently, but more suggestively, by

$$\{a,\, b\} := \mathrm{Lim}\, \frac{a *_h b - b *_h a}{h} \quad \text{as } h \to 0.$$

Using the associativity of the star product $a *_h (b *_h c) = (a *_h b) *_h c$, it is easy to check that B_1 is a Hochschild 2-cocycle for the Hochschild cohomology of A with coefficients in A, i.e., it satisfies the relation

$$aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0$$

for all a, b, c in A. Clearly the bracket $\{\ ,\ \}$ satisfies the Jacobi identity. In short, $(A, \{\ ,\ \})$ is an example of what is sometimes called a *noncommutative Poisson algebra*. If A is a commutative algebra, then it is easy to see that it is indeed a Poisson algebra in the sense of Definition 3.3.3.

The bracket $\{\ ,\ \}$ can be regarded as the *infinitesimal direction* of the deformation, and the deformation problem for a commutative Poisson algebra amounts to finding higher order terms B_i, $i \geq 2$, given B_0 and B_1.

The associativity condition on $*_h$ is equivalent to an infinite system of equations involving the cochains B_i that we derive now. They are given by

$$B_0 \circ B_n + B_1 \circ B_{n-1} + \cdots + B_n \circ B_0 = 0 \quad \text{for all } n \geq 0,$$

or, equivalently,

$$\sum_{i=1}^{n-1} B_i \circ B_{n-i} = \delta B_n. \tag{3.12}$$

Here, the *Gerstenhaber* $\circ$*-product* of 2-cochains f, $g\colon A \otimes A \to A$ is defined as the 3-cochain

$$f \circ g(a,b,c) = f(g(a,b),c) - f(a,g(b,c)).$$

Notice that a 2-cochain f defines an associative product if and only if $f \circ f = 0$. Also notice that the Hochschild coboundary δf can be written as

$$\delta f = -m \circ f - f \circ m,$$

where $m\colon A \otimes A \to A$ is the multiplication of A. These observations lead to the associativity equations (3.12).

To solve these equations starting with $B_0 = m$, by antisymmetrizing we can always assume that B_1 is antisymmetric and hence we can assume $B_1 = \frac{1}{2}\{\,,\,\}$. Assume $B_0, B_1, \ldots, B_n$ have been found so that (3.12) holds. Then one can show that $\sum_{i=1}^{n} B_i \circ B_{n-i}$ is a cocycle. Thus we can find a B_{n+1} satisfying (3.12) if and only if this cocycle is a coboundary, i.e., its class in $H^3(A,A)$ should vanish. The upshot is that the third Hochschild cohomology $H^3(A,A)$ is the *space of obstructions* for the deformation quantization problem. In particular if $H^3(A,A) = 0$ then any Poisson bracket on A can be deformed. Notice, however, that this is only a sufficient condition and is by no means necessary, as will be shown below.

In the most interesting examples, e.g. for $A = C^\infty(M)$, $H^3(A,A) \neq 0$. To see this consider the differential graded Lie algebra $(C^*(A,\,A),\,[\,,\,],\,\delta)$ of continuous Hochschild cochains on A, and the differential graded Lie algebra, with zero differential, $(\bigwedge(TM),\,[\,,\,],\,0)$ of polyvector fields on M. The bracket in the first is the Gerstenhaber bracket and in the second is the Schouten bracket of polyvector fields. By a theorem of Connes (see the resolution in Lemma 44 in [39]), we know that the *antisymmetrization map*

$$\alpha\colon \big(C^\infty\big(\textstyle\bigwedge TM\big),\,0\big) \to \big(C^*(A,\,A),\,\delta\big)$$

sending a polyvector field $X_1 \wedge \cdots \wedge X_k$ to the functional φ defined by

$$\varphi(f^1, \ldots, f^k) = df^1(X_1)df^2(X_2)\ldots df^k(X_k)$$

is a quasi-isomorphism of differential graded algebras. In particular, it induces an isomorphism of graded commutative algebras

$$\bigoplus_k H^k(A,A) \simeq \bigoplus_k C^\infty\big(\textstyle\bigwedge^k TM\big).$$

The map α, however, is not a morphism of Lie algebras and that is where the real difficulty of deforming a Poisson structure is hidden. The *formality theorem* of M. Kontsevich [112] states that as a differential graded Lie algebra, $(C^*(A, A), \delta, [\,,\,])$ is *formal* in the sense that it is quasi-isomorphic to its cohomology. Equivalently, it means that one can perturb the map α, by adding an infinite number of terms, to a morphism of L_∞-algebras. This shows that the original deformation problem of Poisson structures can be transferred to $(C^\infty(\bigwedge TM), 0)$ where it is unobstructed since the differential in the latter DGL is zero. Later in this section we shall give a couple of simple examples where deformations can be explicitly constructed.

There is a much deeper structure hidden in the deformation complex of an associative $(C^*(A, A), \delta)$ than first meets the eye and we can only barely scratch the surface here. The first piece of structure is the cup product. Let $C^* = C^*(A, A)$. The *cup product* $\smile\colon C^p \times C^q \to C^{p+q}$ is defined by

$$(f \smile g)(a^1, \ldots, a^{p+q}) = f(a^1, \ldots, a^p)g(a^{p+1}, \ldots, a^{p+q}).$$

Notice that $\smile$ is associative and one checks that this product is compatible with the differential δ and hence induces an associative graded product on $H^*(A, A)$. What is not so obvious however is that this product is graded commutative for any algebra A [83].

The second piece of structure on $(C^*(A, A), \delta)$ is a graded Lie bracket. It is based on the Gerstenhaber circle product $\circ\colon C^p \times C^q \to C^{p+q-1}$ defined by

$$\begin{aligned}(f \circ g)&(a_1, \ldots, a_{p+q-1})\\ &= \sum_{i=1}^{p-1} (-1)^{|g|(|f|+i-1)} f(a^1, \ldots, g(a^i, \ldots, a^{i+p}), \ldots, a^{p+q-1}).\end{aligned}$$

Notice that $\circ$ is not an associative product. Nevertheless one can show that [83] the corresponding graded bracket $[\,,\,]\colon C^p \times C^q \to C^{p+q-1}$

$$[f, g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$$

defines a graded Lie algebra structure on the deformation cohomology $H^*(A, A)$. Notice that the Lie algebra grading is now shifted by one.

What is most interesting is that the cup product and the Lie algebra structure are compatible in the sense that $[\,,\,]$ is a graded derivation for the cup product; or in short, $(H^*(A, A), \smile, [\,,\,])$ is a *graded Poisson algebra*.

The *fine structure* of the Hochschild cochain complex $(C^*(A, A), \delta)$, e.g. the existence of higher order products and homotopies between them has been the subject of many studies in recent years. While it is relatively easy to write down these higher order products in the form of a brace algebra structure on the Hochschild complex, relating them to known geometric structures such as moduli spaces of curves, as predicted by Deligne's conjecture, is quite hard [114].

Remark 5. A natural question arises from the graded Poisson algebra structure on deformation cohomology $H^*(A, A)$: is $H^*(A, A)$ the semiclassical limit of a 'quantum cohomology' theory for algebras?

We give a couple of examples where deformations can be explicitly constructed.

Example 3.3.6. The simplest non-trivial Poisson manifold is the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. Let $U_h(\mathfrak{g}) = T(\mathfrak{g})/I$, where the ideal I is generated by

$$x \otimes y - y \otimes x - h[x, y], \quad x, y \in \mathfrak{g}.$$

This is simply the enveloping algebra of the rescaled bracket $h[-,-]$. By the Poincaré–Birkhoff–Witt theorem, the antisymmetrization map $\alpha_h \colon S(\mathfrak{g}) \to U_h(\mathfrak{g})$ is a linear isomorphism. We can define a $*$-product on $S(\mathfrak{g})$ by

$$f *_h g = \alpha_h^{-1}(\alpha_h(f)\alpha_h(g)) = \sum_{n=0}^{\infty} h^n B_n(f, g).$$

With some work one can show that the B_n are bidifferential operators and hence the formula extends to a $*$-product on $C^\infty(\mathfrak{g}^*)$.

Example 3.3.7 (Weyl–Moyal quantization)**.** Consider the algebra generated by x and y with relation $xy - yx = \frac{h}{i}1$. Let f, g be polynomials in x and y. Iterated application of the Leibniz rule gives the formula for the product

$$f *_h g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ih}{2} \right)^n B_n(f, g),$$

where $B_0(f, g) = fg$, $B_1(f, g) = \{f,\, g\}$ is the standard Poisson bracket, and for $n \geq 2$,

$$B_n(f,\, g) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g).$$

Notice that this formula makes sense for $f,\, g \in C^\infty(\mathbb{R}^2)$ and defines a deformation of this algebra with its standard Poisson structure. This can be extended to arbitrary *constant* Poisson structures on $\mathbb{R}^2$,

$$\{f,\, g\} = \sum \pi^{ij} \partial_i f \, \partial_j g.$$

The Weyl–Moyal $*$ product is then given by

$$f *_h g = \exp\Bigg(- i\frac{h}{2} \sum \pi^{ij} \partial_i \wedge \partial_j \Bigg)(f,\, g).$$

Finally let us briefly recall Rieffel's strict deformation quantization [158]. Roughly speaking, one demands that formal power series of formal deformation theory should actually be convergent. More precisely, let $(M, \{\ ,\ \})$ be a Poisson manifold.

A *strict deformation quantization* of the Poisson algebra $\mathcal{A} = C^\infty(M)$ is a family of pre-
C^*-algebra structures $(*_h, \|\cdot\|_h)$ on $\mathcal{A}$ for $h \geq 0$ such that the family forms a continuous field of pre-C^*-algebras on $[0, \infty)$ (in particular for any $f \in \mathcal{A}$, $h \mapsto \|f\|_h$ is continuous) and for all $f, g \in \mathcal{A}$,

$$\left\| \frac{f *_h g - g *_h f}{ih} \right\|_h \to \{f, g\}$$

as $h \to 0$. We therefore have a family of C^*-algebras A_h obtained by completing $\mathcal{A}$ with respect to the norm $\|\cdot\|_h$.

Example 3.3.8 (Noncommutative tori)**.** In [157] it is shown that the family of noncommutative tori A_θ form a strict deformation quantization of the Poisson algebra of smooth functions on the 2-torus. This in fact appears as a special case of a more general result. Let α be a smooth action of $\mathbb{R}^n$ on $\mathcal{A} = C^\infty(M)$. Let X_i denote the infinitesimal generators for this action. Each skew-symmetric $n \times n$ matrix J defines a Poisson bracket on $\mathcal{A}$ by

$$\{f, g\} = \sum J_{ij} X_i(f) X_j(g).$$

For each $h \in \mathbb{R}$, define a new product $*_h$ on $\mathcal{A}$ by

$$f *_h g = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{hJu}(f) \alpha_v(g) e^{2\pi i u \cdot v} \, du dv.$$

The $*$-structure is defined by conjugation and is undeformed (see [157] for the definition of $\|f\|_h$). For $\mathcal{A} = C^\infty(\mathbb{T}^2)$ with the natural $\mathbb{R}^2$ action one obtains A_θ.

Remark 6. Does any Poisson manifold admit a strict deformation quantization? This question is still open (even for symplectic manifolds). In [157], Rieffel shows that the canonical symplectic structure on the 2-sphere admits no SO(3)-invariant strict deformation quantization. An intriguing idea proposed in [51] is that the existence of a strict deformation quantization of a Poisson manifold should be regarded as an *integrability condition* for formal deformation quantization. There is a clear analogy with the case of formal and convergent power series solutions of differential equations around singular points. The question is raised of a possible 'theory of ambiguity', i.e., a cohomology theory that could capture the difference between the two cases.

This idea is fully realized by N. P. Landsman in his book [120] in the example of strict deformation quantization of Poisson manifolds dual to Lie algebroids. He shows that these are integrable precisely when they can be deformed, namely by the C^*-algebra of the Lie groupoid integrating the given Lie algebroid (note that the corresponding Poisson manifold is integrable if and only if the Lie algebroid is). On the other hand, as far as we know, $H^3(A, A)$ and indeed all of Hochschild cohomology seems to be irrelevant to strict C^*-deformation quantization.

Exercise 3.3.1. Show that the Weyl algebra A_1 is a simple algebra, i.e., it has no non-trivial two-sided ideals; prove the same for A_n. In a previous exercise we asked to show that any derivation of A_1 is inner. Is it true that any automorphism of A_1 is inner?

Exercise 3.3.2. In Example 3.3.3 show that there is no linear map $q\colon \mathbb{C}[x,y] \to A_1$ such that $q(1) = 1$ and $q\{f, g\} = [q(f), q(g)]$ for all f and g. This is an important special case of the Groenewold–van Hove no-go theorem ([1], [87]).

Exercise 3.3.3. Let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. It is a non-smooth algebra. Describe its algebra of differential operators.

Exercise 3.3.4. Unlike the algebra of differential operators, the algebra of pseudodifferential operators Ψ_1 admits non-inner derivations. Clearly $\log \partial := -\sum_1^\infty \frac{(1-\partial)^n}{n} \notin \Psi_1$, but show that for any $a \in \Psi_1$, we have $[\log \partial, a] \in \Psi_1$ and therefore the map

$$a \mapsto \delta(a) := [\log\, \partial,\, a]$$

defines a non-inner derivation of Ψ_1. The corresponding Lie algebra 2-cocycle

$$\varphi(a,b) = \mathrm{Tr}(a[\log\, \partial,\, b])$$

is the Radul cocycle [116].

3.4 Topological algebras

For applications of Hochschild and cyclic cohomology to noncommutative geometry, it is crucial to consider topological algebras, topological bimodules, topological resolutions, and continuous cochains and chains. For example, while the algebraic Hochschild groups of the algebra of smooth functions on a smooth manifold are not known, and perhaps are hopeless to compute, its continuous Hochschild (co)homology as a topological algebra can be computed as we recall in Example 3.2.3 below. We shall give only a brief outline of the definitions and refer the reader to [39], [41] for more details. A good reference for locally convex topological vector spaces and topological tensor products is [170].

There is no difficulty in defining *continuous* analogues of Hochschild and cyclic cohomology groups for Banach algebras. One simply replaces bimodules by Banach bimodules, that is a bimodule which is also a Banach space where the left and right module actions are bounded operators, and cochains by continuous cochains. Since the multiplication of a Banach algebra is a continuous operation, all operators including the Hochschild boundary and the cyclic operators extend to this continuous setting. The resulting Hochschild and cyclic theory for C^*-algebras, however, is hardly useful and tends to vanish in many interesting examples. This is hardly surprising since the definition of any Hochschild or cyclic cocycle of dimension bigger than zero involves differentiating the elements of the algebra in one way or another. (See Exercise 3.1.3 or, more generally, the Remark below.) This is in sharp contrast with topological K-theory where the right setting, e.g. for Bott periodicity to hold, is the setting of Banach or C^*-algebras.

Remark 7. By combining results of Connes [34] and Haagerup [88], we know that a C^*-algebra is *amenable* if and only if it is *nuclear*. Amenability refers to the property that for all $n \geq 1$,

$$H^n_{\text{cont}}(A, M^*) = 0$$

for an arbitrary Banach dual bimodule M^*. In particular, for any nuclear C^*-algebra
$HH^n_{\text{cont}}(A) = H^n_{\text{cont}}(A, A^*) = 0$ for all $n \geq 1$. Using Connes' long exact sequence (see Section 3.7), we obtain, for any nuclear C^*-algebra A, the vanishing results

$$HC^{2n}_{\text{cont}}(A) = A^* \quad \text{and} \quad HC^{2n+1}_{\text{cont}}(A) = 0$$

for all $n \geq 0$. Nuclear C^*-algebras form a large class which includes commutative algebras, the algebra of compact operators, and reduced group C^*-algebras of amenable groups [15].

The right class of topological algebras for Hochschild and cyclic cohomology turns out to be the class of *locally convex algebras* [39]. An algebra A which is simultaneously a locally convex topological vector space is called a locally convex algebra if its multiplication map $A \otimes A \to A$ is (jointly) continuous. That is, for any continuous seminorm p on A there is a continuous seminorm p' on A such that $p(ab) \leq p'(ab)$ for all a, b in A.

We should mention that there are topological algebras with a locally convex topology for which the multiplication map is only *separately continuous*. But we do not dwell on this more general class in this book as they appear rarely in applications. This distinction between separate and joint continuity of the multiplication map disappears for the class of Fréchet algebras. By definition, a locally convex algebra is called a *Fréchet algebra* if its topology is metrizable and complete. Many examples of 'smooth noncommutative spaces' that one encounters in noncommutative geometry are in fact Fréchet algebras.

Example 3.4.1. Basic examples of Fréchet algebras include the algebra $A = C^\infty(M)$ of smooth functions on a closed smooth manifold and the smooth noncommutative tori $\mathcal{A}_\theta$ and their higher dimensional analogues. We start with a simple down to earth example where $A = C^\infty(S^1)$. We consider the elements of A as smooth periodic functions of period one on the line. Its topology is defined by the sequence of norms

$$p_n(f) = \sup \|f^{(k)}\|_\infty, \quad 0 \leq k \leq n,$$

where $f^{(k)}$ is the k-th derivative of f and $\| \ \|_\infty$ is the sup norm. We can equivalently use the sequence of norms

$$q_n(f) = \sum_{i=0}^{n} \frac{1}{k!} \|f^{(k)}\|_\infty.$$

Notice that q_n's are *submultiplicative*, that is $q_n(fg) \leq q_n(f)q_n(g)$. Locally convex algebras whose topology is induced by a family of submultiplicative seminorms are known to be projective limits of Banach algebras. This is the case in all examples in this section.

In general, the topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|, \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M. The Leibniz rule for derivatives of products shows that the multiplication map is indeed jointly continuous. See the Exercise 3.4.1 for the topology of $\mathcal{A}_\theta$.

Given locally convex topological vector spaces V_1 and V_2, their *projective tensor product* is a locally convex space $V_1 \hat{\otimes} V_2$ together with a universal jointly continuous bilinear map $V_1 \otimes V_2 \to V_1 \hat{\otimes} V_2$ (cf. [86], [170]). That is, for any locally convex space W, we have a natural isomorphism between jointly continuous bilinear maps $V_1 \times V_2 \to W$ and continuous linear maps $V_1 \hat{\otimes} V_2 \to W$. Explicitly, the topology of $V_1 \hat{\otimes} V_2$ is defined by the family of seminorms $p \hat{\otimes} q$, where p, q are continuous seminoms on V_1 and V_2 respectively, and

$$p \hat{\otimes} q(t) := \inf \left\{ \textstyle\sum_i^n p(a_i)q(b_i);\ t = \sum_i^n a_i \otimes b_i,\, a_i \in V_1,\, b_i \in V_2 \right\}.$$

Then $V_1 \hat{\otimes} V_2$ is defined as the completion of $V_1 \otimes V_2$ under the above family of seminorms.

One of the nice properties of the topology of $C^\infty(M)$ is that it is *nuclear* (see [86], [170]). In particular for any other smooth compact manifold N, the natural map

$$C^\infty(M) \hat{\otimes} C^\infty(N) \to C^\infty(M \times N)$$

is an isomorphism of topological algebras. This plays an important role in computing the continuous Hochschild cohomology of $C^\infty(M)$.

Let A be a locally convex topological algebra. A topological left A-module is a locally convex vector space M endowed with a continuous left A-module action $A \hat{\otimes} M \to M$. A *topological free* left A-module is a module of the type $M = A \hat{\otimes} V$ where V is a locally convex space. A *topological projective module* is a topological module which is a direct summand in a free topological module.

Given a locally convex algebra A, let

$$C^n_{\text{cont}}(A) = \text{Hom}_{\text{cont}}(A^{\hat{\otimes} n}, \mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A. All the algebraic definitions and results of this chapter extend to this topological setting. In particular one defines the *continuous Hochschild* and *cyclic cohomology* groups of a locally convex algebra. One must be careful, however, in dealing with resolutions. The right class of topological projective (in particular free) resolutions are those resolutions that admit a continuous linear splitting. This extra condition is needed when one wants to prove comparison theorems for resolutions and, eventually, independence of cohomology from resolutions. We shall not go into details here since this is very well explained in [39].

Exercise 3.4.1. The sequence of norms

$$p_k(a) = \operatorname*{Sup}_{m,n} \{(1 + |n| + |m|)^k |a_{mn}|\}$$

defines a locally convex topology on the smooth noncommutative torus $\mathcal{A}_\theta$. Show that the multiplication of $\mathcal{A}_\theta$ is continuous in this topology.

3.5 Examples: Hochschild (co)homology

We give a few examples of Hochschild (co)homology computations. In particular we shall see that group (co)homology and Lie algebra (co)homology are instances of Hochschild (co)homology. We start by recalling the classical results of Hochschild–Kostant–Rosenberg [100] and Connes [39] which identifies the Hochschild homology of *smooth commutative algebras* with the algebra of differential forms. By a smooth commutative algebra we mean either the topological algebra $A = C^\infty(M)$ of smooth complex-valued functions on a closed smooth manifold M, or the algebra $A = \mathcal{O}[X]$ of regular function on a smooth affine algebraic variety X. We start with the latter case.

Example 3.5.1 (Smooth commutative algebras)**.** Algebras of regular functions on a smooth affine variety can be characterized abstractly through various equivalent conditions (cf. Proposition 3.4.2 in [124]). For example, one knows that a finitely generated commutative algebra A is smooth if and only if it has the *lifting property* with respect to *nilpotent extensions.* More precisely, A is smooth if and only if for any pair (C, I), where C is a commutative algebra and I is an ideal such that $I^2 = 0$, the map

$$\mathrm{Hom}_{\mathrm{alg}}(A, C) \to \mathrm{Hom}_{\mathrm{alg}}(A, C/I)$$

is surjective. Examples of smooth algebras include polynomial algebras $\mathbb{C}[x_1, \ldots, x_n]$, algebras of Laurent polynomials $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, and coordinate rings of affine algebraic groups. The algebra $\mathbb{C}[x, y]/(xy)$ is not smooth.

We recall the definition of the *algebraic de Rham complex* of a commutative, not necessarily smooth, algebra A. The module of 1-forms, or *Kähler differentials*, over A, denoted by Ω^1_A, is by definition a left A-module Ω^1_A endowed with a *universal derivation*

$$d\colon A \to \Omega^1_A.$$

This means that any other derivation $\delta\colon A \to M$ into a left A-module M uniquely factorizes through d. One usually defines $\Omega^1_A := I/I^2$ where I is the kernel of the multiplication map $A \otimes A \to A$. Note that since A is commutative this map is an algebra homomorphism and I is an ideal. The left multiplication defines a left A-module structure on Ω^1_A. The derivation d is defined by

$$d(a) = a \otimes 1 - 1 \otimes a \mod (I^2).$$

Checking its universal property is straightforward. One defines the space of n-forms on A as the n-th exterior power of the A-module Ω^1_A:

$$\Omega^n_A := \bigwedge\nolimits^n_A \Omega^1_A,$$

where the exterior product is over A. There is a unique extension of d to a graded derivation of degree one,

$$d\colon \Omega^*_A \to \Omega^{*+1}_A.$$

It satisfies $d^2 = 0$. The *algebraic de Rham cohomology* of A is the cohomology of the complex $(\Omega^*_A,\, d)$. For some examples of this construction see exercises at the end of this section.

Let us compare the complex of differential forms with trivial differential $(\Omega^*_A,\, 0)$, with the Hochschild complex of A with coefficients in A, $(C_*(A),\, b)$. Consider the *antisymmetrization map*

$$\varepsilon_n\colon \Omega^n_A \to A^{\otimes(n+1)}, \quad n = 0, 1, 2, \dots,$$

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where S_n denotes the symmetric group on n letters. We also have maps

$$\mu_n\colon A^{\otimes(n+1)} \to \Omega^n_A, \quad n = 0, 1, \dots,$$

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n.$$

One checks that both maps are morphisms of complexes, i.e.,

$$b \circ \varepsilon_n = 0 \quad \text{and} \quad \mu_n \circ b = 0.$$

Moreover, one has

$$\mu_n \circ \varepsilon_n = n!\, \mathrm{id}.$$

Assuming the ground field has characteristic zero, it follows that the antisymmetrization map induces an inclusion

$$\Omega^n_A \hookrightarrow HH_n(A), \quad n = 0, 1, 2, \dots. \tag{3.13}$$

In particular, for any commutative algebra A over a field of characteristic zero, the space of differential n-forms on A is always a direct summand of the Hochschild homology group $HH_n(A)$.

This map, however, need not be surjective in general (cf. Exercises below). This has to do with the *singularity* of the underlying geometric space represented by A. The Hochschild–Kostant–Rosenberg theorem [100] states that if A is the algebra of regular functions on a smooth affine variety, then the above map is an isomorphism. Notice that we have verified this fact for polynomial algebras in Example 3.2.2.

Example 3.5.2 (Algebras of smooth functions)**.** This is a continuation of Example 3.1.2. Let M be a smooth closed manifold and $A = C^\infty(M)$ the algebra of smooth complex-valued functions on M. It is a locally convex (in fact, Fréchet) topological algebra as we explained in Section 3.4. Let $\Omega^n M$ (resp. $\Omega_n M$) denote the space of n-forms (resp. n-currents) on M. Consider the map

$$\Omega_n M \to C^n_{\text{cont}}(C^\infty(M)), \quad C \mapsto \varphi_C,$$

where

$$\varphi_C(f_0, f_1, \ldots, f_n) := \langle C,\, f_0 df_1 \ldots df_n \rangle.$$

It is easily checked that this map defines a morphism of complexes

$$(\Omega_* M,\, 0) \to (C^*_{\text{cont}}(C^\infty(M)),\, b).$$

In [39], using an explicit resolution, Connes shows that the induced map on cohomologies is an isomorphism. Thus we have a natural isomorphism between space of de Rham currents on M and (continuous) Hochschild cohomology of $C^\infty(M)$:

$$\boxed{\Omega_i M \simeq HH^i_{\text{cont}}(C^\infty(M)), \quad i = 0, 1, \ldots} \tag{3.14}$$

Without going into details, we shall briefly indicate the resolution introduced in [39]. Let $\bigwedge^k T^*_{\mathbb{C}} M$ denote the bundle of complexified k-forms on M and E_k be its pullback under the projection $\text{pr}_2 \colon M \times M \to M$. Let X be a vector field on $M^2 = M \times M$ such that in a neighborhood of the diagonal $\Delta(M) \subset M \times M$ and in a local geodesic coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ it looks like

$$X = \sum_{i=1}^{n} (x_i - y_i) \frac{\partial}{\partial y_i}.$$

We assume that away from the diagonal X is nowhere zero. Such an X can always be found, provided that M admits a nowhere zero vector field. The latter condition is clearly equivalent to vanishing of the Euler characteristic of M. By replacing M by $M \times S^1$ the general case can be reduced to this special case.

The following is then shown to be a continuous projective resolution of $C^\infty(M)$ as a $C^\infty(M \times M)$-module [39]:

$$\begin{aligned} C^\infty(M) \xleftarrow{\varepsilon} C^\infty(M^2) &\xleftarrow{i_X} C^\infty(M^2, E_1) \\ &\xleftarrow{i_X} C^\infty(M^2, E_2) \xleftarrow{i_X} \cdots \xleftarrow{i_X} C^\infty(M^2, E_n) \leftarrow 0, \end{aligned}$$

where i_X is interior multiplication by X. By applying the Hom functor $\text{Hom}_{A \otimes A}(-, A^*)$ we obtain a complex with zero differentials

$$\Omega_0 M \xrightarrow{0} \Omega_1 M \xrightarrow{0} \cdots \xrightarrow{0} \Omega_n M \xrightarrow{0} 0,$$

and hence the isomorphism (3.14).

The analogous result for Hochschild homology uses the map

$$C^\infty(M) \hat{\otimes} \cdots \hat{\otimes} C^\infty(M) \to \Omega^n M$$

defined by

$$f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \dots df_n.$$

It is easy to check that this defines a morphism of complexes

$$C_*^{\text{cont}}(C^\infty(M), b) \to (\Omega^* M, 0).$$

Using the above resolution and by essentially the same argument, one shows that the induced map on homologies is an isomorphism between continuous Hochschild homology of $C^\infty(M)$ and differential forms on M:

$$\boxed{HH_i^{\text{cont}}(C^\infty(M)) \simeq \Omega^i M, \quad i = 0, 1, \dots}$$

Example 3.5.3 (Group algebras)**.** It is clear from the original definitions that group (co)homology is an example of Hochschild (co)homology. Let G be a group and M be a left G-module. The standard complex for computing the cohomology of G with coefficients in M is the complex (cf. [30], [81], [124])

$$M \xrightarrow{\delta} C^1(G, M) \xrightarrow{\delta} C^2(G, M) \xrightarrow{\delta} \cdots,$$

where

$$C^n(G, M) = \{f \colon G^n \to M\},$$

and the differential δ is defined by

$$(\delta m)(g) = gm - m,$$

$$\begin{aligned}\delta f(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, g_2, \dots, g_n).\end{aligned}$$

Let $A = \mathbb{C}G$ denote the group algebra of G over the complex numbers. Then M is a $\mathbb{C}G$-bimodule via the actions

$$g \cdot m = g(m), \quad m \cdot g = m$$

for all g in G and m in M. It is clear that for all n,

$$C^n(\mathbb{C}G, M) \simeq C^n(G, M),$$

and the isomorphism preserves the differentials. It follows that the group cohomology of G with coefficients in M coincides with the Hochschild cohomology of $\mathbb{C}G$ with coefficients in M:

$$H^n(\mathbb{C}G, M) \simeq H^n(G, M).$$

Conversely, it is easy to see that the Hochschild cohomology of $\mathbb{C}G$ with coefficients in a bimodule M reduces to group cohomology. Let $M^{\mathrm{ad}} = M$ as a vector space and define a left G-action on M^{ad} by

$$g \cdot m = gmg^{-1}.$$

Define a linear isomorphism $i\colon C^n(G, M^{\mathrm{ad}}) \to C^n(\mathbb{C}G, M)$ by

$$(if)(g_1, \ldots, g_n) = f(g_1, \ldots, g_n)g_1g_2 \ldots g_n.$$

It can be checked that i commutes with differentials and hence is an isomorphism of complexes (MacLane isomorphism)

$$C^*(G, M^{\mathrm{ad}}) \xrightarrow{\simeq} C^*(\mathbb{C}G, M).$$

Of course, there is a similar result for homology.

Of particular importance is an understanding of $HH_*(\mathbb{C}G) = H_*(\mathbb{C}G, \mathbb{C}G) = H_*(G, \mathbb{C}G)$, i.e., when $M = \mathbb{C}G$ and G is acting by *conjugation*. By a theorem of Burghelea [25], the Hochschild and cyclic homology groups of $\mathbb{C}G$ decompose over the set of conjugacy classes of G where each summand is the group homology (with trivial coefficients) of a group associated to the conjugacy class. We recall this result for Hochschild homology here and later we shall discuss the corresponding result for cyclic homology.

The whole idea can be traced back to the following simple observation. Let $\tau\colon \mathbb{C}G \to \mathbb{C}$ be a trace on the group algebra. It is clear that τ is constant on each conjugacy class of G and, conversely, the characteristic function of each conjugacy class defines a trace on $\mathbb{C}G$. Thus we have

$$HH^0(\mathbb{C}G) = \prod_{\langle G \rangle} \mathbb{C},$$

where $\langle G \rangle$ denotes the set of conjugacy classes of G. Dually, for homology we have

$$HH_0(\mathbb{C}G) = \bigoplus_{\langle G \rangle} \mathbb{C}.$$

We focus on homology and shall extend the above observation to higher dimensions. Dual cohomological versions are straightforward.

Clearly we have

$$(\mathbb{C}G)^{\otimes(n+1)} = \mathbb{C}G^{n+1}.$$

For each conjugacy class $c \in \langle G \rangle$, let $B_n(G, c)$ be the linear span of all $(n+1)$-tuples $(g_0, g_1, \ldots, g_n) \in G^{n+1}$ such that

$$g_0g_1 \ldots g_n \in c.$$

It is clear that $B_*(G, c)$ is invariant under the Hochschild differential b. We therefore have a decomposition of the Hochschild complex of $\mathbb{C}G$ into subcomplexes indexed by conjugacy classes:

$$C_*(\mathbb{C}G, \mathbb{C}G) = \bigoplus_{c \in \langle G \rangle} B_*(G, c).$$

Identifying the homology of the component corresponding to the conjugacy class of the identity is rather easy. For other components one must work harder. Let $c = \{e\}$ denote the conjugacy class of the identity element of G. The map $(g_0, g_1, \dots, g_n) \mapsto (g_1, g_2, \dots, g_n)$ is easily seen to define an isomorphism of vector spaces

$$B_n(G, \{e\}) \simeq \mathbb{C}G^n.$$

Moreover, under this map the Hochschild differential b goes over to the differential for the group homology of G with trivial coefficients. It follows that

$$H_*(B(G, \{e\})) \simeq H_*(G, \mathbb{C}).$$

Next we describe the Hochschild homology of other components. For an element $g \in G$, let

$$C_g = \{h \in G;\ hg = gh\}$$

denote the *centralizer* of g in G. Notice that the isomorphism class of this group depends only on the conjugacy class of g. One checks that the inclusion $i\colon C_n(C_g, \mathbb{C}) \to B_n(G, c)$ defined by

$$i(g_1, \dots, g_n) = ((g_1 g_2 \dots g_n)^{-1} g, g_1, \dots, g_{n-1})$$

is a chain map. One can in fact show, by an explicit chain homotopy, that i is a quasi-isomorphism. It therefore follows that, for each conjugacy class c and each $g \in c$, we have

$$H_*(B(G,\, c)) = H_*(C_g,\, \mathbb{C}).$$

Putting everything together this shows that the Hochschild homology of $\mathbb{C}G$ decomposes as a direct sum of group homologies of centralizers of conjugacy classes of G, a result due to Burghelea [25] (cf. also [132], [124], [73] for purely algebraic proofs):

$$\boxed{HH_*(\mathbb{C}G) \simeq \bigoplus_{\langle G \rangle} H_*(C_g)} \tag{3.15}$$

The corresponding dual statement for Hochschild cohomology reads as

$$HH^*(\mathbb{C}G) \simeq \prod_{\langle G \rangle} H^*(C_g).$$

Example 3.5.4 (Enveloping algebras)**.** We show that Lie algebra (co)homology is an example of Hochschild (co)homology, a result which goes back to Cartan–Eilenberg [30]. Let $\mathfrak{g}$ be a Lie algebra and M be a (left) $\mathfrak{g}$-module. This simply means that we have a Lie algebra morphism

$$\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(M).$$

The *Lie algebra homology* of $\mathfrak{g}$ with coefficients in M is, by definition, the homology of the *Chevalley–Eilenberg complex*

$$M \xleftarrow{\delta} M \otimes \bigwedge^1 \mathfrak{g} \xleftarrow{\delta} M \otimes \bigwedge^2 \mathfrak{g} \xleftarrow{\delta} M \otimes \bigwedge^3 \mathfrak{g} \xleftarrow{\delta} \cdots,$$

where the differential δ is defined by

$$\delta(m \otimes X) = X(m),$$

$$\begin{aligned}\delta(m \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_n) = \\ \sum_{i<j} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n \\ + \sum_i (-1)^i X_i(m) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n.\end{aligned}$$

(One checks that $\delta^2 = 0$.)

Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. Given a $U(\mathfrak{g})$-bimodule M, we define a left $\mathfrak{g}$-module M^{ad}, where $M^{\mathrm{ad}} = M$ and

$$X \cdot m := Xm - mX$$

for all $X \in \mathfrak{g}$ and $m \in M$. Define a map

$$\varepsilon \colon M^{\mathrm{ad}} \otimes \bigwedge^n \mathfrak{g} \to M \otimes U(\mathfrak{g})^{\otimes n}$$

from the Lie algebra complex to the Hochschild complex by

$$\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}.$$

One checks that $\varepsilon \colon C^{\mathrm{Lie}}(\mathfrak{g}, M^{\mathrm{ad}}) \to C(U(\mathfrak{g}), M)$ is a chain map (prove this!). We claim that it is indeed a quasi-isomorphism, i.e., it induces an isomorphism between the corresponding homology groups:

$$H_*(\mathfrak{g}, M^{\mathrm{ad}}) \simeq H_*(U(\mathfrak{g}), M).$$

We refer to [30], [124] for its standard proof.

Example 3.5.5 (Morita invariance of Hochschild homology)**.** Let A and B be unital Morita equivalent algebras (cf. Section 2.3 for definitions). Let X be an equivalence A–B-bimodule and Y be an inverse B–A-bimodule. Let M be an A-bimodule and $N = Y \otimes_A M \otimes_A X$ the corresponding B-bimodule. *Morita invariance* of Hochschild homology states that there is a natural isomorphism

$$H_n(A, M) \simeq H_n(B, N)$$

for all $n \geq 0$. A proof of this can be found in [127], [124]. There is a similar result, with a similar proof, for cohomology. We sketch a proof of this result for the special case where $B = M_k(A)$ is the algebra of k by k matrices over A. The main idea is to introduce the *generalized trace map*.

Let M be an A-bimodule and $M_k(M)$ be the space of k by k matrices with coefficients in M. It is a bimodule over $M_k(A)$ in an obvious way. The *generalized trace map* is defined by

$$\mathrm{Tr}\colon C_n(M_k(A),\, M_k(M)) \to C_n(A,\, M),$$

$$\mathrm{Tr}(\alpha_0 \otimes m_0 \otimes \alpha_1 \otimes a_1 \otimes \cdots \otimes \alpha_n \otimes a_n) = \mathrm{tr}(\alpha_0 \alpha_1 \ldots \alpha_n) m_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

where $\alpha_i \in M_k(\mathbb{C})$, $a_i \in A$, $m_0 \in M$, and $\mathrm{tr}\colon M_k(\mathbb{C}) \to \mathbb{C}$ is the standard trace of matrices.

As an exercise the reader should show that Tr is a chain map. Let $i\colon A \to M_k(A)$ be the map that sends a in A to the matrix with only one nonzero component in the upper left corner equal to a. There is a similar map $M \to M_k(M)$. These induce a map

$$I\colon C_n(A,\, M) \to C_n(M_k(A),\, M_k(M)),$$

$$I(m \otimes a_1 \otimes \cdots \otimes a_n) = i(m) \otimes i(a_1) \otimes \cdots \otimes i(a_n).$$

We have $\mathrm{Tr} \circ I = \mathrm{id}$, which is easily checked. It is however *not* true that $I \circ \mathrm{Tr} = \mathrm{id}$. There is instead a homotopy between $I \circ \mathrm{Tr}$ and id (cf. [124]). It follows that Tr and I induce inverse isomorphisms between homologies.

As a special case of Morita invariance, by choosing $M = A$, we obtain an isomorphism of Hochschild homology groups

$$HH_n(A) \simeq HH_n(M_k(A))$$

for all n and k.

Example 3.5.6 (Inner derivations and inner automorphisms)**.** We need to know, for example when defining the noncommutative Chern character later in this chapter, that *inner automorphisms* act by the identity on Hochschild homology and *inner derivations* act by zero. Let A be an algebra, $u \in A$ be an invertible element and let $a \in A$ be any element. They induce the chain maps $\Theta\colon C_n(A) \to C_n(A)$ and $L_a\colon C_n(A) \to C_n(A)$ defined by

$$\Theta(a_0 \otimes \cdots \otimes a_n) = u a_0 u^{-1} \otimes \cdots \otimes u a_n u^{-1},$$

and

$$L_a(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} a_0 \otimes \cdots \otimes [a, a_i] \otimes \cdots \otimes a_n.$$

Lemma 3.5.1. *Θ induces the identity map on Hochschild homology and L_a induces the zero map.*

Proof. The maps $h_i \colon A^{\otimes n+1} \to A^{\otimes n+2}$, $i = 0, \dots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0 u^{-1} \otimes u a_1 u^{-1} \otimes \cdots \otimes u \otimes a_{i+1} \otimes \cdots \otimes a_n)$$

define a homotopy

$$h = \sum_{i=0}^{n} (-1)^i h_i$$

between id and Θ.

For the second part one checks that the maps $h_i' \colon A^{\otimes n+1} \to A^{\otimes n+2}$, $i = 0, \dots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0 \otimes \cdots \otimes a_i \otimes a \otimes \cdots \otimes a_n),$$

define a homotopy

$$h' = \sum_{i=0}^{n} (-1)^i h_i'$$

between L_a and 0. □

Exercise 3.5.1. Let $A = S(V)$ be the symmetric algebra of a vector space V. Show that its module of Kähler differentials $\Omega^1_S(V)$ is isomorphic to $S(V) \otimes V$, the free left $S(V)$-module generated by V, where the universal differential is given by

$$d(v_1 v_2 \dots v_n) = \sum_{i=1}^{n} (v_1 \dots \hat{v}_i \dots v_n) \otimes v_i.$$

Exercise 3.5.2 (Additivity of HH_*)**.** Show that for unital algebras A and B, there is a natural isomorphism

$$HH_n(A \oplus B) \simeq HH_n(A) \oplus HH_n(B)$$

for all $n \geq 0$.

Exercise 3.5.3. Show that non-inner automorphisms need not act by the identity on HH_*.

Exercise 3.5.4. Use (3.15) to compute the Hochschild homology HH_* of the algebra $A = \mathbb{C}[z, z^{-1}]$ of Laurent polynomials. Notice that A is a smooth algebra and therefore one can use the Hochschild–Kostant–Rosenberg theorem (see (3.13)) as well. Compare the two computations. Extend to Laurent polynomials in several variables.

Exercise 3.5.5. Use (3.15) to compute the Hochschild homology HH_* of the group algebra of the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$. Do the same for the integral Heisenberg group of unipotent upper triangular matrices with integer coefficients.

Exercise 3.5.6. Compute the de Rham and Hochschild homologies of the algebra of dual numbers $\mathbb{C}[x]/(x^2)$ and show that the map (3.13) is not surjective (this is the simplest example of a non-smooth algebra).

3.6 Cyclic cohomology

Cyclic cohomology is defined in [36], [39] through a remarkable subcomplex of the Hochschild complex. We recall this definition in this section. Later in this chapter we give two other definitions. While these three definitions are equivalent to each other, as we shall see each has its own merits and strengths.

Let A be an algebra over the complex numbers and $(C^*(A), b)$ denote the Hochschild complex of A with coefficients in the A-bimodule A^*. We have, from Section 3.1,

$$C^n(A) = \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{C}), \quad n = 0, 1, \ldots,$$

and

$$\begin{aligned}(bf)(a_0, \ldots, a_{n+1}) = &\sum_{i=0}^{n} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_{n+1}a_0, \ldots, a_n)\end{aligned}$$

for all $f \in C^n(A)$.

The following definition is fundamental and marks our departure from Hochschild cohomology:

Definition 3.6.1. An n-cochain $f \in C^n(A)$ is called *cyclic* if

$$f(a_n, a_0, \ldots, a_{n-1}) = (-1)^n f(a_0, a_1, \ldots, a_n)$$

for all $a_0, \ldots, a_n$ in A. We denote the space of cyclic n-cochains on A by $C^n_\lambda(A)$.

Lemma 3.6.1. *The space of cyclic cochains is invariant under the action of* b*, i.e., for all* n *we have*

$$b\, C^n_\lambda(A) \subset C^{n+1}_\lambda(A).$$

Proof. Define the operators $\lambda \colon C^n(A) \to C^n(A)$ and $b' \colon C^n(A) \to C^{n+1}(A)$ by

$$(\lambda f)(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1}),$$

$$(b'f)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}).$$

One checks that

$$(1-\lambda)b = b'(1-\lambda).$$

Since

$$C^n_\lambda(A) = \mathrm{Ker}(1-\lambda),$$

the lemma is proved. □

We therefore have a subcomplex of the Hochschild complex, called the *cyclic complex* of A:

$$\boxed{C^0_\lambda(A) \xrightarrow{b} C^1_\lambda(A) \xrightarrow{b} C^2_\lambda(A) \xrightarrow{b} \cdots} \qquad (3.16)$$

Definition 3.6.2. The cohomology of the cyclic complex is called the *cyclic cohomology* of A and will be denoted by $HC^n(A)$, $n = 0, 1, 2, \ldots$.

A cocycle for the cyclic cohomology group $HC^n(A)$ is called a *cyclic n-cocycle* on A. It is an $(n+1)$-linear functional f on A which satisfies the two conditions

$$(1-\lambda)f = 0 \quad \text{and} \quad bf = 0.$$

The inclusion of complexes

$$(C_\lambda^*(A),\, b) \hookrightarrow (C^*(A),\, b)$$

induces a map I from the cyclic cohomology of A to the Hochschild cohomology of A with coefficients in the A-bimodule A^*:

$$I\colon HC^n(A) \to HH^n(A), \quad n = 0, 1, 2, \ldots.$$

We shall see that this map is part of a long exact sequence relating Hochschild and cyclic cohomology. For the moment we mention that I need not be injective or surjective (see example below).

Example 3.6.1. Let $A = \mathbb{C}$, the ground field. We have

$$C_\lambda^{2n}(\mathbb{C}) \simeq \mathbb{C}, \quad C_\lambda^{2n+1}(\mathbb{C}) = 0,$$

so the cyclic complex reduces to

$$0 \to \mathbb{C} \to 0 \to \mathbb{C} \to \cdots.$$

It follows that for all $n \geq 0$,

$$HC^{2n}(\mathbb{C}) = \mathbb{C}, \quad HC^{2n+1}(\mathbb{C}) = 0.$$

Since $HH^n(\mathbb{C}) = 0$ for $n \geq 1$, we conclude that the map I need not be injective and the cyclic complex is not a retraction of the Hochschild complex.

Example 3.6.2. It is clear that, for any algebra A, $HC^0(A) = HH^0(A)$ is the space of traces on A.

Example 3.6.3. Let $A = C^\infty(M)$ be the algebra of smooth complex valued functions on a closed smooth oriented manifold M of dimension n. We check that

$$\varphi(f^0, f^1, \ldots, f^n) := \int_M f^0\, df^1 \ldots df^n,$$

is a cyclic n-cocycle on A. We have already checked the cocycle property of φ, $b\varphi = 0$, in Example 3.1.2. The cyclic property of φ

$$\varphi(f^n, f^0, \ldots, f^{n-1}) = (-1)^n \varphi(f^0, \ldots, f^n)$$

is more interesting and is related to *Stokes' formula*. In fact since

$$\int_M (f^n df^0 \dots df^{n-1} - (-1)^n f^0 df^1 \dots df^n) = \int_M d(f^n f^0 df^1 \dots df^{n-1}),$$

we see that the cyclic property of φ follows from a special case of Stokes' formula:

$$\int_M d\omega = 0,$$

valid for any $(n-1)$-form ω on a closed n-manifold M.

The last example can be generalized in several directions. For example, let V be an m-dimensional closed singular chain (a cycle) on M, e.g. V can be a closed m-dimensional submanifold of M. Then integration on V defines an m-dimensional cyclic cocycle on A:

$$\varphi(f^0, f^1, \dots, f^m) = \int_V f^0\, df^1 \dots df^m.$$

We obtain a map

$$H_m(M, \mathbb{C}) \to HC^m(C^\infty(M)), \quad m = 0, 1, \dots,$$

from the singular homology of M (or its equivalents) to the cyclic cohomology of $C^\infty(M)$.

Let

$$\Omega_p M := \mathrm{Hom}_{\mathrm{cont}}(\Omega^p M, \mathbb{C})$$

denote the continuous dual of the space of p-forms on M. Elements of $\Omega_p M$ are *de Rham p-currents* on M as defined in Section 3.1. A p-current is called *closed* if for any $(p-1)$-form ω we have $\langle C, d\omega\rangle = 0$.

It is easy to check that for any m-current C, closed or not, the cochain

$$\varphi_C(f^0, f^1, \dots, f^m) := \langle C,\, f^0 df^1 \dots df^m\rangle,$$

is a Hochschild cocycle on $C^\infty(M)$. Now if C is closed, then one can easily check that φ_C is a cyclic m-cocycle on $C^\infty(M)$. We thus obtain natural maps

$$\Omega_m M \to HH^m(C^\infty(M)) \quad \text{and} \quad Z_m M \to HC^m(C^\infty(M)),$$

where $Z_m(M) \subset \Omega_m M$ is the space of closed m-currents on M.

A noncommutative generalization of this procedure involves the notion of a *cycle on an algebra* due to Connes [39] that we recall now. It gives a geometric and intuitively appealing presentation of cyclic cocycles. It also leads to a definition of cup product in cyclic cohomology and the S operator, as we shall indicate later.

Let $(\Omega,\, d)$ be a differential graded algebra. Thus

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \cdots$$

is a graded algebra and $d\colon \Omega^* \to \Omega^{*+1}$ is a square zero *graded derivation* in the sense that

$$d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1 d(\omega_2) \quad \text{and} \quad d^2 = 0$$

for all homogenous elements ω_1 and ω_2 of Ω.

Definition 3.6.3. A *closed graded trace* of dimension n on a differential graded algebra $(\Omega,\, d)$ is a linear map

$$\int : \Omega^n \to \mathbb{C}$$

such that

$$\int d\omega = 0 \quad \text{and} \quad \int (\omega_1\omega_2 - (-1)^{\deg(\omega_1)\deg(\omega_2)}\omega_2\omega_1) = 0$$

for all ω in Ω^{n-1}, ω_1 in Ω^i, ω_2 in Ω^j and $i + j = n$.

Definition 3.6.4. An *n-cycle* over an algebra A is a triple $(\Omega,\, \int,\, \rho)$ where $\int$ is an n-dimensional closed graded trace on $(\Omega,\, d)$ and $\rho\colon A \to \Omega_0$ is an algebra homomorphism.

Given an n-cycle $(\Omega,\, \int,\, \rho)$ over A its *character* is a cyclic n-cocycle on A defined by

$$\varphi(a^0, a^1, \ldots, a^n) = \int \rho(a^0) d\rho(a^1) \ldots d\rho(a^n). \tag{3.17}$$

Checking the cyclic cocycle conditions $b\varphi = 0$ and $(1 - \lambda)\varphi = 0$ is straightforward but instructive. To simplify the notation we drop the homomorphism ρ and write φ as

$$\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \ldots da^n.$$

We have, using the Leibniz rule for d and the graded trace property of $\int$,

$$\begin{aligned}
&(b\varphi)(a^0, \ldots, a^{n+1}) \\
&\quad = \sum_{i=0}^{n} (-1)^i \int a^0 da^1 \ldots d(a^i a^{i+1}) \ldots da^{n+1} \\
&\qquad\quad + (-1)^{n+1} \int a^{n+1} a^0 da^1 \ldots da^n \\
&\quad = (-1)^n \int a^0 da^1 \ldots da^n \cdot a^{n+1} + (-1)^{n+1} \int a^{n+1} a^0 da^1 \ldots da^n \\
&\quad = 0.
\end{aligned}$$

Notice that we did not need to use the 'closedness' of $\int$ so far. This will be needed however to check the cyclic property of φ:

$$\begin{aligned}(1-\lambda)\varphi(a^0,\ldots,a^n) &= \int a^0 da^1 \ldots da^n - (-1)^n \int a^n da^0 \ldots da^n \\ &= (-1)^{n-1} \int d(a^n a^0 da^1 \ldots da^{n-1}) \\ &= 0.\end{aligned}$$

Conversely, one can show that any cyclic cocycle on A is obtained from a cycle over A via (3.17). To this end, we introduce the algebra $(\Omega A,\, d)$,

$$\Omega A = \Omega^0 A \oplus \Omega^1 A \oplus \Omega^2 A \oplus \cdots ,$$

called the algebra of *noncommutative differential forms* on A as follows. ΩA is the universal (non-unital) differential graded algebra generated by A as a subalgebra. We put $\Omega^0 A = A$, and let $\Omega^n A$ be linearly generated over $\mathbb{C}$ by expressions $a_0 da_1 \ldots da_n$ and $da_1 \ldots da_n$ for $a_i \in A$ (cf. [39] for details). Notice that even if A is unital, ΩA is *not* a unital algebra and in particular the unit of A is only an idempotent in ΩA. The differential d is defined by

$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n \quad \text{and} \quad d(da_1 \ldots da_n) = 0.$$

The universal property of $(\Omega A,\, d)$ is the fact that for any (not necessarily unital) differential graded algebra $(\Omega,\, d)$ and any algebra map $\rho\colon A \to \Omega^0$ there is a unique extension of ρ to a morphism of differential graded algebras,

$$\hat{\rho}\colon \Omega A \to \Omega. \tag{3.18}$$

Now given a cyclic n-cocycle φ on A, define a linear map $\int_\varphi\colon \Omega^n A \to \mathbb{C}$ by

$$\int_\varphi (a_0 + \lambda 1)\, da_1 \ldots da_n = \varphi(a_0, \ldots, a_n).$$

It is easy to check that $\int_\varphi$ is a closed graded trace on ΩA whose character is φ.

Summarizing, we have shown that the relation

$$\int_\varphi (a_0 + \lambda 1)\, da_1 \ldots da_n = \varphi(a_0, a_1, \ldots, a_n)$$

defines a one-to-one correspondence:

$$\boxed{\{\text{cyclic } n\text{-cocycles on } A\} \simeq \{\text{closed graded traces on } \Omega^n A\}} \tag{3.19}$$

Notice that for $n = 0$ we recover the relation in Example 3.6.2 between cyclic 0-cocycles on A and traces on A.

Example 3.6.4 (A 2-cycle on the noncommutative torus)**.** Let δ_1, $\delta_2\colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ denote the canonical derivations of the noncommutative torus and $\tau\colon \mathcal{A}_\theta \to \mathbb{C}$ its canonical trace (cf. Example 1.1.7). It can be shown by a direct computation that the 2-cochain φ defined on $\mathcal{A}_\theta$ by

$$\varphi(a_0, a_1, a_2) = (2\pi i)^{-1}\tau\left(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\right)$$

is a cyclic 2-cocycle. It can also be realized as the character of the following 2-cycle $(\Omega, d, \int)$ on $\mathcal{A}_\theta$ as follows. Let $\Omega = \mathcal{A}_\theta \otimes \bigwedge^* \mathbb{C}^2$ be the tensor product of $\mathcal{A}_\theta$ with the exterior algebra of the vector space $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. The differential d is defined by

$$da = \delta_1(a)e_1 + \delta_2(a)e_2, \quad d(a \otimes e_1) = -\delta_2(a)e_1 \wedge e_2, \quad d(a \otimes e_2) = \delta_1(a)e_1 \wedge e_2.$$

The closed graded trace $\int\colon \Omega^2 \to \mathbb{C}$ is defined by

$$\int a \otimes e_1 \wedge e_2 = (2\pi i)^{-1}\tau(a).$$

The graded trace property of $\int$ is a consequence of the trace property of τ and its closedness follows from the invariance of τ under the infinitesimal automorphisms δ_1 and δ_2, that is, the property $\tau(\delta_i(a)) = 0$ for all a and $i = 1, 2$. Now it is clear that the character of this cycle is the cyclic 2-cocycle φ defined above:

$$\int a_0\, da_1 da_2 = \varphi(a_0, a_1, a_2).$$

In the remainder of this section we indicate a variety of different sources of cyclic cocycles, e.g. from group cocycles or Lie algebra cycles.

Example 3.6.5 (From group cocycles to cyclic cocycles)**.** Let G be a discrete group and $A = \mathbb{C}G$ be its group algebra. Let $c(g_1, \dots, g_n)$ be a group n-cocycle on G. Thus $c\colon G^n \to \mathbb{C}$ satisfies the cocycle condition

$$g_1 c(g_2, \dots, g_{n+1}) - c(g_1 g_2, \dots, g_{n+1}) + \dots + (-1)^{n+1} c(g_1, \dots, g_n) = 0$$

for all $g_1, \dots, g_{n+1}$ in G. Assume c is *normalized* in the sense that

$$c(g_1, \dots, g_n) = 0$$

if $g_i = e$ for some i, or if $g_1 g_2 \dots g_n = e$. (It can be shown that any cocycle is cohomologous to a normalized one). One checks that

$$\varphi_c(g_0, \dots, g_n) = \begin{cases} c(g_1, \dots, g_n), & \text{if } g_0 g_1 \dots g_n = e, \\ 0 & \text{otherwise,} \end{cases}$$

is a cyclic n-cocycle on the group algebra $\mathbb{C}G$ (cf. [38], [41], or exercises below). In this way one obtains a map from the group cohomology of G to the cyclic cohomology of $\mathbb{C}G$,

$$H^n(G, \mathbb{C}) \to HC^n(\mathbb{C}G), \quad c \mapsto \varphi_c.$$

By a theorem of Burghelea [25], the cyclic cohomology group $HC^n(\mathbb{C}G)$ decomposes over the conjugacy classes of G and the component corresponding to the conjugacy class of the identity contains the group cohomology $H^n(G,\mathbb{C})$ as a summand. (See Example 3.10.3 in this chapter.)

Example 3.6.6 (From Lie algebra homology to cyclic cohomology)**.** We start with a simple special case. Let A be an algebra, $\tau\colon A \to \mathbb{C}$ be a trace, and let $\delta\colon A \to A$ be a derivation on A. We assume that the trace is invariant under the action of the derivation in the sense that

$$\tau(\delta(a)) = 0$$

for all $a \in A$. Then one checks that

$$\varphi(a_0, a_1) := \tau(a_0\delta(a_1))$$

is a cyclic 1-cocycle on A. A simple commutative example of this is when $A = C^\infty(S^1)$, τ corresponds to the Haar measure, and $\delta = \frac{d}{dx}$. Then one obtains the fundamental class of the circle

$$\varphi(f_0, f_1) = \int f_0\, df_1.$$

See below for a noncommutative example, with $A = \mathcal{A}_\theta$, the smooth noncommutative torus.

This construction can be generalized. Let $\delta_1, \ldots, \delta_n$ be a *commuting* family of derivations on A, and let τ be a trace on A which is invariant under the action of the δ_i, $i = 1, \ldots, n$. Then one can check that

$$\varphi(a_0, \ldots, a_n) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\tau(a_0\delta_{\sigma(1)}(a_1)\ldots\delta_{\sigma(n)}(a_n)) \tag{3.20}$$

is a cyclic n-cocycle on A. Again we give a commutative example and postpone a noncommutative example to below. Let $A = C_c^\infty(\mathbb{R}^n)$ be the algebra of smooth compactly supported functions on $\mathbb{R}^n$. Let $\tau(f) = \int_{\mathbb{R}^n} f$ and $\delta_i = \frac{\partial}{\partial x_i}$. The corresponding cyclic cocycle is given, using the wedge product, by the formula

$$\varphi(f_0, \ldots, f_n) = \int_{\mathbb{R}^n} f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_n,$$

where $df = \sum_i \frac{\partial f}{\partial x_i}\, dx^i$.

Everything we did so far in this example lends itself to a grand generalization as follows. Let $\mathfrak{g}$ be a Lie algebra acting by derivations on an algebra A. This means that we have a Lie algebra map

$$\mathfrak{g} \to \operatorname{Der}(A, A)$$

from $\mathfrak{g}$ to the Lie algebra of derivations of A. Let $\tau\colon A \to \mathbb{C}$ be a trace which is invariant under the action of $\mathfrak{g}$, i.e.,

$$\tau(X(a)) = 0 \quad \text{for all } X \in \mathfrak{g},\ a \in A.$$

For each $n \geq 0$, define a linear map

$$\bigwedge^n \mathfrak{g} \to C^n(A), \quad c \mapsto \varphi_c,$$

where

$$\varphi_c(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \tau(a_0 X_{\sigma(1)}(a_1) \ldots X_{\sigma(n)}(a_n)) \tag{3.21}$$

if $c = X_1 \wedge \cdots \wedge X_n$ and extended linearly.

It can be shown that φ_c is a Hochschild cocycle for any c, and that it is a cyclic cocycle if c is a Lie algebra cycle. (See Exercise 3.6.5.)

We therefore obtain, for each $n \geq 0$, a map

$$\chi_\tau \colon H_n^{\text{Lie}}(\mathfrak{g}, \mathbb{C}) \to HC^n(A), \quad c \mapsto \varphi_c,$$

from the Lie algebra homology of $\mathfrak{g}$ with trivial coefficients to the cyclic cohomology of A [38].

In particular if $\mathfrak{g}$ is abelian then of course $H_n^{\text{Lie}}(\mathfrak{g}) = \bigwedge^n(\mathfrak{g})$ and we recover our previously defined map (3.20):

$$\bigwedge^n(\mathfrak{g}) \to HC^n(A), \quad n = 0, 1, \ldots.$$

Here is an example of this construction which first appeared in [35]. Let $A = \mathcal{A}_\theta$ denote the "algebra of smooth functions" on the noncommutative torus. Let $X_1 = (1,0), X_2 = (0,1)$. There is an action of the abelian Lie algebra $\mathbb{R}^2$ on $\mathcal{A}_\theta$ defined on generators of $\mathcal{A}_\theta$ by

$$X_1(U) = U, \quad X_1(V) = 0,$$
$$X_2(U) = 0, \quad X_2(V) = V.$$

The induced derivations on $\mathcal{A}_\theta$ are given by

$$X_1\Big(\sum a_{m,n} U^m V^n\Big) = \sum m a_{m,n} U^m V^n,$$
$$X_2\Big(\sum a_{m,n} U^m V^n\Big) = \sum n a_{m,n} U^m V^n.$$

It is easily checked that the trace τ on $\mathcal{A}_\theta$ defined by

$$\tau\Big(\sum a_{m,n} U^m V^n\Big) = a_{0,0}$$

is invariant under the above action of $\mathbb{R}^2$. The generators of $H_*^{\text{Lie}}(\mathbb{R}^2, \mathbb{C})$ are: 1, X_1, X_2, $X_1 \wedge X_2$.

We therefore obtain the following 0-dimensional, 1-dimensional and 2-dimensional cyclic cocycles on $\mathcal{A}_\theta$:

$$\varphi_0(a_0) = \tau(a_0), \quad \varphi_1(a_0, a_1) = \tau(a_0 X_1(a_1)), \quad \varphi_1'(a_0, a_1) = \tau(a_0 X_2(a_1)),$$

$$\varphi_2(a_0, a_1, a_2) = \tau(a_0(X_1(a_1)X_2(a_2) - X_2(a_1)X_1(a_2))).$$

It is shown in [39] that these classes form a basis for the continuous periodic cyclic cohomology of $\mathcal{A}_\theta$.

Example 3.6.7 (Cup product in cyclic cohomology)**.** As we indicated before, the notion of cycle over an algebra can be used to give a natural definition of a cup product for cyclic cohomology. By specializing one of the variables to the ground field, we obtain the S-operation.

Let $(\Omega, \int, \rho)$ be an m-dimensional cycle on an algebra A and $(\Omega', \int', \rho')$ an n-dimensional cycle on an algebra B. Let $\Omega \otimes \Omega'$ denote the (graded) tensor product of the differential graded algebras Ω and Ω'. By definition, we have

$$(\Omega \otimes \Omega')_k = \bigoplus_{i+j=k} \Omega_i \otimes \Omega'_j,$$

$$d(\omega \otimes \omega') = (d\omega) \otimes \omega' + (-1)^{\deg(\omega)} \omega \otimes (d\omega').$$

Let

$$\int'' \omega \otimes \omega' = \int \omega \int' \omega' \quad \text{if } \deg(\omega) = m,\ \deg(\omega') = n.$$

It is easily checked that $\int''$ is a closed graded trace of dimension $m+n$ on $\Omega \otimes \Omega'$.

Using the universal property (3.18) of noncommutative differential forms, applied to the map $\rho \otimes \rho' \colon A \otimes B \to \Omega_0 \otimes \Omega'_0$, one obtains a morphism of differential graded algebras

$$(\Omega(A \otimes B),\, d) \to (\Omega \otimes \Omega',\, d).$$

We therefore obtain a closed graded trace of dimension $m+n$ on $(\Omega(A \otimes B),\, d)$. In [39] it is shown that the resulting *cup product* map in cyclic cohomology,

$$\#\colon HC^m(A) \otimes HC^n(B) \to HC^{m+n}(A \otimes B)$$

is well defined.

We give a couple of simple examples of cup product computations.

Example 3.6.8 (The generalized trace map)**.** Let ψ be a trace on B. Then $\varphi \mapsto \varphi \# \psi$ defines a map

$$HC^m(A) \to HC^m(A \otimes B).$$

Explicitly we have

$$(\varphi \# \psi)(a^0 \otimes b^0, \ldots, a^m \otimes b^m) = \varphi(a^0, \ldots, a^m)\psi(b^0 b^1 \ldots b^m).$$

A special case of this construction plays a very important role in cyclic cohomology and noncommutative geometry. Let $\psi = \mathrm{tr} \colon M_n(\mathbb{C}) \to \mathbb{C}$ be the standard trace. Then 'cupping with trace' defines a map

$$HC^m(A) \to HC^m(M_k(A)).$$

Example 3.6.9 (The periodicity operator S)**.** Another important special case of the cup product is when we choose $B = \mathbb{C}$ and ψ to be the fundamental cyclic 2-cocycle on $\mathbb{C}$ defined by

$$\psi(1, 1, 1) = 1.$$

This leads to an operation of degree 2 on cyclic cohomology:

$$S\colon HC^n(A) \to HC^{n+2}(A), \quad \varphi \mapsto \varphi \# \psi.$$

The formula simplifies to

$$\begin{aligned}(S\varphi)(a^0, \dots, a^{n+2}) = &\int_\varphi a^0 a^1 a^2\, da^3 \dots da^{n+2} \\ &+ \int_\varphi a^0 da^1 (a^2 a^3)\, da^4 \dots da^{n+2} + \cdots \\ &+ \int_\varphi a^0\, da^1 \dots da^{i-1}(a^i a^{i+1}) da^{i+2} \dots da^{n+2} + \cdots \\ &+ \int_\varphi a^0\, da^1 \dots da^n (a^{n+1} a^{n+2}).\end{aligned}$$

In the next section we give a different approach to S via the cyclic bicomplex.

So far we have studied the cyclic cohomology of algebras. There is a 'dual' theory called *cyclic homology* which we introduce now. Let A be an algebra and for $n \geq 0$ let

$$C_n(A) = A^{\otimes(n+1)}.$$

For each $n \geq 0$, define the operators

$$b\colon C_n(A) \to C_{n-1}(A), \quad b' : C_n(A) \to C_{n-1}(A), \quad \lambda\colon C_n(A) \to C_n(A)$$

by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}), \\ b'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), \\ \lambda(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

The relation

$$(1 - \lambda) b' = b(1 - \lambda)$$

can be easily established. Clearly $(C_*(A), b)$ is the Hochschild complex of A with coefficients in the A-bimodule A. Let

$$C_n^\lambda(A) := C_n(A) / \operatorname{Im}(1 - \lambda).$$

The relation $(1-\lambda)b' = b(1-\lambda)$ shows that the operator b is well defined on the quotient complex $C_*^\lambda(A)$. The complex

$$(C_*^\lambda(A),\, b)$$

is called the *cyclic complex* of A and its homology, denoted by $HC_n(A)$, $n = 0, 1, \ldots,$ is called the *cyclic homology* of A.

Example 3.6.10. For $n = 0$,

$$HC_0(A) \simeq HH_0(A) \simeq A/[A, A]$$

is the *commutator quotient* of A. Here $[A, A]$ denotes the subspace of A generated by the commutators $ab - ba$, for a and b in A. In particular if A is commutative then $HC_0(A) = A$.

Exercise 3.6.1. Give a description of Hochschild cocycles on A in terms of linear functionals on ΩA similar to (3.19).

Exercise 3.6.2. Let $\varphi \in HC^0(A)$ be a trace on A. Show that

$$(S\varphi)(a_0, a_1, a_2) = \varphi(a_0 a_1 a_2).$$

Find an explicit formula for $S^n\varphi$ for all n. Let $\varphi \in HC^1(A)$. Express $S\varphi$ in terms of φ.

Exercise 3.6.3 (Area as a cyclic cocycle)**.** Let $f, g\colon S^1 \to \mathbb{R}$ be smooth functions. The map $u \mapsto (f(u), g(u))$ defines a smooth closed curve in the plane. Its signed area is given by $\int f\,dg$. Notice that $\varphi(f, g) = \int f\,dg$ is a cyclic 1-cocycle on $C^\infty(S^1)$.

Exercise 3.6.4. Let $c\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ be the map

$$c((a, b), (c, d)) = ad - bc.$$

Show that c is a normalized group 2-cocycle in the sense of Example 3.6.5. Show that the associated cyclic 2-cocycle on the group algebra $\mathbb{C}\mathbb{Z}^2$ extends to its smooth completion and coincides, up to scale, with the volume form on the two torus.

Exercise 3.6.5. Check that 1) for any c, the cochain φ_c defined in (3.21) is a Hochschild cocycle, i.e., $b\varphi_c = 0$; 2) if c is a Lie algebra cycle, i.e., if $\delta(c) = 0$, then φ_c is a cyclic cocycle.

3.7 Connes' long exact sequence

Our goal in this section is to establish the long exact sequence of Connes relating Hochschild and cyclic cohomology groups. There is a similar sequence relating Hochschild and cyclic homology. Connes' sequence is the long exact sequence of a short exact sequence and the main difficulty in the proof is to identify the cohomology of the quotient as cyclic cohomology, with a shift in dimension, and to identify the maps.

Let A be an algebra and let C_λ and C denote its cyclic and Hochschild cochain complexes, respectively. Consider the short exact sequence of complexes

$$0 \to C_\lambda \to C \xrightarrow{\pi} C/C_\lambda \to 0. \tag{3.22}$$

Its associated long exact sequence is

$$\cdots \to HC^n(A) \to HH^n(A) \to H^n(C/C_\lambda) \to HC^{n+1}(A) \to \cdots. \tag{3.23}$$

We need to identify the cohomology groups $H^n(C/C_\lambda)$. To this end, consider the short exact sequence

$$0 \to C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \to 0, \tag{3.24}$$

where the operator N is defined by

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \to C^n.$$

The relations

$$N(1-\lambda) = (1-\lambda)N = 0 \quad \text{and} \quad bN = Nb'$$

can be verified and they show that $1-\lambda$ and N are morphisms of complexes in (3.24). As for the exactness of (3.24), the only non-trivial part is to show that $\ker(N) \subset \operatorname{im}(1-\lambda)$. But this follows from the relation

$$(1-\lambda)(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n) = N - (n+1)\,\mathrm{id}_{C^n}.$$

Now, assuming A is unital, the middle complex (C, b') in (3.24) can be shown to be exact. In fact we have a contracting homotopy $s\colon C^n \to C^{n-1}$ defined by

$$(s\varphi)(a_0, \ldots, a_{n-1}) = (-1)^{n-1}\varphi(a_0, \ldots, a_{n-1}, 1),$$

which satisfies

$$b's + sb' = \mathrm{id}.$$

The long exact sequence associated to (3.24) looks like

$$\cdots \to H^n(C/C_\lambda) \to H^n_{b'}(C) \to HC^n(A) \to H^{n+1}(C/C_\lambda) \to H^{n+1}_{b'}(C) \to \cdots. \tag{3.25}$$

Since $H^n_{b'}(C) = 0$ for all n, it follows that the connecting homomorphism

$$\delta\colon HC^{n-1}(A) \xrightarrow{\ \sim\ } H^n(C/C_\lambda) \tag{3.26}$$

is an *isomorphism* for all $n \geq 0$. Using this in (3.23), we obtain *Connes' long exact sequence* relating Hochschild and cyclic cohomology:

$$\boxed{\cdots \to HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} \to HC^{n+1}(A) \to \cdots} \tag{3.27}$$

The operators B and S can be made more explicit by finding the connecting homomorphisms in the above long exact sequences. Notice that B is the composition of maps from (3.23) and (3.26):

$$B : HH^n(A) \xrightarrow{\pi} H^n(C/C_\lambda) \xrightarrow{\delta^{-1}} HC^{n-1}(A).$$

We have, on the level of cohomology, $B = (1-\lambda)^{-1} b' N^{-1}$. Remarkably this can be expressed, on the level of cochains, by *Connes' operator* B:

$$B = Ns(1-\lambda).$$

In fact, we have

$$\begin{aligned} Ns(1-\lambda)(1-\lambda)^{-1} b' N^{-1}[\varphi] &= Nsb'N^{-1}[\varphi] \\ &= N(1-b's)N^{-1}[\varphi] \\ &= (1-bNsN^{-1})[\varphi] \\ &= [\varphi]. \end{aligned}$$

We can also write B as

$$B = Ns(1-\lambda) = NB_0,$$

where $B_0\colon C^n \to C^{n-1}$ is defined by

$$B_0\varphi(a_0,\dots,a_{n-1}) = \varphi(1,a_0,\dots,a_{n-1}) - (-1)^n \varphi(a_0,\dots,a_{n-1},1).$$

Using the relations $(1-\lambda)b = b'(1-\lambda)$, $(1-\lambda)N = N(1-\lambda) = 0$, $bN = Nb'$, and $sb' + b's = 1$, it is easy to show that

$$bB + Bb = 0 \quad \text{and} \quad B^2 = 0.$$

Let

$$S'\colon HC^{n-1}(A) \to HC^{n+1}(A)$$

be the composition of connecting homomorphisms in (3.26) and (3.23) associated to the short exact sequences (3.22) and (3.24):

$$S'\colon HC^{n-1}(A) \xrightarrow{\sim} H^n(C/C_\lambda) \to HC^{n+1}(A)$$

Therefore we have

$$S'[\varphi] = [b(1-\lambda)^{-1} b' N^{-1}\varphi].$$

Any cochain $\psi \in (1-\lambda)^{-1} b' N^{-1}\varphi$ has the property that $b\psi$ is cyclic, as can be easily checked, and $B[\psi] = [\varphi]$. For the latter notice that

$$\begin{aligned} B(1-\lambda)^{-1} b' N^{-1}\varphi &= Ns(1-\lambda)(1-\lambda)^{-1} b' N^{-1}\varphi \\ &= N(1-b's)N^{-1}\varphi \\ &= \varphi - bNs\varphi. \end{aligned}$$

This gives us the formula

$$S'[\varphi] = [b\psi] = [bB^{-1}\varphi].$$

So far we have a long exact sequence

$$\cdots \to HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S'} HC^{n+1}(A) \to \cdots . \quad (3.28)$$

At this point an important remark is in order. The operator S' as defined above, coincides, *up to scale*, with the periodicity operator S defined in Example 3.6.9. In fact, using the explicit formulae for both S and S', one shows (cf. also [38], Lemma 4.34) that for any cyclic $(n-1)$-cocycle $[\varphi] \in HC^{n-1}(A)$,

$$S[\varphi] = n(n+1)S'[\varphi].$$

Thus in the exact sequence (3.28) we can replace S' with its scalar multiple S and this of course will give Connes' exact sequence (3.27). For future use we record the new formula for $S\colon HC^{n-1}(A) \to HC^{n+1}(A)$,

$$S[\varphi] = n(n+1)\, bB^{-1}[\varphi] = n(n+1)[b(1-\lambda)^{-1}b'N^{-1}\varphi]. \quad (3.29)$$

Using the periodicity operator S, the *periodic cyclic cohomology* of an algebra A is defined as the *direct limit* under the operator S of cyclic cohomology groups:

$$HP^i(A) := \varinjlim HC^{2n+i}(A), \quad i = 0, 1.$$

Notice that since S has degree 2 there are only two periodic groups.

Typical applications of Connes' *IBS* long exact sequence are to extract information on cyclic cohomology from Hochschild cohomology. We list some of them:

1) Let $f\colon A \to B$ be an algebra homomorphism and suppose that the induced maps on Hochschild groups

$$f^*\colon HH^n(B) \to HH^n(A)$$

are isomorphisms for all $n \geq 0$. Then

$$f^*\colon HC^n(B) \to HC^n(A)$$

is an isomorphism for all $n \geq 0$ as well. This simply follows by comparing the *IBS* sequences for A and B and applying the Five Lemma. For example, using Lemma 3.5.1, it follows that inner automorphisms act as identity on (periodic) cyclic cohomology.

Maps between cohomology groups need not be induced by algebra maps. For example if $f\colon (C^*(B), b) \to (C^*(A), b)$ is a morphism of Hochschild complexes and if f commutes with the cyclic operator λ, then it induces a map $(C^*_\lambda(B), b) \to (C^*_\lambda(A), b)$ between cyclic complexes. Using the *IBS* sequence, we conclude that if the induced maps between Hochschild cohomology groups are isomorphisms then the induced maps between cyclic groups are isomorphisms as well. For example using Lemma 3.5.1 we conclude that derivations act trivially on (periodic) cyclic cohomology groups.

2) (Morita invariance of cyclic cohomology) Let A and B be Morita equivalent unital algebras. The Morita invariance property of cyclic cohomology states that there is a natural isomorphism

$$HC^n(A) \simeq HC^n(B), \quad n = 0, 1, \ldots.$$

For a proof of this fact in general see [124]. In the special case where $B = M_k(A)$ a simple proof can be given as follows. Indeed, by Morita invariance of Hochschild cohomology, we know that the inclusion $i\colon A \to M_k(A)$ induces isomorphisms on Hochschild groups and therefore on cyclic groups by 1) above.

3) (Normalization) A cochain $f\colon A^{\otimes(n+1)} \to \mathbb{C}$ is called *normalized* if

$$f(a_0, a_1, \ldots, a_n) = 0$$

whenever $a_i = 1$ for some $i \geq 1$. It is clear that normalized cyclic cochains form a subcomplex $(C^*_{\lambda,\mathrm{norm}}(A), b)$ of the cyclic complex of A. Since the corresponding inclusion for Hochschild complexes is a quasi-isomorphism (Exercise 3.2.3), using the *IBS* sequence we conclude that the inclusion of cyclic complexes is a quasi-isomorphism as well.

Exercise 3.7.1. Show that (3.24) is exact (the interesting part is to show that $\mathrm{Ker}\, N \subset \mathrm{Im}(1-\lambda)$).

Exercise 3.7.2. Prove the relations $bB + Bb = 0$ and $B^2 = 0$. (They will be used later, together with $b^2 = 0$, to define the $(b,\, B)$-bicomplex).

Exercise 3.7.3. Let $A = \mathbb{C}$ be the ground field. Compute the operators $B\colon C^n(\mathbb{C}) \to C^{n+1}(\mathbb{C})$ and $S\colon HC^n(\mathbb{C}) \to HC^{n+2}(\mathbb{C})$. Conclude that $HP^{2n}(\mathbb{C}) = \mathbb{C}$ and $HP^{2n+1}(\mathbb{C}) = 0$.

Exercise 3.7.4. Let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers, or, equivalently, the exterior algebra of a vector space of dimension 1. Compute the cyclic and periodic cyclic groups of A. (Hint: Use the complex of normalized cochains.)

Exercise 3.7.5. Let $A = M_n(\mathbb{C})$. Show that the cochains $\varphi_{2n}\colon A^{\otimes(2n+1)} \to \mathbb{C}$ defined by

$$\varphi_{2n}(a_0, \ldots, a_{2n}) = \mathrm{Tr}(a_0 a_1 \ldots a_{2n})$$

are cyclic cocycles on A. We have $S[\varphi_{2n}] = \lambda_{2n}[\varphi_{2n+2}]$. Compute the constants λ_{2n}.

Exercise 3.7.6. Give examples of algebras whose Hochschild groups are isomorphic in all dimensions but whose cyclic groups are not isomorphic. In other words, an 'accidental' isomorphism of Hochschild groups does not imply cyclic cohomologies are isomorphic (despite the long exact sequence).

3.8 Connes' spectral sequence

The cyclic complex (3.16) and the long exact sequence (3.27), useful as they are, are not powerful enough for computations. A much deeper relation between Hochschild and cyclic cohomology groups is encoded in Connes' (b, B)-*bicomplex* and the associated spectral sequence that we shall briefly recall now, following closely the original paper [39].

Let A be a unital algebra. The (b, B)-bicomplex of A, denoted by $\mathcal{B}(A)$, is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
C^1(A) & \xrightarrow{B} & C^0(A) & & \\
\uparrow{\scriptstyle b} & & & & \\
C^0(A). & & & &
\end{array}$$

Of the three relations

$$b^2 = 0, \quad bB + Bb = 0, \quad B^2 = 0,$$

only the middle relation is not obvious. But this follows from the relations $b's + sb' = 1$, $(1 - \lambda)b = b'(1 - \lambda)$ and $Nb' = bN$, already used in the previous section.

The *total complex* of a bicomplex $(C^{*,*}, d^1, d^2)$ is defined as the complex $(\operatorname{Tot} C, d)$, where $(\operatorname{Tot} C)^n = \bigoplus_{p+q=n} C^{p,q}$ and $d = d^1 + d^2$. The following result is fundamental. It shows that the resulting *Connes' spectral sequence* obtained by filtration by rows which has Hochschild cohomology for its E^1 terms, converges to cyclic cohomology.

Theorem 3.8.1 ([39])**.** *The map* $\varphi \mapsto (0, \dots, 0, \varphi)$ *is a quasi-isomorphism of complexes*

$$(C^*_\lambda(A),\, b) \to (\operatorname{Tot} \mathcal{B}(A),\, b + B).$$

This is a consequence of the vanishing of the E^2 term of the second spectral sequence (filtration by columns) of $\mathcal{B}(A)$. To prove this consider the short exact sequence of b-complexes

$$0 \to \operatorname{Im} B \to \operatorname{Ker} B \to \operatorname{Ker} B / \operatorname{Im} B \to 0$$

By a hard lemma of Connes ([39], Lemma 41), the induced map

$$H_b(\operatorname{Im} B) \to H_b(\operatorname{Ker} B)$$

is an isomorphism. It follows that $H_b(\operatorname{Ker} B / \operatorname{Im} B)$ vanishes. To take care of the first column one appeals to the fact that

$$\operatorname{Im} B \simeq \operatorname{Ker}(1 - \lambda)$$

is the space of cyclic cochains.

We give an alternative proof of Theorem 3.8.1 above. To this end, consider the *cyclic bicomplex* $\mathcal{C}(A)$ defined by

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2(A) & \xrightarrow{N} & C^2(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1(A) & \xrightarrow{N} & C^1(A)(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0(A) & \xrightarrow{N} & C^0(A)(A) & \xrightarrow{1-\lambda} & \cdots .
\end{array}$$

The total cohomology of $\mathcal{C}(A)$ is isomorphic to cyclic cohomology:

$$H^n(\operatorname{Tot}\mathcal{C}(A)) \simeq HC^n(A), \quad n \geq 0.$$

This is a consequence of the fact that the rows of $\mathcal{C}(A)$ are exact except in degree zero. To see this, define the homotopy operator

$$H = \frac{1}{n+1}(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n) \colon C^n(A) \to C^n(A). \tag{3.30}$$

We have $(1-\lambda)H = \frac{1}{n+1}N - \mathrm{id}$, which of course implies the exactness of rows in positive degrees and for the first column we are left with the cyclic complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow & & \uparrow & & \\
C^1_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow & & \uparrow & & \\
C^0_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

We are therefore done with the proof of Theorem 3.8.1 provided we can prove that $\operatorname{Tot}\mathcal{B}(A)$ and $\operatorname{Tot}\mathcal{C}(A)$ are quasi-isomorphic. The next proposition proves this by an explicit formula:

Proposition 3.8.1. *The complexes* $\operatorname{Tot}\mathcal{B}(A)$ *and* $\operatorname{Tot}\mathcal{C}(A)$ *are homotopy equivalent.*

Proof. We define explicit chain maps between these complexes and show that they are chain homotopic via explicit homotopies. Define

$$\begin{aligned}
I &\colon \operatorname{Tot}\mathcal{B}(A) \to \operatorname{Tot}\mathcal{C}(A), \quad I = \mathrm{id} + Ns, \\
J &\colon \operatorname{Tot}\mathcal{C}(A) \to \operatorname{Tot}\mathcal{B}(A), \quad J = \mathrm{id} + sN.
\end{aligned}$$

One checks that I and J are chain maps.

Now consider the operators

$$g\colon \operatorname{Tot}\mathcal{B}(A) \to \operatorname{Tot}\mathcal{B}(A), \quad g = Ns^2 B_0,$$
$$h\colon \operatorname{Tot}\mathcal{C}(A) \to \operatorname{Tot}\mathcal{C}(A), \quad h = s,$$

where $B_0 = s(1-\lambda)$.

We have, by direct computation:

$$I \circ J = \mathrm{id} + h\delta + \delta h,$$
$$J \circ I = \mathrm{id} + g\delta' + \delta' g,$$

where δ (resp. δ') denotes the differential of $\operatorname{Tot}\mathcal{C}(A)$ (resp. $\operatorname{Tot}\mathcal{B}(A)$). □

There is a similar result for cyclic homology.

Exercise 3.8.1. Use the (b, B)-bicomplex definition of cyclic cohomology to obtain an alternative proof of the *IBS* sequence (3.27). Find an expression for the periodicity operator S in this picture of cyclic cohomology.

Exercise 3.8.2. Use Theorem 3.8.1 to show that periodic cyclic cohomology is isomorphic to the cohomology of the 2-periodic complex

$$\bigoplus_n C^{2n}(A) \rightleftarrows \bigoplus_n C^{2n+1}(A).$$

with differential $b + B$. Show that the cohomology of the complex

$$\prod_n C^{2n}(A) \rightleftarrows \prod_n C^{2n+1}(A)$$

is trivial.

3.9 Cyclic modules

In the previous sections we gave three alternative definitions for the cyclic cohomology of an algebra. Here we present yet another approach to cyclic cohomology due to Connes [38], which defines it as a kind of derived functor. In fact this comprises an extension of cyclic cohomology beyond the category of algebras by introducing the notion of a *cyclic object* in an abelian category and its cyclic cohomology [38]. Later developments proved that this extension was of great significance. Apart from earlier applications, we should mention the recent work [43] where the abelian category of cyclic modules plays the role of the category of motives in noncommutative geometry. Another recent example is the cyclic cohomology of Hopf algebras [55], [56], [89], [90], which cannot be defined as the cyclic cohomology of an algebra or a coalgebra but only as the cyclic cohomology of a cyclic module naturally attached to the given Hopf algebra.

The original motivation of [38] was to define cyclic cohomology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not even an additive category (for the simple reason that the sum of two algebra homomorphisms is not an algebra homomorphism in general), the standard (abelian) homological algebra is not applicable. Let k be a unital commutative ring. In [38], the category Λ_k of cyclic k-modules appears as an 'abelianization' of the category of k-algebras. Cyclic cohomology is then shown to be the derived functor of the *functor of traces*, as we explain in this section.

Recall that the *simplicial category* Δ is a small category whose objects are the totally ordered sets (cf. e.g. [81], [124], [179])

$$[n] = \{0 < 1 < \cdots < n\}, \quad n = 0, 1, 2, \ldots.$$

A morphism $f\colon [n] \to [m]$ of Δ is an order preserving, i.e., monotone non-decreasing, map $f\colon \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$. Of particular interest among the morphisms of Δ are *faces* δ_i and *degeneracies* σ_j,

$$\begin{aligned} \delta_i &\colon [n-1] \to [n], \quad i = 0, 1, \ldots, n, \\ \sigma_j &\colon [n+1] \to [n], \quad j = 0, 1, \ldots, n. \end{aligned}$$

By definition δ_i is the unique injective morphism missing i and σ_j is the unique surjective morphism identifying j with $j+1$. It can be checked that they satisfy the following *simplicial identities*:

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1} \quad \text{if } i < j, \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \quad \text{if } i \le j, \\ \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j, \\ \mathrm{id}_{[n]} & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1} \sigma_j & \text{if } i > j+1. \end{cases} \end{aligned}$$

Every morphism of Δ can be uniquely decomposed as a product of faces followed by a product of degeneracies.

The *cyclic category* Λ has the same set of objects as Δ and in fact contains Δ as a subcategory. Morphisms of Λ are generated by simplicial morphisms and new morphisms $\tau_n\colon [n] \to [n], n \ge 0$, defined by $\tau_n(i) = i + 1$ for $0 \le i < n$ and $\tau_n(n) = 0$. We have the following extra relations:

$$\begin{aligned} \tau_n \delta_i &= \delta_{i-1} \tau_{n-1}, \quad 1 \le i \le n, \\ \tau_n \delta_0 &= \delta_n, \\ \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1}, \quad 1 \le i \le n, \\ \tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2, \\ \tau_n^{n+1} &= \mathrm{id}\,. \end{aligned}$$

It can be shown that the classifying space $B\Lambda$ of the small category Λ is homotopy equivalent to the classifying space $BS^1 = \mathbb{C}P^\infty$ [38].

A *cyclic object* in a category $\mathcal{C}$ is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$. A *cocyclic object* in $\mathcal{C}$ is a functor $\Lambda \to \mathcal{C}$. For any commutative unital ring k, we denote the category of cyclic k-modules by Λ_k. A morphism of cyclic k-modules is a natural transformation between the corresponding functors. Equivalently, a morphism $f\colon X \to Y$ consists of a sequence of k-linear maps $f_n\colon X_n \to Y_n$ compatible with the face, degeneracy, and cyclic operators. It is clear that Λ_k is an abelian category. The kernel and cokernel of a morphism f are defined pointwise: $(\mathrm{Ker}\, f)_n = \mathrm{Ker}\, f_n\colon X_n \to Y_n$ and $(\mathrm{Coker}\, f)_n = \mathrm{Coker}\, f_n\colon X_n \to Y_n$. More generally, if $\mathcal{A}$ is any abelian category then the category $\Lambda\mathcal{A}$ of cyclic objects in $\mathcal{A}$ is itself an abelian category.

Let Alg_k denote the category of unital k-algebras and unital algebra homomorphisms. There is a functor

$$\natural\colon \mathrm{Alg}_k \to \Lambda_k$$

defined as follows. To an algebra A we associate the cyclic module $A^\natural$ defined by

$$A^\natural_n = A^{\otimes(n+1)}, \quad n \geq 0,$$

with face, degeneracy and cyclic operators given by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,\\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},\\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n,\\
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.
\end{aligned}$$

A unital algebra map $f\colon A \to B$ induces a morphism of cyclic modules $f^\natural\colon A^\natural \to B^\natural$ by $f^\natural(a_0 \otimes \cdots \otimes a_n) = f(a_0) \otimes \cdots \otimes f(a_n)$, and this defines the functor $\natural$.

Example 3.9.1. We have

$$\mathrm{Hom}_{\Lambda_k}(A^\natural, k^\natural) \simeq T(A),$$

where $T(A)$ is the space of traces from $A \to k$. To a trace τ we associate the cyclic map $(f_n)_{n\geq 0}$, where

$$f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1 \ldots a_n), \quad n \geq 0.$$

It can be easily shown that this defines a one-to-one correspondence.

Now we can state the following fundamental theorem of Connes [38] which greatly extends the above example:

Theorem 3.9.1. *Let k be a field of characteristic zero. For any unital k-algebra A there is a canonical isomorphism*

$$HC^n(A) \simeq \mathrm{Ext}^n_{\Lambda_k}(A^\natural, k^\natural) \quad \textit{for all } n \geq 0.$$

Before sketching its proof, we mention that combined with the above example the theorem implies that cyclic cohomology is, in some sense, the non-abelian derived functor of the functor of traces

$$A \rightsquigarrow T(A)$$

from the category of k-algebras to the category of k-modules.

Sketch of proof. The main step in the proof of Theorem 3.9.1 is to find an *injective resolution* of $k^\natural$ in Λ_k. The required injective cyclic modules will be the dual of some projective cyclic modules that we define first. For each integer $m \geq 0$, let us define a cyclic module $\boldsymbol{C}^m$ where

$$(\boldsymbol{C}^m)_n = k\,\mathrm{Hom}_\Lambda([m],[n])$$

is the free k-module generated by the set of all cyclic maps from $[m] \to [n]$. Composition in Λ defines a natural cyclic module structure on each $\boldsymbol{C}^k$. For any cyclic module M we clearly have $\mathrm{Hom}_{\Lambda_k}(\boldsymbol{C}^m, M) = M_m$. This of course implies that each $\boldsymbol{C}^m$ is a projective cyclic module. (Recall that an object P of an abelian category is called projective if the functor $M \mapsto \mathrm{Hom}(P, M)$ is *exact* in the sense that it sends any short exact sequence in the category into a short exact sequence of abelian groups.) The corresponding projective resolution of $k^\natural$ is defined as the total complex of the following double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\boldsymbol{C}^2 & \xrightarrow{1-\lambda} & \boldsymbol{C}^2 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^2 & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle \boldsymbol{b}} & & \uparrow{\scriptstyle \boldsymbol{b}'} & & \uparrow{\scriptstyle \boldsymbol{b}} & & \\
\boldsymbol{C}^1 & \xrightarrow{1-\lambda} & \boldsymbol{C}^1 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^1 & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle \boldsymbol{b}} & & \uparrow{\scriptstyle \boldsymbol{b}'} & & \uparrow{\scriptstyle \boldsymbol{b}} & & \\
\boldsymbol{C}^0 & \xrightarrow{1-\lambda} & \boldsymbol{C}^0 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^0 & \xrightarrow{1-\lambda} & \cdots,
\end{array} \tag{3.31}$$

where the cyclic module maps $\boldsymbol{b}, \boldsymbol{b'}, \boldsymbol{\lambda}$ and $\boldsymbol{N}$ are defined by

$$\boldsymbol{b}(f) = \sum_{i=0}^{k} (-1)^i f \circ \delta_i, \quad \boldsymbol{b'}(f) = \sum_{i=0}^{k-1} (-1)^i f \circ \delta_i,$$

$$\boldsymbol{\lambda}(f) = (-1)^k f \circ \tau_k, \quad \text{and} \quad \boldsymbol{N} = 1 + \lambda + \cdots + \lambda^k$$

Now by a direct argument one shows that the row homologies of the above bicomplex (3.31) are trivial in positive dimensions. Thus to compute its total homology it remains to compute the homology of the complex of complexes:

$$\boldsymbol{C^0}/(1-\lambda) \xleftarrow{\boldsymbol{b}} \boldsymbol{C^1}/(1-\lambda) \xleftarrow{\boldsymbol{b}} \cdots .$$

It can be shown that for each fixed m, the complex of vector spaces

$$\boldsymbol{C}^{\boldsymbol{0}}_m/(1-\lambda) \xleftarrow{\boldsymbol{b}} \boldsymbol{C}^{\boldsymbol{1}}_m/(1-\lambda) \xleftarrow{\boldsymbol{b}} \cdots$$

coincides with the complex that computes the simplicial homology of the simplicial set Δ^m. The simplicial set Δ^m is defined by $\Delta^m_n = \mathrm{Hom}_\Delta([m],[n])$ for all $n \geq 0$. The geometric realization of Δ^m is the closed unit ball in $\mathbb{R}^m$ which is of course contractible. It follows that the total homology of the above bicomplex is the cyclic module $k^\natural$. We note that for this argument K need not be a field. Now if k is a field of characteristic zero the cyclic modules $\boldsymbol{C}_m$, $m \geq 0$, defined by $(\boldsymbol{C}_m)_n = \mathrm{Hom}_k((\boldsymbol{C}^m)_n, k)$ are injective cyclic modules. Dualizing the bicomplex (3.31), finally we obtain an injective resolution of $k^\natural$ as a cyclic module. To compute the $\mathrm{Ext}^*_{\Lambda_k}(A^\natural, k^\natural)$ groups, we apply the functor $\mathrm{Hom}_{\Lambda_k}(A^\natural, -)$ to this resolution. We obtain the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2(A) & \xrightarrow{N} & C^2(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1(A) & \xrightarrow{N} & C^1 & \xrightarrow{1-\lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0(A) & \xrightarrow{N} & C^0(A) & \xrightarrow{1-\lambda} & \cdots .
\end{array} \tag{3.32}$$

We are done with the proof of Theorem 3.9.1, provided we can show that the cohomology of (3.32) is isomorphic to the cyclic cohomology of A. But this we have already shown in the last section. This finishes the proof of the theorem. □

A remarkable property of the cyclic category Λ, not shared by the simplicial category, is its *self-duality* in the sense that there is a natural isomorphism of categories $\Lambda \simeq \Lambda^{\mathrm{op}}$ [38]. Roughly speaking, the duality functor $\Lambda^{\mathrm{op}} \to \Lambda$ acts as identity on objects of Λ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying the duality isomorphism.

Example 3.9.2 (Hopf cyclic cohomology)**.** We give a very non-trivial example of a cocyclic module. Let H be a Hopf algebra. A character of H is a unital algebra map $\delta\colon H \to \mathbb{C}$. A group-like element is a nonzero element $\sigma \in H$ such that $\Delta\sigma = \sigma \otimes \sigma$. Following [55], [56], we say (δ, σ) is a *modular pair* if $\delta(\sigma) = 1$, and a *modular pair in involution* if

$$\tilde{S}^2_\delta(h) = \sigma h \sigma^{-1}$$

for all h in H. Here the *δ-twisted antipode* $\tilde{S}_\delta\colon H \to H$ is defined by

$$\tilde{S}_\delta(h) = \sum \delta(h^{(1)}) S(h^{(2)}).$$

Now let (H, δ, σ) be a Hopf algebra endowed with a modular pair in involution. In [55] Connes and Moscovici attach a cocyclic module $H^{\natural}_{(\delta,\sigma)}$ to this data as follows. Let

$$H^{\natural,0}_{(\delta,\sigma)} = \mathbb{C} \quad \text{and} \quad H^{\natural,n}_{(\delta,\sigma)} = H^{\otimes n} \qquad \text{for } n \geq 1.$$

Its face, degeneracy and cyclic operators δ_i, σ_i, and τ_n are defined by

$$\begin{aligned} \delta_0(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n, \\ \delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \quad \text{for } 1 \leq i \leq n, \end{aligned}$$

$$\begin{aligned}
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma, \\
\sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_i \varepsilon(h_{i+1}) \otimes \cdots \otimes h_n \quad \text{for } 0 \le i < n, \\
\tau_n(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1} \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).
\end{aligned}$$

Checking the cyclic property of τ_n, i.e., $\tau_n^{n+1} = 1$ is a highly non-trivial task. The cyclic cohomology of the cocyclic module $H^\natural_{(\delta,\sigma)}$ is the Hopf cyclic cohomology of the triple (H, δ, σ). (cf. also [2], [89], [90] for more examples of cyclic modules arising from actions and coactions of Hopf algebras on algebras and coalgebras.)

3.10 Examples: cyclic cohomology

Cyclic cohomology has been computed for many algebras, most notably algebras of smooth functions, group algebras and crossed product algebras, groupoid algebras, noncommutative tori, universal enveloping algebras, and almost commutative algebras. Equipped with these core examples, one can then use general results like additivity, Morita invariance, homotopy invariance, Künneth formulae, and excision [62], to compute the (periodic) cyclic cohomology of even larger classes of algebras. The main technique to deal with core examples is to find a suitable resolution for the Hochschild complex to compute the Hochschild cohomology first and then find the action of the operator B on the Hochschild complex. In good cases, the E^2-term of the spectral sequence associated with the (b, B)-bicomplex vanishes and one ends up with a computation of cyclic cohomology. To illustrate this idea, we recall some of these computations in this section.

Example 3.10.1 (Algebras of smooth functions)**.** Let $A = C^\infty(M)$ denote the algebra of smooth complex-valued functions on a closed smooth manifold M with its natural Fréchet algebra topology, and let $(\Omega M, d)$ denote the de Rham complex of M. Let $C_n(A) = A^{\hat{\otimes}(n+1)}$ denote the space of continuous n-chains on A. We saw in Example 3.5.2 that the map $\mu\colon C_n(A) \to \Omega^n M$ defined by

$$\mu(f_0 \otimes \cdots \otimes f_n) = \frac{1}{n!} f_0 df_1 \wedge \cdots \wedge df_n,$$

induces an isomorphism between the continuous Hochschild homology of A and differential forms on M:

$$HH_n^{\text{cont}}(A) \simeq \Omega^n M.$$

To compute the continuous cyclic homology of A, we first show that under the map μ the operator B corresponds to the de Rham differential d. More precisely, for each integer $n \ge 0$ we have a commutative diagram:

$$\begin{array}{ccc}
C_n(A) & \xrightarrow{\mu} & \Omega^n M \\
\downarrow B & & \downarrow d \\
C_{n+1}(A) & \xrightarrow{\mu} & \Omega^{n+1} M.
\end{array}$$

We have

$$\begin{aligned}
&\mu B(f_0 \otimes \cdots \otimes f_n) \\
&\quad = \mu \sum_{i=0}^{n} (-1)^{ni} (1 \otimes f_i \otimes \cdots \otimes f_{i-1} - (-1)^n f_i \otimes \cdots \otimes f_{i-1} \otimes 1) \\
&\quad = \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{ni} df_i \dots df_{i-1} \\
&\quad = \frac{1}{(n+1)!} (n+1) df_0 \dots df_n = d\mu(f_0 \otimes \cdots \otimes f_n).
\end{aligned}$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(A) \to \Omega(A),$$

where $\Omega(A)$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Omega^2 M & \xleftarrow[d]{} & \Omega^1 M & \xleftarrow[d]{} & \Omega^0 M \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^1 M & \xleftarrow[d]{} & \Omega^0 M & & \\
\downarrow{\scriptstyle 0} & & & & \\
\Omega^0 M. & & & &
\end{array}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have [39]:

$$HC_n^{\text{cont}}(C^\infty(M)) \simeq \Omega^n M / \operatorname{im} d \oplus H_{\text{dR}}^{n-2}(M) \oplus \cdots \oplus H_{\text{dR}}^{k}(M),$$

where $k = 0$ if n is even and $k = 1$ if n is odd. Notice that the top part, for $n \leq \dim(M)$, consists of the so called *co-closed* differential n-forms on M.

Using the corresponding periodic complexes, one concludes that the continuous periodic cyclic homology of $C^\infty(M)$ is given by

$$HP_k^{\text{cont}}(C^\infty(M)) \simeq \bigoplus_i H_{\text{dR}}^{2i+k}(M), \quad k = 0, 1.$$

There are of course dual results relating continuous cyclic cohomology of $C^\infty(M)$ and de Rham homology of M. Let $(\Omega_* M, d)$ denote the complex of de Rham currents on M. Recall from Example 3.5.2 that the map $\Omega_n M \to C^n_{\text{cont}}(A)$ defined by sending a current $C \in \Omega_n M$ to the cochain φ_C, where

$$\varphi_C(f^0, f^1, \dots, f^n) = \langle C, f^0 df^1 \wedge \cdots \wedge df^n \rangle$$

is a quasi-isomorphism. By basically following the same route as above we obtain the following theorem of Connes [39]:

$$\boxed{HC^n_{\text{cont}}(C^\infty(M)) \simeq Z^n(M) \oplus H^{\text{dR}}_{n-2}(M) \oplus \cdots \oplus H^{\text{dR}}_k(M)} \tag{3.33}$$

where $Z^n(M)$ is the space of closed de Rham n-currents on M and $k = 0$ if n is even and $k = 1$ if n is odd. Finally, for the continuous periodic cyclic cohomology we obtain:

$$\boxed{HP^k_{\text{cont}}(C^\infty(M)) \simeq \bigoplus_i H^{\text{dR}}_{2i+k}(M), \quad k = 0, 1} \tag{3.34}$$

Now (3.33) shows that cyclic cohomology is not *homotopy invariant*. In fact while the de Rham cohomology components are homotopy invariant the top component $Z^n(M)$ cannot be homotopy invariant. Formula (3.34) on the other hand shows that the periodic cyclic cohomology of $C^\infty(M)$ is homotopy invariant. This is a special case of the homotopy invariance of periodic cyclic cohomology [39], [124]. More precisely, for any algebras A and B and *smoothly homotopic* algebra maps $f_0, f_1 \colon A \to B$, we have $f_0^* = f_1^* \colon HP^*(B) \to HP^*(A)$. In particular the algebras A and $A[x]$ have isomorphic periodic cyclic cohomologies.

Example 3.10.2 (Smooth commutative algebras)**.** Let $A = \mathcal{O}(X)$ be the coordinate ring of an affine smooth variety over $\mathbb{C}$ and let (Ω^*_A, d) denote the de Rham complex of A. As we saw in Example 3.5.1, by Hochschild–Kostant–Rosenberg's theorem, the map

$$\mu \colon C_n(A) \to \Omega^n_A$$

induces an isomorphism between Hochschild homology of A and the differential forms on X. By the same method as in the previous example one then arrives at the isomorphisms

$$HC_n(\mathcal{O}(X)) \simeq \Omega^n_A / \operatorname{im} d \oplus H^{n-2}_{\text{dR}}(X) \oplus \cdots \oplus H^k_{\text{dR}}(X),$$

$$HP_k(\mathcal{O}(X)) \simeq \bigoplus_i H^{\text{dR}}_{2i+k}(X), \quad k = 0, 1.$$

Notice that the de Rham cohomology $H^n_{\text{dR}}(X)$ appearing on the right side is isomorphic to the singular cohomology $H^n(X_{top}, \mathbb{C})$ of the underlying topological space of X.

When X is singular the relations between the cyclic homology of $\mathcal{O}(X)$ and the topology of X_{top} can be quite complicated. The situation for periodic cyclic homology however is quite straightforward, as the following theorem of Feigin and Tsygan [79] indicates:

$$HP_k(\mathcal{O}(X)) \simeq \bigoplus_i H^{2i+k}(X_{\text{top}}, \mathbb{C}). \tag{3.35}$$

Notice that X need not be smooth and the cohomology on the right-hand side is the singular cohomology.

Example 3.10.3 (Group algebras)**.** Let $\mathbb{C}G$ denote the group algebra of a discrete group G. Here, to be concrete, we work over $\mathbb{C}$, but results hold over any field of characteristic zero. As we saw in Example 3.5.3, the Hochschild complex of $\mathbb{C}G$ decomposes over the set $\langle G\rangle$ of conjugacy classes of G and the homology of each summand is isomorphic to the group homology of a group associated to the conjugacy class:

$$HH_n(\mathbb{C}G) \simeq \bigoplus_{\langle G\rangle} H_n(C_g),$$

where C_g is the centralizer of a representative g of a conjugacy class of G [25], [41], [124]. Recall the decomposition

$$C_*(\mathbb{C}G, b) = \bigoplus_{c\in\langle G\rangle} B(G, c)$$

of the Hochschild complex of $\mathbb{C}G$ from Example 3.5.3, where for each conjugacy class $c \in \langle G\rangle$, $B_n(G, c)$ is the linear span of all $(n+1)$-tuples $(g_0, g_1, \ldots, g_n) \in G^{n+1}$ such that

$$g_0 g_1 \ldots g_n \in c.$$

It is clear that $B_n(G, c)$, $n = 0, 1, 2, \ldots$, are invariant not only under the Hochschild differential b, but also under the cyclic operator λ. Let

$$B_n^\lambda(G, c) = B_n(G, c)/\operatorname{im}(1-\lambda).$$

We then have a decomposition of the cyclic complex of $\mathbb{C}G$ into subcomplexes indexed by conjugacy classes:

$$C_*^\lambda(\mathbb{C}G, b) = \bigoplus_{c\in\hat{G}} B_*^\lambda(G, c).$$

The homology of $B_*^\lambda(G, \{e\})$ was first computed by Karoubi [104], [124] in terms of the group homology of G. The result is

$$H_n(B_*^\lambda(G, \{e\}) = \bigoplus_i H_{n-2i}(G).$$

Burghelea's computation of the cyclic homology of $\mathbb{C}G$ [25] (cf. also [132], [73] for a purely algebraic proof) can be described as follows. Let $\langle G\rangle^{\text{fin}}$ and $\langle G\rangle^\infty$ denote the set of conjugacy classes of elements of finite, and infinite orders, respectively. For an element $g \in G$, let $N_g = C_g/\langle g\rangle$, where $\langle g\rangle$ is the group generated by g and C_g is the centralizer of g. Notice that the isomorphism type of N_g only depends on the conjugacy class of g. In each conjugacy class c we pick a representative $g \in c$ once and for all. Now if g is an element of finite order we have

$$H_n(B_*^\lambda(G, c)) = \bigoplus_{i\geq 0} H_{n-2i}(C_g).$$

On the other hand, if g is of infinite order we have

$$H_n(B_*^\lambda(G, c)) = H_n(N_g).$$

Putting these results together we obtain:

$$\boxed{HC_n(\mathbb{C}G) \simeq \bigoplus_{\langle G\rangle^{\text{fin}}} \big(\bigoplus_{i\geq 0} H_{n-2i}(C_g)\big) \bigoplus_{\langle G\rangle^{\infty}} H_n(N_g)} \tag{3.36}$$

In particular, the Hochschild group has $H_n(G)$ as a direct summand, while the cyclic homology group has $\bigoplus_i H_{n-2i}(G)$ as a direct summand (corresponding to the conjugacy class of the identity of G).

Example 3.10.4 (Noncommutative torus)**.** We shall briefly recall Connes' computation of the Hochschild and cyclic cohomology groups of smooth noncommutative tori [39]. In Example 1.1.7 we showed that when θ is rational the smooth noncommutative torus $\mathcal{A}_\theta$ is Morita equivalent to $C^\infty(T^2)$, the algebra of smooth functions on the torus. One can then use the Morita invariance of Hochschild and cyclic cohomology to reduce the computation of these groups to those for the algebra $C^\infty(T^2)$. This takes care of the computation for rational θ. So, through the rest of this example we assume that θ is irrational and we denote the generators of $\mathcal{A}_\theta$ by U_1 and U_2 with the relation $U_2U_1 = \lambda U_1U_2$, where $\lambda = e^{2\pi i\theta}$.

Let $\mathcal{B} = \mathcal{A}_\theta \hat{\otimes} \mathcal{A}_\theta^{\text{op}}$. There is a topological free resolution of $\mathcal{A}_\theta$ as a left $\mathcal{B}$-module

$$\mathcal{A}_\theta \xleftarrow{\varepsilon} \mathcal{B}\otimes\Omega_0 \xleftarrow{b_1} \mathcal{B}\otimes\Omega_1 \xleftarrow{b_2} \mathcal{B}\otimes\Omega_2 \leftarrow 0,$$

where $\Omega_i = \bigwedge^i\mathbb{C}^2, i = 0,1,2$ is the i-th exterior power of $\mathbb{C}^2$. The differentials are given by

$$\begin{aligned} b_1(1\otimes e_j) &= 1\otimes U_j - U_j\otimes 1, \quad j=1,2,\\ b_2(1\otimes(e_1\wedge e_2)) &= (U_2\otimes 1 - \lambda\otimes U_2)\otimes e_1 - (\lambda U_1\otimes 1 - 1\otimes U_1)\otimes e_2,\\ \varepsilon(a\otimes b) &= ab. \end{aligned}$$

The following result completely settles the question of continuous Hochschild cohomology of A_θ when θ is irrational. Recall that an irrational number θ is said to satisfy a Diophantine condition if $|1-\lambda^n|^{-1} = O(n^k)$ for some positive integer k.

Proposition 3.10.1 ([39])**.** *Let $\theta \notin \mathbb{Q}$. Then the following holds.*

a) One has $HH^0(\mathcal{A}_\theta) = \mathbb{C}$.

b) If θ satisfies a Diophantine condition then $HH^i(\mathcal{A}_\theta)$ is 2-dimensional for $i = 1$ and is 1-dimensional for $i = 2$.

c) If θ does not satisfy a Diophantine condition, then $HH^i(\mathcal{A}_\theta)$ are infinite dimensional non-Hausdorff spaces for $i = 1, 2$.

Remarkably, for all values of θ, the periodic cyclic cohomology is finite dimensional and is given by

$$HP^0(\mathcal{A}_\theta) = \mathbb{C}^2, \quad HP^1(\mathcal{A}_\theta) = \mathbb{C}^2.$$

An explicit basis for these groups is given by cyclic 1-cocycles

$$\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1)) \quad \text{and} \quad \varphi_1(a_0, a_1) = \tau(a_0\delta_2(a_1)),$$

and by cyclic 2-cocycles

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \quad \text{and} \quad S\tau,$$

where $\delta_1, \delta_2 \colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ are the canonical derivations defined by

$$\delta_1\Big(\sum a_{mn}U_1^m U_2^n\Big) = \sum m a_{mn} U_1^m U_2^n \quad \text{and} \quad \delta_2(U_1^m U_2^n) = \sum n a_{mn} U_1^m U_2^n,$$

and $\tau \colon \mathcal{A}_\theta \to \mathbb{C}$ is the canonical trace (cf. Example 1.1.7). Note that $S\tau(a_0, a_1, a_2) = \tau(a_0 a_1 a_2)$.

Let $\mathcal{O}(T_\theta^2)$ denote the (dense) subalgebra of $\mathcal{A}_\theta$ generated by U_1 and U_2. In Exercise 3.2.7 we ask the reader to show that the (algebraic) Hochschild groups of $\mathcal{O}(T_\theta^2)$ are finite dimensional for *all* values of θ.

Exercise 3.10.1. Since $\mathbb{C}\mathbb{Z} = \mathbb{C}[z, z^{-1}]$ is both a group algebra and a smooth algebra, we have two descriptions of its Hochschild and cyclic homologies. Compare the two descriptions and show that they are the same.

Exercise 3.10.2. Let G be a finite group. Use Burghelea's theorem in Example 3.10.3 to compute the Hochschild and cyclic homology of $\mathbb{C}G$. Alternatively, one knows that $\mathbb{C}G$ is a direct sum of matrix algebras and one can use the Morita invariance of Hochschild and cyclic theory. Compare the two approaches.

Exercise 3.10.3. Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ be the infinite dihedral group. Use (3.36) to compute the cyclic homology of the group algebra $\mathbb{C}D_\infty$.

Exercise 3.10.4. Let $X = \{(x, y) \in \mathbb{C}^2;\ xy = 0\}$, and let $A = \mathcal{O}(X)$ be the coordinate ring of X. Verify (3.35) for A.

Exercise 3.10.5. Prove directly, without using Proposition 3.10.1, that when θ is irrational $\mathcal{A}_\theta$ has a unique trace and therefore $HH^0(\mathcal{A}_\theta) = \mathbb{C}$. Describe the traces on $\mathcal{A}_\theta$ for rational θ.

Chapter 4

Connes–Chern character

The classical commutative Chern character relates the K-theory of a space to its ordinary cohomology. In noncommutative geometry, in addition to K-theory there is also a very important dual *K-homology* theory built out of abstract elliptic operators on the noncommutative space. In this chapter we study the noncommutative analogues of Chern character maps for both K-theory and K-homology, with values in cyclic homology and cyclic cohomology, respectively. As was indicated in the introduction to Chapter 3, it was the search for a noncommutative analogue of the Chern character that eventually led to the discovery of cyclic cohomology. K-theory, K-homology, cyclic homology and cohomology, via their allied Connes–Chern character maps, enter into a beautiful index formula of Connes which plays an important role in applications of noncommutative geometry.

4.1 Connes–Chern character in K-theory

The *classical Chern character* is a natural transformation from K-theory to ordinary cohomology theory with rational coefficients [138]. More precisely, for each compact Hausdorff space X we have a natural homomorphism

$$\mathrm{Ch}\colon K^0(X) \to \bigoplus_{i\geq 0} H^{2i}(X, \mathbb{Q}),$$

where K^0 (resp. H) denotes the K-theory (resp. Čech cohomology with rational coefficients). It satisfies certain axioms and these axioms completely characterize Ch. We shall not recall these axioms here, however, since they are not very useful for finding the noncommutative analogue of Ch. What turned out to be most useful in this regard was the *Chern–Weil* definition of the Chern character for smooth manifolds,

$$\mathrm{Ch}\colon K^0(X) \to \bigoplus_{i\geq 0} H^{2i}_{\mathrm{dR}}(X),$$

using differential geometric notions of connection and curvature on vector bundles over smooth manifolds. (cf. [138], and Example 4.1.4 in this section). Now let us describe the situation in the noncommutative case.

In [35], [39], [41], Connes shows that Chern–Weil theory admits a vast generalization to a noncommutative setting. For example, for a not necessarily commutative algebra A and each integer $n \geq 0$ there are natural maps, called *Connes–Chern character* maps,

$$\mathrm{Ch}_0^{2n} \colon K_0(A) \to HC_{2n}(A),$$
$$\mathrm{Ch}_1^{2n+1} \colon K_1(A) \to HC_{2n+1}(A)$$

from the K-theory of A to its cyclic homology.

Alternatively, these maps can be defined as a *pairing* between cyclic cohomology and K-theory:

$$HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}, \quad HC^{2n+1}(A) \otimes K_1(A) \to \mathbb{C}. \tag{4.1}$$

These pairings are compatible with the periodicity operator S in cyclic cohomology in the sense that

$$\langle [\varphi],\, [e] \rangle = \langle S[\varphi], [e] \rangle$$

for all cyclic cocycles φ and K-theory classes $[e]$, and thus induce a pairing

$$HP^i(A) \otimes K_i(A) \to \mathbb{C}, \quad i = 0, 1,$$

between periodic cyclic cohomology and K-theory.

We start by briefly recalling the definitions of the functors K_0 and K_1. Let A be a unital algebra and let $\mathcal{P}(A)$ denote the set of isomorphism classes of finitely generated projective right A-modules. Under the operation of direct sum, $\mathcal{P}(A)$ is an abelian monoid. The group $K_0(A)$ is, by definition, the *Grothendieck group* of the monoid $\mathcal{P}(A)$ in the sense that there is a universal additive map $\mathcal{P}(A) \to K_0(A)$. Thus elements of $K_0(A)$ can be written as $[P]-[Q]$ for $P,\, Q \in \mathcal{P}(A)$, with $[P]-[Q] = [P'] - [Q']$ if and only if there is an $R \in \mathcal{P}(A)$ such that $P \oplus Q' \oplus R \simeq P' \oplus Q \oplus R$.

There is an alternative description of $K_0(A)$ in terms of idempotents in matrix algebras over A that is often convenient. An idempotent $e \in M_n(A)$ defines a right A-module map

$$e \colon A^n \to A^n$$

by left multiplication by e. Let $P_e = eA^n$ be the image of e. The relation

$$A^n = eA^n \oplus (1-e)A^n$$

shows that P_e is a finite projective right A-module. Different idempotents may define isomorphic modules. This happens, for example, if e and f are *equivalent idempotents* (sometimes called *similar*) in the sense that

$$e = ufu^{-1}$$

for some invertible $u \in \mathrm{GL}(n, A)$. Let $M(A) = \bigcup M_n(A)$ be the direct limit of the matrix algebras $M_n(A)$ under the embeddings $M_n(A) \to M_{n+1}(A)$ defined by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Similarly let $\mathrm{GL}(A)$ be the direct limit of the groups $\mathrm{GL}(n, A)$. It acts on $M(A)$ by conjugation.

Definition 4.1.1. Two idempotents $e \in M_k(A)$ and $f \in M_l(A)$ are called *stably equivalent* if their images in $M(A)$ are equivalent under the action of $\mathrm{GL}(A)$.

The following is easy to prove and answers our original question.

Lemma 4.1.1. *The projective modules P_e and P_f are isomorphic if and only if the idempotents e and f are stably equivalent.*

Let $\mathrm{Idem}(M(A))/\,\mathrm{GL}(A)$ denote the set of stable equivalence classes of idempotents over A. This is an abelian monoid under the operation

$$(e,\, f) \mapsto e \oplus f := \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

It is clear that any finite projective module is of the type P_e for some idempotent e. In fact writing $P \oplus Q \simeq A^n$, one can let e be the idempotent corresponding to the projection map $(p, q) \mapsto (p, 0)$. These observations prove the following lemma.

Lemma 4.1.2. *For any unital ring A, the map $e \mapsto P_e$ defines an isomorphism of monoids*

$$\mathrm{Idem}(M(A))/\,\mathrm{GL}(A) \simeq \mathcal{P}(A).$$

Given an idempotent $e = (e_{ij}) \in M_n(A)$, its image under a homomorphism $f\colon A \to B$ is the idempotent $f_*(e) = (f(e_{ij}))$. This is our formula for $f_*\colon K_0(A) \to K_0(B)$ in the idempotent picture of K-theory. It turns $A \to K_0(A)$ into a functor from unital algebras to abelian groups.

For a unital Banach algebra A, $K_0(A)$ can be described in terms of connected components of the space of idempotents of $M(A)$ under its inductive limit topology (a subset $V \subset M(A)$ is open in the inductive limit topology if and only if $V \cap M_n(A)$ is open for all n). It is based on the following important observation: Let e and f be idempotents in a unital Banach algebra A and assume $\|e - f\| < 1/\|2e - 1\|$. Then $e \sim f$. In fact with

$$v = (2e - 1)(2f - 1) + 1 \tag{4.2}$$

and $u = \frac{1}{2}v$, we have $ueu^{-1} = f$. To see that u is invertible note that $\|u - 1\| < 1$. One consequence of this fact is that if e and f are in the same path component of the space of idempotents in A, then they are equivalent. As a result we have, for any Banach algebra A, an isomorphism of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}(M(A))),$$

where π_0 is the functor of path components.

For C^*-algebras, instead of idempotents it suffices to consider only the *projections*. A projection is a self-adjoint idempotent ($p^2 = p = p^*$). The reason is that every idempotent in a C^*-algebra is similar to a projection [14]: let e be an idempotent and set $z = 1 + (e - e^*)(e^* - e)$. Then z is invertible and positive and one shows that $p = ee^*z^{-1}$ is a projection and is similar to e. In fact, it can be shown that the set of projections of a C^*-algebra is a retraction of its set of idempotents. Let $\mathrm{Proj}(M(A))$ denote the space of projections in $M(A)$. We have established isomorphisms of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}(M(A))) \simeq \pi_0(\mathrm{Proj}(M(A)))$$

which reflects the coincidence of stable equivalence, Murray–von Neumann equivalence, and homotopy equivalence in Proj $(M(A))$.

Starting with K_1, algebraic and topological K-theory begin to differ from each other. We shall briefly indicate the definition of algebraic K_1, and the necessary modification needed for topological K^1. Let A be a unital algebra. The algebraic K_1 of A is defined as the *abelianization of* the group $\mathrm{GL}(A)$:

$$K_1^{\mathrm{alg}}(A) := \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)],$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator subgroup. Applied to $A = C(X)$, this definition does not reproduce the topological $K^1(X)$. For example for $A = \mathbb{C} = C(\mathrm{pt})$ we have $K_1^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}^\times$ where the isomorphism is induced by the determinant map

$$\det\colon \mathrm{GL}(\mathbb{C}) \to \mathbb{C}^\times,$$

while $K^1(\mathrm{pt}) = 0$. It turns out that, to obtain the right result, one should divide $\mathrm{GL}(A)$ by a bigger subgroup, i.e., by the *closure* of its commutator subgroup. This works for all Banach algebras and will give the right definition of topological K_1. A better approach however is to define the higher K groups in terms of K_0 and the *suspension functor* [14].

After this quick and brief introduction to K-theory, we come now to the main topic of this section. We start by defining the pairings (4.1). Let φ be a *cyclic* $2n$*-cocycle* on an algebra A. For each integer $k \geq 1$, the formula

$$\tilde{\varphi}(m_0 \otimes a_0, \ldots, m_{2n} \otimes a_{2n}) = \mathrm{tr}(m_0 \ldots m_{2n})\varphi(a_0, \ldots, a_{2n}) \tag{4.3}$$

defines a cyclic $2n$-cocycle $\tilde{\varphi} \in Z_\lambda^{2n}(M_k(A))$. Let $e \in M_k(A)$ be an idempotent representing a class in $K_0(A)$. Define a bilinear map

$$HC^{2n}(A) \otimes K_0(A) \to \mathbb{C} \tag{4.4}$$

by the following formula:

$$\boxed{\langle[\varphi], [e]\rangle = (n!)^{-1}\,\tilde{\varphi}(e, \ldots, e)} \tag{4.5}$$

Let us first check that the value of the pairing depends only on the cyclic cohomology class of φ in $HC^{2n}(A)$. It suffices to assume $k = 1$ (why?). Let $\varphi = b\psi$ with $\psi \in C_\lambda^{2n-1}(A)$, be a coboundary. Then we have

$$\begin{aligned}\varphi(e, \ldots, e) &= b\psi(e, \ldots, e) \\ &= \psi(ee, e, \ldots, e) - \psi(e, ee, \ldots, e) + \cdots + (-1)^{2n}\psi(ee, e, \ldots, e) \\ &= \psi(e, \ldots, e) \\ &= 0,\end{aligned}$$

where the last relation follows from the cyclic property of ψ.

To verify that the value of $\langle[\varphi], [e]\rangle$, for fixed φ, only depends on the class of $[e] \in K_0(A)$ we have to check that for $u \in \mathrm{GL}_k(A)$ an invertible matrix, we have $\langle[\varphi], [e]\rangle = \langle[\varphi], [ueu^{-1}]\rangle$. It again suffices to show this for $k = 1$. But this is exactly the fact, proved in Section 3.7, that inner automorphisms act by the identity on cyclic cohomology. Formula (4.5) can be easily seen to be additive in $[e]$ under the direct sum $e \oplus f$ of idempotents. This shows that the pairing (4.4) is well defined.

Proposition 4.1.1. *For any cyclic cocycle $\varphi \in Z_\lambda^{2n}(A)$ and idempotent $e \in M_k(A)$ we have*

$$\langle[\varphi], [e]\rangle = \langle S[\varphi], [e]\rangle.$$

Proof. Without loss of generality we can assume that $k = 1$. Using our explicit formula (3.29) for the S-operator, we have

$$S[\varphi] = (2n+1)(2n+2)[b(1-\lambda)^{-1}b'N^{-1}\varphi],$$

where $N^{-1}\varphi = \frac{1}{2n+1}\varphi$ (since φ is cyclic), and

$$(1-\lambda)^{-1}b'\varphi = \frac{-1}{2n+2}(1 + 2\lambda + 3\lambda^2 + \cdots + (2n+2)\lambda^{2n+1})b'\varphi.$$

Thus we have

$$\begin{aligned}S\varphi(e, \ldots, e) &= -b(1 + 2\lambda + 3\lambda^2 + \cdots + (2n+2)\lambda^{2n+1})b'\varphi(e, \ldots, e) \\ &= (n+1)b'\varphi(e, \ldots, e) \\ &= (n+1)\varphi(e, \ldots, e).\end{aligned}$$

Now we have

$$\langle S[\varphi], [e]\rangle = \frac{1}{(n+1)!}(S\varphi)(e, \ldots, e) = \frac{1}{n!}\varphi(e, \ldots, e) = \langle[\varphi], [e]\rangle. \qquad \square$$

Example 4.1.1 ($n = 0$)**.** $HC^0(A)$ is the space of traces on A. Therefore the Connes–Chern pairing for $n = 0$ reduces to a map

$$\{\text{traces on } A\} \times K_0(A) \to \mathbb{C},$$

$$\langle \tau, [e] \rangle = \sum_{i=1}^{k} \tau(e_{ii}),$$

where $e = [e_{ij}] \in M_k(A)$ is an idempotent. The induced function on $K_0(A)$ is called the *dimension function* and denoted by $\dim_\tau$. This terminology is suggested by the commutative case. In fact if X is a compact connected topological space, then $\tau(f) = f(x_0)$, $x_0 \in X$, defines a trace on $C(X)$, and for a vector bundle E on X, $\dim_\tau(E)$ is the rank of the vector bundle E and is an integer. One of the striking features of noncommutative geometry is the existence of noncommutative vector bundles with non-integral dimensions. A beautiful example of this phenomenon is shown is Example 1.2.3 through the Powers–Rieffel projection $e \in \mathcal{A}_\theta$ with $\tau(e) = \theta$, where τ is the canonical trace on the noncommutative torus (cf. also [41]).

Here is a slightly different approach to this dimension function. Let E be a finite projective right A-module. A trace τ on A induces a trace on the endomorphism algebra of E,

$$\mathrm{Tr}\colon \mathrm{End}_A(E) \to \mathbb{C}$$

as follows. First assume that $E = A^n$ is a free module. Then $\mathrm{End}_A(E) \simeq M_n(A)$ and our trace map is defined by

$$\mathrm{Tr}(a_{i,j}) = \sum_i a_{ii}.$$

It is easy to check that the above map is a trace. In general, there is an A-module F such that $E \oplus F \simeq A^n$ is a free module and $\mathrm{End}_A(E)$ embeds in $M_n(A)$. One can check that the induced trace on $\mathrm{End}_A(E)$ is independent of the choice of splitting. Now, from our description of Tr in terms of τ, it is clear that

$$\langle \tau, [E] \rangle = \dim_\tau(E) = \mathrm{Tr}(\mathrm{id}_E)$$

for any finite projective A-module E.

The topological information hidden in an idempotent is much more subtle than just its 'rank', as two idempotents, say vector bundles, can have the same rank but still be non-isomorphic. In fact traces can only capture the 0-dimensional information. To know more about idempotents and K-theory we need the higher dimensional analogues of traces, which are cyclic cocycles, and the pairing (4.5).

As we saw in Section 3.8 cyclic cocycles can also be realized in the (b, B)-bicomplex picture of cyclic cohomology. Given an even cocycle

$$\varphi = (\varphi_0, \varphi_2, \ldots, \varphi_{2n})$$

in the (b, B)-bicomplex, its pairing with an idempotent $e \in M_k(\mathcal{A})$ can be shown to be given by

$$\boxed{\langle [\varphi], [e] \rangle = \sum_{k=1}^{n} (-1)^k \frac{k!}{(2k)!} \varphi_{2k}\left(e - \frac{1}{2}, e, \ldots, e\right)} \tag{4.6}$$

(cf. Exercise 4.1.2).

When A is a Banach (or at least a suitable topological) algebra, to verify that the pairing (4.5) is well defined, it suffices to check that for a smooth family of idempotents e_t, $0 \leq t \leq 1$, $\varphi(e_t, \ldots, e_t)$ is constant in t. There is an alternative "infinitesimal proof" of this fact which is worth recording [41]:

Lemma 4.1.3. *Let e_t, $0 \leq t \leq 1$, be a smooth family of idempotents in a Banach algebra A. There exists a smooth family x_t, $0 \leq t \leq 1$, of elements of A such that*

$$\dot{e_t} := \frac{d}{dt}(e_t) = [x_t,\, e_t] \quad \textit{for } 0 \leq t \leq 1.$$

Proof. Let

$$x_t = [\dot{e_t},\, e_t] = \dot{e_t}e_t - e_t\dot{e_t}.$$

Differentiating the idempotent condition $e_t^2 = e_t$ with respect to t we obtain

$$\frac{d}{dt}(e_t^2) = \dot{e_t}e_t + e_t\dot{e_t} = \dot{e_t}.$$

Multiplying this last relation on the left by e_t yields

$$e_t\dot{e_t}e_t = 0.$$

Now we have

$$[x_t,\, e_t] = [\dot{e_t}e_t - e_t\dot{e_t},\, e_t] = \dot{e_t}e_t + e_t\dot{e_t} = \dot{e_t}. \qquad \square$$

It follows that if $\tau\colon A \to \mathbb{C}$ is a trace (= a cyclic zero cocycle), then

$$\frac{d}{dt}\langle \tau,\, e_t\rangle = \frac{d}{dt}\tau(e_t) = \tau(\dot{e_t}) = \tau([x_t,\, e_t]) = 0.$$

Hence the value of the pairing, for a fixed τ, depends only on the homotopy class of the idempotent. This shows that the pairing

$$\{\text{traces on } A\} \times K_0(A) \to \mathbb{C}$$

is well defined.

This is generalized in

Lemma 4.1.4. *Let $\varphi(a_0, \ldots, a_{2n})$ be a cyclic $2n$-cocycle on A and let e_t be a smooth family of idempotents in A. Then the number*

$$\langle [\varphi],\, [e_t]\rangle = \varphi(e_t, \ldots, e_t)$$

is constant in t.

Proof. Differentiating with respect to t and using the above lemma, we obtain

$$\begin{aligned}\frac{d}{dt}\varphi(e_t,\ldots,e_t) &= \varphi(\dot{e_t},\ldots,e_t)+\varphi(e_t,\dot{e_t},\ldots,e_t)+\cdots\\ &\qquad\cdots+\varphi(e_t,\ldots,e_t,\dot{e_t})\\ &=\sum_{i=0}^{2n}\varphi(e_t,\ldots,[x_t,e_t],\ldots,e_t)\\ &= L_{x_t}\varphi(e_t,\ldots,e_t).\end{aligned}$$

We saw in Section 3.7 that inner derivations act trivially on Hochschild and cyclic cohomology. This means that for each t there is a cyclic $(2n-1)$-cochain ψ_t such that the *Lie derivative* $L_{x_t}\varphi = b\psi_t$. We then have

$$\frac{d}{dt}\varphi(e_t,\ldots,e_t) = (b\psi_t)(e_t,\ldots,e_t)=0. \qquad \square$$

The formulas in the *odd case* are as follows. Given an invertible matrix $u \in M_k(A)$, representing a class in $K_1^{\mathrm{alg}}(A)$, and an odd cyclic $(2n-1)$-cocycle φ on A, we define

$$\langle[\varphi],\,[u]\rangle := \frac{2^{-(2n+1)}}{(n-\frac{1}{2})\ldots\frac{1}{2}}\,\tilde{\varphi}(u^{-1}-1,u-1,\ldots,u^{-1}-1,u-1), \tag{4.7}$$

where the cyclic cocycle $\tilde{\varphi}$ is defined in (4.3). As we saw in Section 3.7, any cyclic cocycle can be represented by a *normalized* cocycle for which $\varphi(a_0,\ldots,a_1)=0$ if $a_i = 1$ for some i. When φ is normalized, formula (4.7) reduces to

$$\boxed{\langle[\varphi],\,[u]\rangle = \frac{2^{-(2n+1)}}{(n-\frac{1}{2})\ldots\frac{1}{2}}\,\tilde{\varphi}(u^{-1},u,\ldots,u^{-1},u)} \tag{4.8}$$

As in the even case, the induced pairing $HC^{2n+1}(A)\otimes K_1^{\mathrm{alg}}(A)\to\mathbb{C}$ is compatible with the periodicity operator: for any odd cyclic cocycle $\varphi \in Z_\lambda^{2n+1}(A)$ and an invertible $u\in \mathrm{GL}_k(A)$, we have

$$\langle[\varphi],\,[u]\rangle = \langle S[\varphi],[u]\rangle.$$

These pairings are just manifestations of perhaps more fundamental maps that define the even and odd *Connes–Chern characters*

$$\mathrm{Ch}_0^{2n}\colon K_0(A)\to HC_{2n}(A),$$

$$\mathrm{Ch}_1^{2n+1}\colon K_1(A)\to HC_{2n+1}(A),$$

as we describe them now. In the even case, given an idempotent $e = (e_{ij}) \in M_k(A)$, we define for each $n \geq 0$,

$$\begin{aligned} \mathrm{Ch}_0^{2n}(e) &= (n!)^{-1}\,\mathrm{Tr}(\underbrace{e \otimes e \otimes \cdots \otimes e}_{2n+1}) \\ &= \sum_{i_0, i_1, \ldots, i_{2n}} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2n} i_0}, \end{aligned} \tag{4.9}$$

where on the right-hand side the class of the tensor in $A^{\otimes(2n+1)}/\,\mathrm{Im}(1-\lambda)$ is understood. In low dimensions we have

$$\begin{aligned} \mathrm{Ch}_0^0(e) &= \sum_{i=1}^{k} e_{ii}, \\ \mathrm{Ch}_0^2(e) &= \sum_{i_0=1}^{k} \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes e_{i_2 i_0}, \end{aligned}$$

etc. To check that $\mathrm{Ch}_0^{2n}(e)$ is actually a cycle, notice that

$$b(\mathrm{Ch}_0^{2n}(e)) = \frac{1}{2}(1-\lambda)\,\mathrm{Tr}(\underbrace{e \otimes \cdots \otimes e}_{2n}),$$

which shows its class is zero in the quotient.

In the odd case, given an invertible matrix $u \in M_k(A)$, we define

$$\mathrm{Ch}_1^{2n+1}([u]) = \mathrm{Tr}(\underbrace{(u^{-1}-1) \otimes (u-1) \otimes \cdots \otimes (u^{-1}-1) \otimes (u-1)}_{2n+2}).$$

Example 4.1.2. Let $A = C^\infty(S^1)$ denote the algebra of smooth complex-valued functions on the circle. One knows that $K_1(A) \simeq K^1(S^1) \simeq \mathbb{Z}$ and $u(z) = z$ is a generator of this group. Let

$$\varphi(f_0, f_1) = \int_{S^1} f_0\, df_1$$

denote the cyclic cocycle on A representing the fundamental class of S^1 in de Rham homology. Notice that this is a normalized cocycle since $\varphi(1, f) = \varphi(f, 1) = 0$ for all $f \in A$. We have

$$\langle [\varphi], [u] \rangle = \varphi(u, u^{-1}) = \int_{S^1} u\, du^{-1} = -2\pi i.$$

Alternatively, the Connes–Chern character

$$\mathrm{Ch}_1^1([u]) = u \otimes u^{-1} \in HC_1(A) \simeq H^1_{\mathrm{dR}}(S^1)$$

is the class of the differential form $\omega = z^{-1} dz$, representing the fundamental class of S^1 in de Rham cohomology.

Example 4.1.3. Let $A = C^\infty(S^2)$ and let $e \in M_2(A)$ denote the idempotent representing the Hopf line bundle on S^2:

$$e = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1+ix_2 \\ x_1-ix_2 & 1-x_3 \end{pmatrix}.$$

Let us check that under the map

$$HC_2(A) \to \Omega^2 S^2, \quad a_0 \otimes a_1 \otimes a_2 \mapsto a_0 da_1 da_2,$$

the Connes–Chern character of e corresponds to the fundamental class of S^2. We have

$$\begin{aligned} \mathrm{Ch}_0^2(e) &= \mathrm{Tr}(e \otimes e \otimes e) \mapsto \mathrm{Tr}(edede) \\ &= \frac{1}{8}\,\mathrm{Tr}\begin{pmatrix} 1+x_3 & x_1+ix_2 \\ x_1-ix_2 & 1-x_3 \end{pmatrix}\begin{pmatrix} dx_3 & dx_1+idx_2 \\ dx_1-idx_2 & -dx_3 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} dx_3 & dx_1+idx_2 \\ dx_1-idx_2 & -dx_3 \end{pmatrix}. \end{aligned}$$

Performing the computation one obtains

$$\mathrm{Ch}_0^2(e) \mapsto \frac{-i}{2}(x_1dx_2dx_3 - x_2dx_1dx_3 + x_3dx_1dx_2).$$

One can then integrate this 2-form on the two-sphere S^2. The result is $-2\pi i$. Notice that for the unit of the algebra $1 \in A$, representing the trivial rank one line bundle on S^2, we have $\mathrm{Ch}_0^0(1) = 1$ and $\mathrm{Ch}_0^{2n}(1) = 0$ for all $n > 0$. Thus e and 1 represent different K-theory classes in $K_0(A)$. A fact which cannot be proved using just $\mathrm{Ch}_0^0(e) = \mathrm{Tr}(e) = 1$.

Example 4.1.4. For smooth *commutative* algebras, the noncommutative Chern character reduces to the classical Chern character. We verify this only in the even case. The verification hinges on two things: the Chern–Weil approach to characteristic classes via connections and curvatures, and the general fact, valid even in the noncommutative case, that an idempotent $e \in M_n(A)$ is more than just a (noncommutative) vector bundle as it carries with it a god-given connection:

idempotent = noncommutative vector bundle + connection

Let X be a smooth closed manifold, $A = C^\infty(X)$, and let $\Omega^\bullet X$ denote the de Rham complex of X. The alternative definition of the classical Chern character Ch, called the *Chern–Weil theory*, uses the differential geometric notions of connection and curvature on vector bundles as we briefly recall now [138]. Let E be a complex vector bundle on X and let ∇ be a *connection* on E. Thus by definition,

$$\nabla \colon C^\infty(E) \to C^\infty(E) \otimes_A \Omega^1 X$$

is a $\mathbb{C}$-linear map satisfying the Leibniz rule

$$\nabla(f\xi) = f\nabla(\xi) + \xi \otimes df$$

for all smooth sections ξ of E and smooth functions f on X. Let

$$\hat{\nabla}\colon C^\infty(E) \otimes_A \Omega^\bullet X \to C^\infty(E) \otimes_A \Omega^{\bullet+1} X$$

denote the natural extension of ∇ to E-valued differential forms. It is uniquely defined by virtue of the graded Leibniz rule

$$\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi} \xi d\omega$$

for all $\xi \in C^\infty(E) \otimes_A \Omega^\bullet X$ and $\omega \in \Omega^\bullet X$. The *curvature* of ∇ is the operator

$$\hat{\nabla}^2 \in \mathrm{End}_{\Omega^\bullet X}(C^\infty(E) \otimes_A \Omega^\bullet X) = C^\infty(\mathrm{End}(E)) \otimes \Omega^\bullet X,$$

which can be easily checked to be ΩX-linear. Thus it is completely determined by its restriction to $C^\infty(E)$. This gives us the *curvature form* of ∇ as a *'matrix-valued 2-form'*

$$R \in C^\infty(\mathrm{End}(E)) \otimes \Omega^2 X.$$

Let

$$\mathrm{Tr}\colon C^\infty(\mathrm{End}(E)) \otimes_A \Omega^{\mathrm{ev}} X \to \Omega^{\mathrm{ev}} X$$

denote the canonical trace. The Chern character of E is then defined to be the class of the non-homogeneous even form

$$\mathrm{Ch}(E) = \mathrm{Tr}(e^R).$$

(We have omitted the normalization factor of $\frac{1}{2\pi i}$ to be multiplied by R.) One shows that $\mathrm{Ch}(E)$ is a closed form and that its cohomology class is independent of the choice of connection.

Now let $e \in M_n(C^\infty(X))$ be an idempotent representing the smooth vector bundle E on X. Smooth sections of E are in one-to-one correspondence with smooth map $\xi\colon X \to \mathbb{C}^n$ such that $e\xi = \xi$. One can check that the following formula defines a connection on E, called the *Levi-Civita* or *Grassmannian* connection:

$$\nabla(\xi) = ed\xi \in C^\infty(E) \otimes_A \Omega^1 X.$$

Computing the curvature form, we obtain

$$R(\xi) = \hat{\nabla}^2(\xi) = ed(ed\xi) = eded\xi.$$

Differentiating the relation $\xi = e\xi$, we obtain $d\xi = (de)\xi + ed\xi$. Also, by differentiating the relation $e^2 = e$, we obtain $ede \cdot e = 0$. If we use these two relations in the above formula for R, we obtain

$$R(\xi) = edede\xi,$$

and hence the following formula for the matrix valued curvature 2-form R:

$$R = edede.$$

Using $ede \cdot e = 0$, we can easily compute powers of R. They are given by

$$R^n = (edede)^n = e\underbrace{dede \dots dede}_{2n}.$$

The classical Chern–Weil formula for the Chern character $\mathrm{Ch}(E)$ is

$$\mathrm{Ch}(E) = \mathrm{Tr}(e^R) = \mathrm{Tr}\Big(\sum_{n\geq 0} \frac{R^n}{n!}\Big) \in \Omega^{\mathrm{even}}(X),$$

so that its n-th component is given by

$$\mathrm{Tr}\,\frac{R^n}{n!} = \frac{1}{n!}\,\mathrm{Tr}((edede)^n) = \frac{1}{n!}\,\mathrm{Tr}(ede \dots de) \in \Omega^{2n}X.$$

The Connes–Chern character of e defined in (4.9) is

$$\mathrm{Ch}_0^{2n}(e) := (n!)^{-1}\,\mathrm{Tr}(e \otimes \dots \otimes e).$$

We see that under the canonical map

$$HC_{2n}(A) \to H_{\mathrm{dR}}^{2n}(M), \quad a_0 \otimes \dots \otimes a_{2n} \mapsto a_0 da_1 \dots da_{2n},$$

$\mathrm{Ch}_0^{2n}(e)$ is mapped to the component of $\mathrm{Ch}(E)$ of degree $2n$.

Example 4.1.5 (Noncommutative Chern–Weil theory)**.** It may happen that an element of $K_0(A)$ is represented by a finite projective module, rather than by an explicit idempotent. It is then important to have a formalism that would give the value of its pairing with cyclic cocycles. This is in fact possible and is based on a noncommutative version of Chern–Weil theory developed by Connes in [35], [39] that we sketch next.

Let A be an algebra. By a *noncommutative differential calculus* on A we mean a triple (Ω, d, ρ) such that (Ω, d) is a differential graded algebra and $\rho\colon A \to \Omega^0$ is an algebra homomorphism. Thus

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^1 \oplus \Omega^2 \oplus \cdots$$

is a graded algebra, and we assume that the differential $d\colon \Omega^i \to \Omega^{i+1}$ increases the degree, and d is a graded derivation in the sense that

$$d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1 d(\omega_2) \quad \text{and} \quad d^2 = 0.$$

Given a differential calculus on A and a right A-module $\mathcal{E}$, a *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1$$

satisfying the Leibniz rule

$$\nabla(\xi a) = \nabla(\xi)a + \xi \otimes da$$

for all $\xi \in \mathcal{E}$ and $a \in A$.

Let

$$\hat{\nabla} \colon \mathcal{E} \otimes_{\mathcal{A}} \Omega^{\bullet} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^{\bullet+1}$$

be the (necessarily unique) extension of ∇ which satisfies the graded Leibniz rule

$$\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi}\xi d\omega$$

with respect to the right Ω-module structure on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$. It is defined by

$$\hat{\nabla}(\xi \otimes \omega) = \nabla(\xi)\omega + (-1)^{\deg \omega}\xi \otimes d\omega.$$

The *curvature* of ∇ is the operator $\hat{\nabla}^2 \colon E \otimes_A \Omega^{\bullet} \to E \otimes_A \Omega^{\bullet}$, which can be easily checked to be Ω-linear:

$$\hat{\nabla}^2 \in \operatorname{End}_{\Omega}(E \otimes_A \Omega) = \operatorname{End}_A(E) \otimes \Omega.$$

Let $\int \colon \Omega^{2n} \to \mathbb{C}$ be a closed graded trace representing a cyclic $2n$-cocycle φ on A (cf. Definition 3.6.4). Now since E is finite projective over A it follows that $E \otimes_A \Omega$ is finite projective over Ω and therefore the trace $\int \colon \Omega \to \mathbb{C}$ extends to a trace, denoted again by $\int$, on $\operatorname{End}_A(E) \otimes \Omega$. The following result of Connes relates the value of the pairing as defined above to its value computed through the Chern–Weil formalism:

$$\boxed{\langle [\varphi], [\mathcal{E}] \rangle = \frac{1}{n!} \int \hat{\nabla}^{2n}}$$

The next example is a concrete illustration of this method.

Example 4.1.6. Let $\mathcal{E} = \mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on the real line. The operators

$$(\xi \cdot U)(x) = \xi(x + \theta), \quad (\xi \cdot V)(x) = e^{2\pi i x}\xi(x)$$

turn $\mathcal{S}(\mathbb{R})$ into a right $\mathcal{A}_\theta$-module for all $\xi \in \mathcal{S}(\mathbb{R})$. It is the simplest of a series of modules $\mathcal{E}_{p,q}$ on the noncommutative torus, defined by Connes in [35]. It turns out that $\mathcal{E}$ is finite projective, and for the canonical trace τ on $\mathcal{A}_\theta$ we have

$$\langle \tau, \mathcal{E} \rangle = -\theta.$$

In Example 3.6.4 we defined a differential calculus, in fact a 2-cycle, on $\mathcal{A}_\theta$. It is easy to see that a connection on $\nabla \colon \mathcal{E} \to E \otimes_A \Omega^1$ with respect to this calculus is simply given by a pair of operators $\nabla_1, \nabla_2 \colon \mathcal{E} \to \mathcal{E}$ ('covariant derivatives' with respect to noncommutative vector fields δ_1 and δ_2) satisfying

$$\nabla_j(\xi a) = (\nabla_j \xi)a + \xi\delta_j(a), \quad j = 0, 1,$$

for all $\xi \in \mathcal{E}$ and $a \in \mathcal{A}_\theta$.

One can now check that the following formulae define a connection on $\mathcal{E}$ [35], [41]:

$$\nabla_1(\xi)(s) = -\frac{s}{\theta}\,\xi(s), \quad \nabla_2(\xi)(s) = \frac{d\xi}{ds}(s).$$

The curvature of this connection is constant and is given by

$$\nabla^2 = [\nabla_1,\, \nabla_2] = \frac{1}{\theta}\,\mathrm{id} \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}).$$

Exercise 4.1.1. Let E be a finite projective right A-module. Show that $\mathrm{End}_A(E) \simeq E \otimes_A E^*$ where $A^* = \mathrm{Hom}_A(E, A)$. The canonical pairing $E \otimes_A E^* \to A$ defined by $\xi \otimes f \mapsto f(\xi)$ induces a map $\mathrm{End}_A(E) \to A/[A, A] = HC_0(A)$. In particular if $\tau\colon A \to \mathbb{C}$ is a trace on A, the induced trace on $\mathrm{End}_A(E)$ is simply obtained by composing τ with the canonical pairing between E and E^*.

Exercise 4.1.2. Verify that under the natural quasi-isomorphism between (b, B) and cyclic complexes, formula (4.6) corresponds to (4.5).

Exercise 4.1.3. Show that a right A-module E admits a connection with respect to the universal differential calculus $(\Omega A,\, d)$, if and only if E is projective.

4.2 Connes–Chern character in K-homology

By K-homology for spaces we mean the theory which is dual to topological K-theory. While such a theory can be constructed using general techniques of algebraic topology, a beautiful and novel idea of Atiyah [6] (in the even case), and Brown–Douglas–Fillmore [22] (in the odd case) was to use techniques of index theory, functional analysis and operator algebras to define a K-homology theory (cf. also [9]). What is even more interesting is that the resulting theory can be extended to noncommutative algebras and pairs with K-theory. This extension, in full generality, then paved the way for Kasparov's bivariant KK-theory which unifies both K-theory and K-homology into a single bivariant theory (cf. [106] and [14]). Unfortunately the name K-homology is used even when one is dealing with algebras, despite the fact that the resulting functor is in fact contravariant for algebras, while K-theory for algebras is covariant. We hope this will cause no confusion for the reader.

To motivate the discussions, we start this section by recalling the notion of an *abstract elliptic operator* over a compact space [6]. This will then be extended to the noncommutative setting by introducing the notion of an, even or odd, Fredholm module over an algebra [39]. The Connes–Chern character of a Fredholm module is introduced next. We shall then define the index pairing between K-theory and K-homology, which indicates the sense in which these theories are dual to each other. The final result of this section is an index formula of Connes which computes the index pairing in terms of Connes–Chern characters for K-theory and K-homology.

Let X be a compact Hausdorff space. The even cycles for Atiyah's theory in [6] are *abstract elliptic operators* (H, F) over $C(X)$. This means that $H = H^+ \oplus H^-$ is a $\mathbb{Z}_2$-graded Hilbert space, $\pi\colon C(X) \to \mathcal{L}(H)$ with

$$\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix}$$

is an *even* representation of $C(X)$ in the algebra of bounded operators on H, and $F\colon H \to H$ with

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \tag{4.10}$$

is an *odd* bounded operator with $F^2 - I \in \mathcal{K}(H)$ being compact. This data must satisfy the crucial condition

$$[F, \pi(a)] \in \mathcal{K}(H)$$

for all $a \in C(X)$. We shall make no attempt at turning these cycles into a homology theory. Suffice it to say that the homology theory is defined as the quotient of the set of these cycles by a homotopy equivalence relation (cf. [96], [63] for a recent account).

When X is a smooth closed manifold, the main examples of abstract elliptic operators in the above sense are given by elliptic pseudodifferential operators of order 0, $D\colon C^\infty(E^+) \to C^\infty(E^-)$ acting between sections of vector bundles E^+ and E^- on X. Let $P\colon H^+ \to H^-$ denote the natural extension of D to a bounded operator where $H^+ = L^2(E^+)$ and $H^- = L^2(E^-)$, and let $Q\colon H^- \to H^+$ denote a parametrix of P. Define F by (4.10). Then with $C(X)$ acting as multiplication operators on H^+ and H^-, basic elliptic theory shows that (H, F) is an elliptic operator in the above sense on $C(X)$.

If $e \in M_n(C(X))$ is an idempotent representing a vector bundle on X, then the formula

$$\langle (H, F)\,, [e] \rangle := \operatorname{index} F_e^+$$

with the Fredholm operator $F_e^+ := eFe\colon eH^+ \to eH^-$ can be shown to define a pairing between the K-theory of X and abstract elliptic operators on X. This is the duality between K-homology and K-theory.

A modification of the above notion of abstract elliptic operator, which makes sense over noncommutative algebras, both in the even and odd case, is the following notion of Fredholm module in [39] which is an important variation of a related notion from [6], [22], [107] (cf. the remark below).

Definition 4.2.1. An *odd Fredholm module* over an algebra A is a pair (H, F) where

1) H is a Hilbert space endowed with a representation

$$\pi\colon A \to \mathcal{L}(H);$$

2) $F \in \mathcal{L}(H)$ is a bounded selfadjoint operator with $F^2 = I$;

3) for all $a \in A$ we have

$$[F, \pi(a)] = F\pi(a) - \pi(a)F \in \mathcal{K}(H). \tag{4.11}$$

For $1 \leq p < \infty$, let $\mathcal{L}^p(H)$ denote the Schatten ideal of p-summable operators. A Fredholm module (H, F) is called *p-summable* if, instead of (4.11), we have the stronger condition:

$$[F, \pi(a)] \in \mathcal{L}^p(H) \tag{4.12}$$

for all $a \in A$. Since $\mathcal{L}^p(H) \subset \mathcal{L}^q(H)$ for $p \leq q$, a p-summable Fredholm module is clearly q-summable for all $q \geq p$.

Definition 4.2.2. An *even Fredholm module* over an algebra A is a triple (H, F, γ) such that (H, F) is a Fredholm module over A in the sense of the above definition and $\gamma \colon H \to H$ is a bounded selfadjoint operator with $\gamma^2 = I$ and such that

$$F\gamma = -\gamma F, \quad \pi(a)\gamma = \gamma\pi(a) \tag{4.13}$$

for all $a \in A$.

Let H^+ and H^- denote the $+1$ and -1 eigenspaces of γ. They define an orthogonal decomposition $H = H^+ \oplus H^-$. With respect to this decomposition, equations (4.13) are equivalent to saying that π is an even representation and F is an odd operator, so that we can write

$$\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix},$$

where π^+ and π^- are representations of A on H^+ and H^-, respectively. The notion of a p-summable even Fredholm module is defined as in the odd case above.

Remark 8. Notice that in the preceding example with F defined by (4.10), and in general in [6], [22], [107], the condition $F^2 = I$ for a Fredholm module only holds modulo compact operators. Similarly for the two equalities in equation (4.13). It is shown in [41] that with simple modifications one can replace such (H, F) by an equivalent Fredholm module in which these equations, except (4.11), hold exactly. The point is that as far as pairing with K-theory is concerned the set up in [6], [22], [107] is enough. It is for the definition of the Chern character and pairing with cyclic cohomology that one needs the exact equalities $F^2 = I$ and (4.13), as well as the finite summability assumption (4.12).

Example 4.2.1. Let $A = C(S^1)$ $(S^1 = \mathbb{R}/\mathbb{Z})$ act on $H = L^2(S^1)$ as multiplication operators. Let $F(e_n) = e_n$ if $n \geq 0$ and $F(e_n) = -e_n$ for $n < 0$, where $e_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$, denotes the standard basis of H. Clearly F is selfadjoint and $F^2 = I$. To show that $[F, \pi(f)]$ is a compact operator for all $f \in C(S^1)$, notice that if $f = \sum_{|n| \leq N} a_n e_n$ is a finite trigonometric sum then $[F, \pi(f)]$ is a finite rank operator and hence is compact. In general we can uniformly approximate a continuous function by a trigonometric sum and show that the commutator is compact for any continuous f. This shows that (H, F) is an odd Fredholm module over $C(S^1)$.

This Fredholm module is not p-summable for any $1 \leq p < \infty$. If we restrict it to the subalgebra $C^\infty(S^1)$ of smooth functions, then it can be checked that (H, F) is in fact p-summable for all $p > 1$, but is not 1-summable even in this case.

Let (H, F) be an odd p-summable Fredholm module over an algebra A and let n be an integer such that $2n \geq p$. To simplify the notation, from now on in our formulae the operator $\pi(a)$ will be denoted by a. Thus an expression like $a_0[F, a_1]$ stands for the operator $\pi(a_0)[F, \pi(a_1)]$, etc. We define a $(2n-1)$-cochain on A by

$$\boxed{\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \mathrm{Tr}(F[F, a_0][F, a_1] \ldots [F, a_{2n-1}])} \tag{4.14}$$

where Tr denotes the operator trace. Notice that by our p-summability assumption, each commutator is in $\mathcal{L}^p(H)$ and hence, by Hölder inequality for Schatten class operators (cf. Appendix B), their product is in fact a trace class operator as soon as $2n \geq p$.

Proposition 4.2.1. *φ_{2n-1} is a cyclic $(2n-1)$-cocycle on A.*

Proof. For $a \in A$, let $da := [F, a]$. The following relations are easily established: for all $a, b \in A$ we have

$$d(ab) = ad(b) + da \cdot b \quad \text{and} \quad F da = -da \cdot F \tag{4.15}$$

Notice that for the second relation the assumption $F^2 = 1$ is essential. Now φ_{2n-1} can be written as

$$\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \mathrm{Tr}(F da_0 da_1 \ldots da_{2n-1}),$$

and therefore

$$\begin{aligned}(b\varphi_{2n-1})(a_0, \ldots, a_{2n}) = \mathrm{Tr}\Big(\sum_{i=0}^{2n} (-1)^i F da_0 \ldots d(a_i da_{i+1}) \ldots da_{2n}\Big)\\ + (-1)^{2n+1} \mathrm{Tr}(F d(a_{2n}a_0) da_1 \ldots da_{2n-1}).\end{aligned}$$

Using the derivation property of d, we see that most of the terms cancel and we are left with just four terms

$$\begin{aligned}&= \mathrm{Tr}(F a_0 da_1 \ldots da_{2n}) - \mathrm{Tr}(F da_0 \ldots da_{2n-1} a_{2n})\\ &\quad + \mathrm{Tr}(F a_{2n} da_0 \ldots da_{2n-1}) + \mathrm{Tr}(F da_{2n} a_0 da_1 \ldots da_{2n-1}).\end{aligned}$$

Using the relation $F da = -da \cdot F$ and the trace property of Tr we see that the second and third terms cancel. By the same argument the first and last terms cancel as well. This shows that φ_{2n-1} is a Hochschild cocycle.

To check the cyclic property of φ_{2n-1}, again using the relation $Fda = -da \cdot F$, and the trace property of Tr, we have

$$\begin{aligned}\varphi_{2n-1}(a_{2n-1}, a_0, \ldots, a_{2n-2}) &= \mathrm{Tr}(F da_{2n-1} da_0 \ldots da_{2n-2}) \\ &= -\mathrm{Tr}(da_{2n-1} F da_0 \ldots da_{2n-2}) \\ &= -\mathrm{Tr}(F da_0 \ldots da_{2n-2} da_{2n-1}) \\ &= -\varphi_{2n-1}(a_0, \ldots, a_{2n-2}, a_{2n-1}). \qquad \square\end{aligned}$$

Notice that if $2n \geq p-1$, then the cyclic cocycle (4.14) can be written as

$$\begin{aligned}\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) &= 2\,\mathrm{Tr}(a_0[F,\, a_1] \ldots [F,\, a_{2n-1}]) \\ &= 2\,\mathrm{Tr}(a_0 da_1 \ldots da_{2n-1})\end{aligned}$$

which looks remarkably like a noncommutative analogue of the integral

$$\int_M f_0\, df_1 \ldots df_{2n-1}.$$

Now the products $[F,\, a_0][F,\, a_1] \ldots [F,\, a_{2m-1}]$ are trace class for all $m \geq n$. Therefore we obtain a sequence of odd cyclic cocycles

$$\varphi_{2m-1}(a_0, a_1, \ldots, a_{2m-1}) = \mathrm{Tr}(F[F, a_0][F, a_1] \ldots [F, a_{2m-1}]), \quad m \geq n.$$

The next proposition shows that these cyclic cocycles are related to each other via the periodicity S-operator of cyclic cohomology:

Proposition 4.2.2. *For all $m \geq n$ we have*

$$S\varphi_{2m-1} = -\left(m + \frac{1}{2}\right) \varphi_{2m+1}.$$

Proof. Let us define a $2m$-cochain ψ_{2m} on A by the formula

$$\psi_{2m}(a_0, a_1, \ldots, a_{2m}) = \mathrm{Tr}(F a_0 da_1 \ldots da_{2m}).$$

We claim that

$$B\psi_{2m} = (2m)\, \varphi_{2m-1} \quad \text{and} \quad b\psi_{2m} = -\frac{1}{2}\, \varphi_{2m+1}.$$

In fact, we have $B\psi_{2m} = NB_0\psi_{2m}$, where

$$\begin{aligned}B_0\psi_{2m}(a_0, \ldots, a_{2m-1}) &= \psi_{2m}(1, a_0, \ldots, a_{2m-1}) \\ &\quad - (-1)^{2m-1}\psi_{2m}(a_0, \ldots, a_{2m-1}, 1) \\ &= \mathrm{Tr}(F da_0 \ldots da_{2m-1}),\end{aligned}$$

and hence

$$\begin{aligned}(NB_0)\psi_{2m}(a_0,\ldots,a_{2m-1}) &= \mathrm{Tr}(Fda_0\ldots da_{2m-1}) - \mathrm{Tr}(Fda_{2m-1}da_0\ldots da_{2m-2}) \\ &\quad + \cdots - \mathrm{Tr}(Fda_1\ldots da_0) \\ &= (2m)\,\mathrm{Tr}(Fda_0\ldots da_{2m-1}) \\ &= (2m)\varphi_{2m-1}(a_0,\ldots,a_{2m-1})\end{aligned}$$

and

$$\begin{aligned}(b\psi_{2m})(a_0,\ldots,a_{2m+1}) &= \mathrm{Tr}(Fa_0a_1da_2\ldots da_{2m+1}) - \mathrm{Tr}(Fa_0d(a_1a_2)\ldots da_{2m+1}) \\ &\quad + \cdots + \mathrm{Tr}(Fa_0da_1\ldots d(a_{2m}a_{2m+1})) \\ &\quad - \mathrm{Tr}(Fa_{2m+1}a_0da_1\ldots da_{2m}).\end{aligned}$$

After cancelations, only two terms remain which can be collected into a single term:

$$\begin{aligned}&= \mathrm{Tr}(Fa_0da_1\ldots da_{2m}\cdot a_{2m+1}) - \mathrm{Tr}(Fa_{2m+1}a_0da_1\ldots da_{2m}) \\ &= -\frac{1}{2}\,\mathrm{Tr}(Fda_0da_1\ldots da_{2m+1}) \\ &= -\frac{1}{2}\varphi_{2m+1}(a_0,\ldots,a_{2m+1}).\end{aligned}$$

The above computation shows that

$$bB^{-1}\varphi_{2m-1} = -\frac{1}{2(2m)}\varphi_{2m+1}.$$

Now using formula (3.29) for the operator S, we have

$$S\varphi_{2m-1} = (2m)(2m+1)bB^{-1}\varphi_{2m-1} = -\left(m+\frac{1}{2}\right)\varphi_{2m+1}. \qquad \square$$

The odd Connes–Chern characters $\mathrm{Ch}^{2m-1} = \mathrm{Ch}^{2m-1}(H,\,F)$, are defined by rescaling the cocycles φ_{2m-1} appropriately. Let

$$\begin{aligned}&\mathrm{Ch}^{2m-1}(a_0,\ldots,a_{2m-1}) \\ &\quad := (-1)^m 2\left(m-\frac{1}{2}\right)\cdots\frac{1}{2}\;\mathrm{Tr}(F[F,a_0][F,a_1]\ldots[F,a_{2m-1}]).\end{aligned}$$

The following is an immediate corollary of the above proposition:

Corollary 4.2.1. *We have*

$$S(\mathrm{Ch}^{2m-1}) = \mathrm{Ch}^{2m+1} \quad \textit{for all } m \geq n.$$

Definition 4.2.3. The *Connes–Chern character* of an odd p-summable Fredholm module (H,F) over an algebra A is the class of the cyclic cocycle Ch^{2m-1} in the odd periodic cyclic cohomology group $HP^{\mathrm{odd}}(A)$.

By the above corollary, the class of Ch^{2m-1} in $HP^{\mathrm{odd}}(A)$ is independent of the choice of m.

Example 4.2.2. Let us compute the character of the Fredholm module of Example 4.2.1 with $A = C^\infty(S^1)$. By the above definition, $\mathrm{Ch}^1(H, F) = [\varphi_1]$ is the class of the following cyclic 1-cocycle in $HP^{\mathrm{odd}}(A)$:

$$\varphi_1(f_0, f_1) = \mathrm{Tr}(F[F, f_0][F, f_1]),$$

and the question is if we can identify this cocycle with some local formula. We claim that

$$\mathrm{Tr}(F[F, f_0][F, f_1]) = \frac{4}{2\pi i} \int f_0 \, df_1 \quad \text{for all } f_0, f_1 \in A.$$

To verify the claim, it suffices to check it for the basis elements $f_0 = e_m, f_1 = e_n$ for all $m, n \in \mathbb{Z}$. The right-hand side is easily computed:

$$\frac{4}{2\pi i} \int e_m \, de_n = \begin{cases} 0 & \text{if } m + n \neq 0, \\ 4n & \text{if } m + n = 0. \end{cases}$$

To compute the left-hand side, notice that

$$[F, e_n](e_k) = \begin{cases} 0 & \text{if } k \geq 0, \, n + k \geq 0, \\ -2e_{n+k} & \text{if } k \geq 0, \, n + k < 0, \\ 2e_{n+k} & \text{if } k < 0, \, n + k \geq 0, \\ 0 & \text{if } k < 0, \, n + k < 0. \end{cases}$$

From this we conclude that $Fe_m[F, e_n] = 0$ if $m + n \neq 0$. To compute the operator trace for $m + n = 0$, we use the formula $\mathrm{Tr}(T) = \sum_k \langle T(e_k), e_k \rangle$ for the trace of a trace class operator T. Using the above information we have

$$F[F, e_{-n}][F, e_n](e_k) = \begin{cases} 0 & \text{if } k \geq 0, \, n + k \geq 0, \\ 4e_k & \text{if } k \geq 0, \, n + k < 0, \\ 4e_k & \text{if } k < 0, \, n + k \geq 0, \\ 0 & \text{if } k < 0, \, n + k < 0, \end{cases}$$

from which we readily obtain

$$\mathrm{Tr}(F\,[F, e_{-n}][F, e_n]) = 4n \quad \text{for all } n \in \mathbb{Z}.$$

This finishes the proof.

Next we turn to the even case. Let (H, F, γ) be an even p-summable Fredholm module over an algebra A and let n be an integer such that $2n + 1 \geq p$. Define a $2n$-cochain on A by the formula

$$\varphi_{2n}(a_0, a_1, \ldots, a_{2n}) = \mathrm{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2n}]). \tag{4.16}$$

Proposition 4.2.3. *φ_{2n} is a cyclic $2n$-cocycle on A.*

Proof. The proof is similar to the odd case and is left to the reader. Apart from relations (4.15), one needs the auxiliary relation $\gamma\, da = -da\, \gamma$ for the proof as well. □

Notice that if we have the stronger condition $2n \geq p$, then the cyclic cocycle (4.16) can be written as

$$\begin{aligned}\varphi_{2n}(a_0, a_1, \ldots, a_{2n}) &= \mathrm{Tr}(\gamma a_0[F, a_1] \ldots [F, a_{2n}]) \\ &= \mathrm{Tr}(\gamma a_0 da_1 \ldots da_{2n}).\end{aligned}$$

As in the odd case, we obtain a sequence of even cyclic cocycles φ_{2m}, defined by

$$\varphi_{2m}(a_0, a_1, \ldots, a_{2m}) = \mathrm{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2m}]), \quad m \geq n.$$

Proposition 4.2.4. *For all $m \geq n$ we have*

$$S\varphi_{2m} = -(m+1)\varphi_{2m+2}.$$

Proof. Define a $(2m+1)$-cochain ψ_{2m+1} on A by the formula

$$\psi_{2m+1}(a_0, a_1, \ldots, a_{2m+1}) = \mathrm{Tr}(\gamma\, F a_0 da_1 \ldots da_{2m+1}).$$

The following relations can be proved as in the odd case:

$$B\psi_{2m+1} = (2m+1)\varphi_{2m} \quad \text{and} \quad b\psi_{2m+1} = -\frac{1}{2}\varphi_{2m+2}.$$

It shows that

$$bB^{-1}\varphi_{2m} = -\frac{1}{2(2m+1)}\varphi_{2m+2},$$

so that using formula (3.29) for the operator S we obtain

$$S\varphi_{2m} = (2m+1)(2m+2)bB^{-1}\varphi_{2m} = -(m+1)\varphi_{2m+2}.$$ □

The even Connes–Chern characters $\mathrm{Ch}^{2m} = \mathrm{Ch}^{2m}(H, F, \gamma)$ are now defined by rescaling the even cyclic cocycles φ_{2m}:

$$\boxed{\mathrm{Ch}^{2m}(a_0, a_1, \ldots, a_{2m}) := \tfrac{(-1)^m m!}{2}\ \mathrm{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2m}])} \tag{4.17}$$

The following is an immediate corollary of the above proposition:

Corollary 4.2.2. *We have*

$$S(\mathrm{Ch}^{2m}) = \mathrm{Ch}^{2m+2} \quad \textit{for all } m \geq n.$$

Definition 4.2.4. The *Connes–Chern character* of an even p-summable Fredholm module (H, F, γ) over an algebra A is the class of the cyclic cocycle $\mathrm{Ch}^{2m}(H, F, \gamma)$ in the even periodic cyclic cohomology group $HP^{\mathrm{even}}(A)$.

By the above corollary, the class of Ch^{2m} in $HP^{\mathrm{even}}(A)$ is independent of m.

Example 4.2.3 (A noncommutative example)**.** Following [39], we construct an even Fredholm module over $A = C_r^*(F_2)$, the reduced group C^*-algebra of the free group on two generators. This Fredholm module is not p-summable for any p, but by restricting it to a properly defined dense subalgebra of A (which plays the role of 'smooth functions' on the underlying noncommutative space), we shall obtain a 1-summable Fredholm. We shall also identify the character of this 1-summable module. It is known that a group is free if and only if it has a free action on a tree. Let then T be a tree with a free action of F_2, and let T^0 and T^1 denote the set of vertices and 1-simplices of T, respectively. Let

$$H^+ = \ell^2(T^0) \quad \text{and} \quad H^- = \ell^2(T^1) \oplus \mathbb{C},$$

and let the canonical basis of $\ell^2(T^0)$ (resp. $\ell^2(T^1)$) be denoted by ε_q, $q \in T^0$ (resp. $q \in T^1$). Fixing a vertex $p \in T^0$, we can define a one-to-one correspondence

$$\varphi\colon T^0 - \{p\} \to T^1$$

by sending $q \in T^0 - \{p\}$ to the unique 1-simplex containing q and lying between p and q. This defines a unitary operator $P\colon H^+ \to H^-$ by

$$P(\varepsilon_q) = \varepsilon_{\varphi(q)} \quad \text{if } q \neq p, \quad \text{and} \quad P(\varepsilon_p) = (0, 1).$$

The action of F_2 on T^0 and T^1 induces representations of $C_r^*(F_2)$ on $\ell^2(T^0)$ and $\ell^2(T^1)$ and on $H^- = \ell^2(T^1) \oplus \mathbb{C}$ by the formula $a(\xi, \lambda) = (a\xi, 0)$. Let $H = H^+ \oplus H^-$ and

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To check that (H, F, γ) is a Fredholm module over A we need to verify that $[F, a] \in \mathcal{K}(H)$ for all $a \in C_r^*(F_2)$. Since the group algebra $\mathbb{C}F_2$ is dense in $C_r^*(F_2)$, it suffices to check that for all $a = g \in F_2$, the commutator $[F, g]$ is a finite rank operator. This in turn is a consequence of the easily established fact that for all $g \in F_2$,

$$\varphi(gq) = g\varphi(q) \quad \text{for all } q \neq g^{-1}p.$$

In fact, for $q \in T^0$ we have

$$[F, g](\varepsilon_q) = F(\varepsilon_{gq}) - gF(\varepsilon_q) = \varepsilon_{\varphi(gq)} - \varepsilon_{g\varphi(q)} = 0$$

if $q \neq g^{-1}p$, and $[F, g](\varepsilon_{g^{-1}p}) = (0, 1) - g\varepsilon_{\varphi(g^{-1}p)}$. A similar argument works for the basis elements ε_q, $q \in T^1$. This shows that $[F, g]$ is a rank one operator.

Let

$$\mathcal{A} = \{a \in A;\, [F, a] \in \mathcal{L}^1(H)\}.$$

Using the relation $[F, ab] = a[F, b] + [F, a]b$, it is clear that $\mathcal{A}$ is a subalgebra of A. It is also dense in A as it contains the group algebra $\mathbb{C}F_2$. Though we do not need it now, it can also be shown that $\mathcal{A}$ is *stable under holomorphic functional calculus* and in particular the inclusion $\mathcal{A} \subset A$ induces an isomorphism $K_0(\mathcal{A}) \to K_0(A)$ in K-theory (cf. Section 4.3 for more on this). By its very definition, we now have an even 1-summable Fredholm module (H, F, γ) over $\mathcal{A}$ and it remains to compute its character. Let $\tau\colon A \to \mathbb{C}$ denote the canonical trace on A. We claim that

$$\frac{1}{2}\,\mathrm{Tr}(\gamma F[F, a]) = \tau(a) \quad \text{for all } a \in \mathcal{A}, \tag{4.18}$$

so that

$$\mathrm{Ch}(H, F, \gamma)(a) = \tau(a).$$

To verify the claim, notice that

$$\gamma F[F, a] = \begin{bmatrix} a - P^{-1}aP & 0 \\ 0 & -a + PaP^{-1} \end{bmatrix},$$

so that

$$\frac{1}{2}\,\mathrm{Tr}(\gamma F[F, a]) = \mathrm{Tr}(a - P^{-1}aP).$$

Now for the operator $a - P^{-1}aP\colon H^+ \to H^+$ we have

$$\langle a\varepsilon_q, \varepsilon_q\rangle = \tau(a) \quad \text{for all } q \in T^0,$$

and

$$\langle (P^{-1}aP)(\varepsilon_p), \varepsilon_p\rangle = 0 \quad \text{and} \quad \langle (P^{-1}aP)(\varepsilon_q), \varepsilon_q\rangle = \tau(a)$$

for all $q \neq p$, from which (4.18) follows.

Our next goal is to define the *index pairing* between Fredholm modules over A and the K-theory of A. Notice that for this we do not need to assume that the Fredholm module is finitely summable. We start with the even case. Let (H, F, γ) be an even Fredholm module over an algebra A and let $e \in A$ be an idempotent. Let

$$F_e^+\colon eH_0 \to eH_1$$

denote the restriction of the operator eFe to the subspace eH_0. It is a Fredholm operator. To see this, let $F_e^-\colon eH_1 \to eH_0$ denote the restriction of eFe to the subspace eH_1. We claim that the operators $F_e^+F_e^- - 1$ and $F_e^-F_e^+ - 1$ are compact. Atkinson's theorem then shows that F_e^+ is Fredholm. The claim follows from the following computation:

$$eFeeFe = e(Fe - eF + eF)Fe = e[F, e]Fe + e,$$

and the fact that $[F, e]$ is a compact operator.

For an idempotent $e \in A$ let us define a pairing:

$$\boxed{\langle (H, F, \gamma), [e]\rangle := \mathrm{index}\, F_e^+}$$

More generally, if $e \in M_n(A)$ is an idempotent in the algebra of n by n matrices over A, we define

$$\langle (H,\, F, \gamma),\, [e] \rangle := \langle (H^n,\, F^n, \gamma),\, [e] \rangle,$$

where $(H^n,\, F^n)$ is the n*-fold inflation* of (H, F) defined by $H^n = H \otimes \mathbb{C}^n$, $F^n = F \otimes I_{\mathbb{C}^n}$. It is easily seen that $(H^n,\, F^n)$ is a Fredholm module over $M_n(A)$ and if (H, F) is p-summable then so is $(H^n,\, F^n)$. It is easily checked that the resulting map is additive with respect to direct sum of idempotents and is conjugation invariant. This shows that each even Fredholm module, which need not be finitely summable, induces an additive map on K-theory:

$$\langle (H,\, F,\, \gamma),\, - \rangle \colon K_0(A) \to \mathbb{C}.$$

There is a similar index pairing between odd Fredholm modules over A and the algebraic K-theory group $K_1^{\mathrm{alg}}(A)$. Let $(H,\, F)$ be an odd Fredholm module over A and let $U \in A^\times$ be an invertible element in A. Let $P = \frac{F+1}{2} \colon H \to H$ be the projection operator defined by F. Let us check that the operator

$$PUP \colon PH \to PH$$

is a Fredholm operator. Again the proof hinges on Atkinson's theorem and noticing that $PU^{-1}P$ is an inverse for PUP modulo compact operators. We have

$$\begin{aligned} PUPPU^{-1}P - I_{PH} &= PUPU^{-1}P - I_{PH} = P(UP - PU + PU)U^{-1}P - I_{PH} \\ &= P[U,\, P]U^{-1}P + P - I_{PH} = \frac{1}{2} P[U,\, F]U^{-1}P. \end{aligned}$$

But $[F, U]$ is a compact operator by our definition of Fredholm modules and hence the last term is compact too. Similarly one checks that $PU^{-1}PPUP - I_{PH}$ is a compact operator as well. We can thus define the index pairing:

$$\boxed{\langle (H,\, F),\, [U] \rangle := \operatorname{index}(PUP)}$$

If the invertible U happens to be in $M_n(A)$ we can apply the above definition to the n-fold iteration of $(H,\, F)$, as in the even case above, to define the pairing. The resulting map can be shown to induce a well-defined additive map

$$\langle (H,\, F),\, - \rangle \colon K_1^{\mathrm{alg}}(A) \to \mathbb{C}.$$

Example 4.2.4. Let $(H,\, F)$ be the Fredholm module of Example 4.2.1 and let $f \in C(S^1)$ be a nowhere zero continuous function on S^1 representing an element of $K_1^{\mathrm{alg}}(C(S^1))$. We want to compute the index pairing $\langle [(H,\, F)],\, [f] \rangle = \operatorname{index}(PfP)$. The operator $PfP \colon H^+ \to H^+$ is called a *Toeplitz operator*. The following standard result, known as the Gohberg–Krein index theorem, computes the index of a Toeplitz operator in terms of the winding number of f:

$$\langle [(H,\, F)],\, [f] \rangle = \operatorname{index}(PfP) = -W(f, 0).$$

To prove this formula notice that both sides are homotopy invariant. For the left-hand side this is a consequence of the homotopy invariance of the Fredholm index while for the right-hand side it is a standard fact about the winding number. Also, both sides are additive. Therefore it suffices to show that the two sides coincide on the generator of $\pi_1(S^1)$, i.e., for $f(z) = z$. Then PzP is easily seen to be the forward shift operator given by $PzP(e_n) = e_{n+1}$ in the given basis. Clearly then $\text{index}(PzP) = -1 = -W(z, 0)$.

When f is smooth we have the following well-known formula for the winding number:

$$W(f, 0) = \frac{1}{2\pi i} \int f^{-1}\, df = \frac{1}{2\pi i} \varphi(f^{-1}, f),$$

where φ is the cyclic 1-cocycle on $C^\infty(S^1)$ defined by $\varphi(f, g) = \int f\, dg$. Since this cyclic cocycle is the Connes–Chern character of the Fredholm module (H, F), the above equation can be written as

$$\langle [(H,\, F)],\, [f] \rangle = \frac{1}{2\pi i} \langle \text{Ch}^{\text{odd}}(H, F),\, \text{Ch}_{\text{odd}}(f) \rangle,$$

where the pairing on the right-hand side is between cyclic cohomology and homology. As we shall prove next, this is a special case of a very general index formula of Connes.

Now what makes the Connes–Chern character in K-homology useful is the fact that it can capture the analytic index by giving a topological formula for the index. More precisely we have the following index formula due to Connes [39]:

Theorem 4.2.1. *Let $(H,\, F,\, \gamma)$ be an even p-summable Fredholm module over A and n be an integer such that $2n + 1 \geq p$. If e is an idempotent in A then*

$$\text{index}(F_e^+) = \frac{(-1)^n}{2}\, \varphi_{2n}(e, e, \dots, e),$$

where the cyclic $2n$-cocycle φ_{2n} is defined by

$$\varphi_{2n}(a_0, a_1, \dots, a_{2n}) = \text{Tr}(\gamma F [F, a_0][F, a_1] \dots [F, a_{2n}]).$$

Proof. We use the following fact from the theory of Fredholm operators (cf. Proposition B.2 for a proof): let $P' \colon H' \to H''$ be a Fredholm operator and let $Q' \colon H'' \to H'$ be such that for an integer $n \geq 0$, $1 - P'Q' \in \mathcal{L}^{n+1}(H'')$ and $1 - Q'P' \in \mathcal{L}^{n+1}(H')$. Then

$$\text{index}\,(P') = \text{Tr}(1 - Q'P')^{n+1} - \text{Tr}(1 - P'Q')^{n+1}.$$

We can also write the above formula as a supertrace

$$\text{index}(P') = \text{Tr}(\gamma'(1 - F'^2)^{n+1}), \tag{4.19}$$

where the operators F' and γ' acting on $H' \oplus H'$ are defined by

$$F' = \begin{pmatrix} 0 & Q' \\ P' & 0 \end{pmatrix} \quad \text{and} \quad \gamma' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We apply this result to $P' = F_e^+$, $Q' = F_e^-$, $H' = e_0 H_0$ and $H'' = e_1 H_1$. By our summability assumption, both operators $1 - P'Q'$ and $1 - Q'P'$ are in $\mathcal{L}^{n+1}$ and we can apply (4.19). We have

$$\operatorname{index}(F_e^+) = \operatorname{Tr}(\gamma' (1 - F'^2)^{n+1}) = \operatorname{Tr}(\gamma (e - (eFe)^2)^{n+1}).$$

As in the proof of Proposition 4.2.1 let $de := [F, e]$. Using the relations $e^2 = e$, $ede \cdot e = 0$, and $edede = de \cdot de \cdot e$, we have

$$e - (eFe)^2 = eF(Fe - eF)e = (eF - Fe + Fe)(Fe - eF)e = -edede$$

and hence $(e - (eFe)^2)^{n+1} = (-1)^{n+1}(edede)^{n+1} = (-1)^{n+1}e(de)^{2n+2}$. Thus the index can be written as

$$\operatorname{index}(F_e^+) = (-1)^{n+1} \operatorname{Tr}(\gamma e (de)^{2n+2}).$$

On the other hand, using $de = ede + de \cdot e$, we have

$$\begin{aligned}
\varphi_{2n}(e, e, \ldots, e) &= \operatorname{Tr}(\gamma F (de)^{2n+1}) \\
&= \operatorname{Tr}(\gamma (Fe) e (de)^{2n+1} + \gamma F de \cdot ee (de)^{2n} \\
&= \operatorname{Tr}(Fe - eF + eF) e (de)^{2n+1}) + \gamma (eF - Fe + Fe) de \cdot e (de)^{2n} \\
&= \operatorname{Tr}(\gamma de \cdot e (de)^{2n+1} - \operatorname{Tr}(\gamma dede \cdot e (de)^{2n}) \\
&= - \operatorname{Tr}(\gamma dede \cdot e (de)^{2n}) - \operatorname{Tr}(\gamma dede \cdot e (de)^{2n}) \\
&= -2 \operatorname{Tr}(\gamma e (de)^{2n+2})
\end{aligned}$$

which of course proves the theorem. In the above computation we used the fact that $\operatorname{Tr}(\gamma eFe(de)^{2n+1}) = \operatorname{Tr}(\gamma F ede \cdot e (de)^{2n}) = 0$. □

Using the pairing $HC^{2n}(A) \otimes K^0(A) \to \mathbb{C}$ between cyclic cohomology and K-theory defined in (4.5), and the definition of the Connes–Chern character of (H, F, γ) in (4.17), the above index formula can be written as

$$\operatorname{index}(F_e^+) = \langle \operatorname{Ch}^{2n}(H,\, F,\, \gamma),\, [e] \rangle,$$

or in its stable form

$$\operatorname{index}(F_e^+) = \langle \operatorname{Ch}^{\text{even}}(H,\, F,\, \gamma),\, [e] \rangle.$$

There is yet another way to interpret the index formula as

$$\langle (H,\, F,\, \gamma),\, [e] \rangle = \langle \operatorname{Ch}^{2n}(H,\, F,\, \gamma),\, \operatorname{Ch}_{2n}[e] \rangle,$$

where on the left-hand side we have the pairing between K-homology and K-theory and on the right-hand side the pairing between cyclic cohomology and homology.

The corresponding index formula in the odd case is as follows:

Proposition 4.2.5. *Let* (H, F) *be an odd* p*-summable Fredholm module over* A *and let* n *be an integer such that* $2n \geq p$*. If* u *is an invertible element in* A*, then*

$$\operatorname{index}(PuP) = \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}(u^{-1}, u, \ldots, u^{-1}, u),$$

where the cyclic cocycle φ_{2n-1} *is defined by*

$$\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \operatorname{Tr}(F[F, a_0][F, a_1] \ldots [F, a_{2n-1}]).$$

Proof. Let $P = \frac{1+F}{2}$, $H' = PH$, $P' = PuP \colon H' \to H'$, $Q' = Pu^{-1}P \colon H' \to H'$, and $du = [F, u]$. We have

$$\begin{aligned}
1 - Q'P' &= 1 - Pu^{-1}PPuP \\
&= 1 - pu^{-1}(Pu - uP + uP)P \\
&= 1 - Pu^{-1}[P, u]P - P \\
&= -\frac{1}{2}Pu^{-1}du \cdot P \\
&= \frac{1}{2}Pdu^{-1} \cdot uP = \frac{1}{2}Pdu^{-1}(uP - Pu + Pu) \\
&= -\frac{1}{4}Pdu^{-1}du + Pdu^{-1}Pu \\
&= -\frac{1}{4}Pdu^{-1}du,
\end{aligned}$$

where in the last step the relation $Pdu^{-1}P = 0$ was used. This relation follows from

$$[P,\ u^{-1}] = [P^2,\ u^{-1}] = P[P,\ u^{-1}] + [P,\ u^{-1}]P.$$

Since, by our summability assumption, $du = [F, u] \in \mathcal{L}^{2n}(H)$ and similarly $du^{-1} \in \mathcal{L}^{2n}(H)$, we have $1 - Q'P' \in \mathcal{L}^n(H')$.

A similar computation shows that

$$1 - P'Q' = -\frac{1}{4}Pdudu^{-1},$$

and hence $1 - P'Q' \in \mathcal{L}^n(H')$.

Using formula (4.19) for the index, we obtain

$$\begin{aligned}
\operatorname{index}(PuP) &= \operatorname{Tr}((1-Q'P')^n) - \operatorname{Tr}((1-P'Q')^n) \\
&= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}((Pdu^{-1}du)^n) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}((Pdudu^{-1})^n) \\
&= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(P(du^{-1}du)^n) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(P(dudu^{-1})^n) \\
&= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}\left(\frac{1+F}{2}(du^{-1}du)^n\right) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}\left(\frac{1+F}{2}(dudu^{-1})^n\right) \\
&= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(F\,(du^{-1}du)^n) \\
&= \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}(u^{-1}, u, \ldots, u^{-1}, u),
\end{aligned}$$

where in the last step we used the relations

$$\operatorname{Tr}((du^{-1}du)^n) = \operatorname{Tr}((dudu^{-1})^n) \quad \text{and} \quad \operatorname{Tr}(F(du^{-1}du)^n) = -\operatorname{Tr}(F(dudu^{-1})^n).$$

□

Using the pairing $HC^{2n-1}(A) \otimes K_1^{\mathrm{alg}}(A) \to \mathbb{C}$ and the definition of $\mathrm{Ch}^{2n-1}(H, F)$, the above index formula can be written as

$$\operatorname{index}(PuP) = \langle \mathrm{Ch}^{2n-1}(H, F), [u] \rangle,$$

or in its stable form

$$\operatorname{index}(PuP) = \langle \mathrm{Ch}^{\mathrm{odd}}(H, F), [u] \rangle.$$

There is yet another way to interpret the index formula as

$$\langle (H, F), [u] \rangle = \langle \mathrm{Ch}^{2n-1}(H, F), \mathrm{Ch}_{2n-1}[u] \rangle,$$

where on the left-hand side we have the pairing between K-homology and K-theory and on the right-hand side the pairing between cyclic cohomology and homology.

Example 4.2.5 (A noncommutative connected space)**.** A *projection* in a $*$-algebra is an element e satisfying $e^2 = e = e^*$. It is called a trivial projection if $e = 0$ or $e = 1$. It is clear that a compact space X is connected if and only if the algebra $C(X)$ has no non-trivial projections. Let us agree to call a noncommutative space represented by a C^*-algebra A connected if A has no non-trivial projections. The *Kadison conjecture* states that the reduced group C^*-algebra of a torsion-free discrete group is connected. This conjecture, in its full generality, is still open although it has now been verified for various classes of groups [172]. Methods of noncommutative geometry play an important role in these proofs. The validity of the conjecture for free groups was first established by Pimsner and Voiculescu [149] using techniques of K-theory. Here we reproduce Connes' proof of this conjecture for free groups. Note that the conjecture is obviously true for the finitely generated free abelian groups $\mathbb{Z}^n$, since by Fourier theory, or the Gelfand–Naimark theorem, $C^*(\mathbb{Z}^n) \simeq C(\mathbb{T}^n)$, and the n-torus T^n is of course connected.

Let $\tau\colon C_r^*(F_2) \to \mathbb{C}$ be the canonical normalized trace. It is positive and faithful in the sense that for all $a \in A$, $\tau(aa^*) \geq 0$ and $\tau(aa^*) = 0$ if and only if $a = 0$. Thus if we can show that for a projection e, $\tau(e)$ is an integer then we can deduce that $e = 0$ or $e = 1$. In fact since e is a projection we have $0 \leq e \leq 1$ and therefore $0 \leq \tau(e) \leq 1$, and by integrality we have $\tau(e) = 1$ or $\tau(e) = 0$. Since τ is faithful from $0 = \tau(e) = \tau(ee^*)$ we have $e = 0$. A similar argument works for $\tau(e) = 1$.

Now the proof of the *integrality* of $\tau(e)$ is based on Connes' index formula in Theorem 4.2.1 and is remarkably similar to proofs of classical integrality theorems for characteristic numbers in topology using an index theorem: to show that a number $\tau(e)$ is an integer it suffices to shows that it is the index of a Fredholm operator. Let (H, F, γ) be the even 1-summable Fredholm module over the dense subalgebra $\mathcal{A} \subset C_r^*(F_2)$ defined in Example 4.2.3. The index formula combined with (4.18), shows that if $e \in \mathcal{A}$ is a projection then

$$\tau(e) = \frac{1}{2}\,\mathrm{Tr}(\gamma F e[F,\, e]) = \mathrm{index}(F_e^+) \in \mathbb{Z}$$

is an integer and we are done. To prove the integrality result for idempotents in A which are not necessarily in $\mathcal{A}$, we make use of the fact that $\mathcal{A}$ is stable under holomorphic functional calculus. Let $e \in A$ be an idempotent. For any $\epsilon > 0$ there is an idempotent $e' \in \mathcal{A}$ such that $\|e - e'\| < \epsilon$. In fact, since $\mathcal{A}$ is dense in A we can first approximate it by an element $g \in \mathcal{A}$. Since $\mathrm{sp}(e) \subset \{0, 1\}$, $\mathrm{sp}(g)$ is concentrated around 0 and 1. Let f be a holomorphic function defined on an open neighborhood of $\mathrm{sp}(g)$ which is identically equal to 0 around 0 and identically equal to 1 around 1. Then

$$e' = f(g) = \frac{1}{2\pi i}\int_\gamma f(z)(z1 - e)^{-1}\,dz,$$

is an idempotent in $\mathcal{A}$ which is close to e. As we showed before (cf. formula (4.2)), close idempotents are equivalent in the sense that $e = ue'u^{-1}$ for an appropriate $u \in A$. In particular we conclude that $\tau(e) = \tau(ue'u^{-1}) = \tau(e')$ is an integer.

In connection with Exercise 4.2.6 it is appropriate to mention that there is a refinement of the notion of Fredholm module to that of a *spectral triple* that plays a very important role in further developments of noncommutative geometry. Broadly speaking, going from Fredholm modules to spectral triples is like passing from the conformal class of a metric to the Riemannian metric itself. Spectral triples simultaneously provide a notion of *Dirac operator* in noncommutative geometry, as well as a Riemannian type *distance function* for noncommutative spaces.

To motivate the definition of a spectral triple, we recall that the Dirac operator $\not{D}$ on a compact Riemannian spin^c manifold acts as an unbounded selfadjoint operator on the Hilbert space $L^2(M, S)$ of L^2-spinors on the manifold M. If we let

$C^\infty(M)$ act on $L^2(M,S)$ by multiplication operators, then one can check that for any smooth function f, the commutator $[D,f]=Df-fD$ extends to a bounded operator in $L^2(M,S)$. Now the geodesic distance d on M can be recovered from the following *distance formula* of Connes [41]:

$$d(p,q)=\sup\{|f(p)-f(q)|;\, \| \,[D,f]\, \|\leq 1\} \quad \text{for all } p,q\in M. \tag{4.20}$$

The triple $(C^\infty(M), L^2(M,S), \not{D})$ is a commutative example of a spectral triple.

In general, in the odd case, a spectral triple is a triple $(\mathcal{A},\mathcal{H},D)$, where $\mathcal{A}$ is a $*$-algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D, encoding the Dirac operator and metric, is an unbounded selfadjoint operator on $\mathcal{H}$. It is required that D interacts with the algebra in a bounded fashion, i.e., that for all $a\in\mathcal{A}$ the commutators $[D,a]=Da-aD$ are well defined on the domain of D and extend to bounded operators on $\mathcal{H}$. It is further postulated that the operator D should have *compact resolvent* in the sense that $(D+\lambda)^{-1}\in\mathcal{K}(\mathcal{H})$ for all $\lambda\notin\mathbb{R}$. This last condition implies that the spectrum of D consists of a discrete set of eigenvalues with finite multiplicity. A spectral triple is called finitely summable if $(D^2+1)^{-1}\in\mathcal{L}^p(\mathcal{H})$ for some $1\leq p<\infty$.

Given a spectral triple as above and assuming that D is invertible, one checks that with $F:=D|D|^{-1}$, the phase of D, $(\mathcal{A},\mathcal{H},F)$ is a Fredholm module. By passing from the spectral triple to the corresponding Fredholm module, we lose the metric structure, but still retain the topological information, in particular the index pairing, encoded by the triple. For examples of spectral triples arising in physics and number theory the reader should consult [52].

Exercise 4.2.1. The Fredholm module of Example 4.2.3 can be defined over any free group. For $\Gamma=\mathbb{Z}$ one obtains an even Fredholm module over $C^*(\mathbb{Z})\simeq C(S^1)$. Identify this Fredholm module and its character.

Exercise 4.2.2. Give an example of a discrete group Γ and a projection $e\in\mathbb{C}\Gamma$ such that $\tau(e)$ is not an integer (τ is the canonical trace).

Exercise 4.2.3. Let $(H,\,F)$ be an odd p-summable Fredholm module over an algebra A. What happens if in the cochain (4.14) we replace $(2n-1)$ by an even integer. Similarly for even Fredholm modules.

Exercise 4.2.4. Show that the Fredholm module in Example 4.2.2 is p-summable for any $p>1$ but is not 1-summable. If we consider it as a Fredholm module over the algebra $C^k(S^1)$ of k-times continuously differentiable functions then (H,F) is p-summable for some $p>1$. Find a relation between k and p.

Exercise 4.2.5. Show that the Fredholm module over $C_r^*(F_2)$ in Example 4.2.3 is not 1-summable.

Exercise 4.2.6. Let $D=-i\frac{d}{dx}\colon C^\infty(S^1)\to C^\infty(S^1)$. It has an extension to a selfadjoint unbounded operator $D\colon \operatorname{Dom}(D)\subset L^2(S^1)\to L^2(S^1)$. Show that the arc distance on S^1 can be recovered from D via the formula

$$\operatorname{dist}(p,q)=\sup\{|f(p)-f(q)|;\ \|[D,\pi(f)\|\leq 1\}, \tag{4.21}$$

where $\pi(f)$ is the multiplication by f operator. The triple $(C^\infty(S^1), L^2(S^1), D)$ is an example of a *spectral triple* and (4.21) is a prototype of a very general *distance formula* of Connes that recovers the distance on a Riemannian spin^c manifold from its Dirac operator (cf. formula (4.20) in this section and the last chapter of [41]).

4.3 Algebras stable under holomorphic functional calculus

The pairing $HC^*(A) \otimes K_*(A) \to \mathbb{C}$ between cyclic cohomology and K-theory poses a challenge. As we saw in Section 3.4 cyclic cohomology is non-trivial and useful on 'smooth algebras' which are typically a dense subalgebra of some C^*-algebra. Topological K-theory on the other hand is most naturally defined for Banach and C^*-algebras. Thus the natural domains of the two functors HC^* and K_* are different. How can we reconcile the two categories of algebras here? One approach is to identify those dense subalgebras $\mathcal{A} \subset A$ of Banach algebras whose K-theory is isomorphic to the K-theory of A and work with $\mathcal{A}$ instead of A. A classical, commutative, example is the algebra of smooth functions as a subalgebra of the algebra of continuous functions. It is well known that the inclusion $C^\infty(M) \hookrightarrow C(M)$ induces an isomorphism between K_0 groups [103].

We are going to describe a situation where many features of the embedding $C^\infty(M) \subset C(M)$ are captured and extended to the noncommutative world. In particular we shall identify a class of dense subalgebras of Banach algebras $\mathcal{A} \subset A$ where the induced map on K-theory is an isomorphism. These subalgebras are called stable under holomorphic functional calculus.

Let A be a unital Banach algebra and let f be a holomorphic function defined on a neighborhood of $\mathrm{sp}(a)$, the spectrum of $a \in A$. Let

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z1 - a)^{-1}\, dz, \tag{4.22}$$

where the contour γ goes around the spectrum of a only once (counter clockwise). Thanks to the holomorphicity of f, the integral is independent of the choice of the contour and it can be shown that for a fixed a, the map

$$f \mapsto f(a)$$

is a unital algebra map from the algebra of holomorphic functions on a neighborhood of $\mathrm{sp}(a)$ to A. It is called the *holomorphic functional calculus.* (Cf. e.g. [122] for more details.)

If f happens to be holomorphic in a disc containing $\mathrm{sp}(a)$ with power series expansion $f(z) = \sum c_i z^i$, then one shows, using the Cauchy integral formula, that $f(a) = \sum c_i a^i$, so that the two definitions of $f(a)$ coincide. If A is a C^*-algebra and a is a normal element then, thanks to the Gelfand–Naimark theorem, applied to the commutative C^*-algebra generated by a, we have the much more powerful *continuous functional calculus* from $C(\mathrm{sp}(a)) \to A$. It extends the holomorphic functional calculus.

Definition 4.3.1. Let $B \subset A$ be a unital subalgebra of a unital Banach algebra A. We say B is stable under holomorphic functional calculus if for all $a \in B$ and all holomorphic functions on $\mathrm{sp}(a)$, we have $f(a) \in B$.

Example 4.3.1. 1. The algebra $C^\infty(M)$ of smooth functions on a closed smooth manifold M is stable under holomorphic functional calculus in $C(M)$. This is clear because the spectrum of a function $a \in C(M)$ is simply its range and formula (4.22) simplifies to $f(a) = f \circ a$ (composition of functions). The same can be said about the algebra $C^k(M)$ of k-times differentiable functions. The algebra $\mathbb{C}[X]$ of polynomial functions is not stable under holomorphic functional calculus in $C[0,1]$.

2. The smooth noncommutative torus $\mathcal{A}_\theta \subset A_\theta$ is stable under holomorphic functional calculus. (cf. Example 4.3.2 below.)

Let $\mathrm{sp}_B(a)$ denote the spectrum of $a \in B$ with respect to the subalgebra B. Clearly $\mathrm{sp}_A(a) \subset \mathrm{sp}_B(a)$ but the reverse inclusion holds if and only if invertibility in A implies invertibility in B. A good example to keep in mind is $\mathbb{C}[x] \subset C[0,1]$. It is easy to see that if B is stable under holomorphic functional calculus, then we have the *spectral permanence property*

$$\mathrm{sp}_A(a) = \mathrm{sp}_B(a).$$

The converse need not be true (cf. Exercises below), but under some extra conditions on the subalgebra B the above spectral permanence property can be shown to imply that B is stable under holomorphic functional calculus. In fact, in this case for all $z \in \mathrm{sp}_A(a)$, $(z1 - a)^{-1} \in B$ and if there is a suitable topology in B, stronger than the topology induced from A, in which B is complete, one can then show that the integral (4.22) converges in B and hence $f(a) \in B$. We give two instances where this technique works.

Let (H, F) be a Fredholm module over a Banach algebra A and assume that the action of A on H is continuous.

Proposition 4.3.1 ([41])**.** *For each $p \in [1, \infty)$, the subalgebra*

$$\mathcal{A} = \{a \in A;\ [F, a] \in \mathcal{L}^p(H)\}$$

is stable under holomorphic functional calculus.

Another source of examples is *smooth vectors* for actions of Lie groups. Let G be a Lie group acting continuously on a Banach algebra A. Continuity here means that for any $a \in A$, the map $g \mapsto g(a)$ from $G \to A$ is continuous. An element $a \in A$ is called *smooth* if the map $g \mapsto g(a)$ is smooth. It can be shown that smooth vectors form a dense subalgebra of A which is stable under holomorphic functional calculus (cf. Proposition 3.4.5 in [85]).

Example 4.3.2. The formulas

$$U \mapsto \lambda_1 U, \quad V \mapsto \lambda_2 V,$$

where $(\lambda_1, \lambda_2) \in \mathbb{T}^2$, define an action of the two-torus $\mathbb{T}^2$ on the noncommutative torus A_θ. Its set of smooth vectors can be shown to coincide with the smooth noncommutative torus $\mathcal{A}_\theta$ [37]. It follows that $\mathcal{A}_\theta$ is stable under holomorphic functional calculus in A_θ.

For applications to K-theory, the following result is crucial [162].

Proposition 4.3.2. *If B is a dense subalgebra of a Banach algebra A which is stable under holomorphic functional calculus then so is $M_n(B)$ in $M_n(A)$ for all $n \geq 1$.*

Now let $e \in A$ be an idempotent in A. For any $\epsilon > 0$ there is an idempotent $e' \in B$ such that $\|e - e'\| < \epsilon$. In fact, since B is dense in A we can first approximate it by an element $g \in B$. Since $\mathrm{sp}(e) \subset \{0, 1\}$, $\mathrm{sp}(g)$ is concentrated around 0 and 1. Let f be a holomorphic function which is locally constant on $\mathrm{sp}(g)$, and is equal to 0 at 0 and equal to 1 at 1. Then

$$e' = f(g) = \frac{1}{2\pi i} \int_\gamma f(z)(z1 - e)^{-1}\, dz$$

is an idempotent in B which is close to e. In particular $[e] = [f(g)]$ in $K_0(A)$ (see Section 5.1). Thanks to the above proposition, we can repeat this argument for $M_n(B) \subset M_n(A)$ for all n. It follows that if B is dense is A and is stable under holomorphic functional calculus, the natural embedding $B \to A$ induces an isomorphism $K_0(B) \simeq K_0(A)$ in K-theory (cf. also the article of J.-B. Bost [17] where a more general density theorem along these lines is proved).

Example 4.3.3 (Toeplitz algebras)**.** The original Toeplitz algebra $\mathcal{T}$ is defined as the universal unital C^*-algebra generated by an *isometry*, i.e., an element S with

$$S^*S = I.$$

It can be concretely realized as the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by the unilateral forward shift operator $S(e_i) = e_{i+1}$, $i = 0, 1, \ldots$. Since the algebra $C(S^1)$ of continuous functions on the circle is the universal algebra defined by a unitary u, the map $S \mapsto u$ defines a C^*-algebra surjection

$$\sigma \colon \mathcal{T} \to C(S^1),$$

called the *symbol map*. It is an example of the symbol map for pseudodifferential operators of order zero over a closed manifold (see below).

The rank one projection $I - SS^*$ is in the kernel of σ. Since the closed ideal generated by $I - SS^*$ is the ideal $\mathcal{K}$ of compact operators, we have $\mathcal{K} \subset \operatorname{Ker}\sigma$. With some more work one shows that in fact $\mathcal{K} = \operatorname{Ker}\sigma$ and therefore we have a short exact sequence of C^*-algebras, called the *Toeplitz extension* (due to Coburn, cf. [70], [80])

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0. \tag{4.23}$$

There is an alternative description of the Toeplitz algebra and extension (4.23) that makes its relation with pseudodifferential operators and index theory more transparent. Let $H = L^2(S^1)^+$ denote the Hilbert space of square integrable functions on the circle whose negative Fourier coefficients vanish (the Hardy space) and let $P\colon L^2(S^1) \to L^2(S^1)^+$ denote the canonical projection. Any continuous function $f \in C(S^1)$ defines a *Toeplitz operator*

$$T_f\colon L^2(S^1)^+ \to L^2(S^1)^+, \quad T_f(g) = P(gf).$$

It can be shown that the C^*-algebra generated by the set of Toeplitz operators $\{T_f;\ f \in C(S^1)\}$ is isomorphic to the Toeplitz algebra $\mathcal{T}$. The relation

$$T_f T_g - T_{fg} \in \mathcal{K}(H)$$

shows that any element T of this C^*-algebra can be written as

$$T = T_f + K,$$

where K is a compact operator. In fact this decomposition is unique and gives another definition of the symbol map σ by $\sigma(T_f + K) = f$. It is also clear from extension (4.23) that a Toeplitz operator T is Fredholm if and only if its symbol $\sigma(T)$ is an invertible function on S^1.

The algebra generated by Toeplitz operators T_f for $f \in C^\infty(S^1)$ is called the *smooth Toeplitz algebra* $\mathcal{T}^\infty \subset \mathcal{T}$. Similar to (4.23) we have an extension

$$0 \to \mathcal{K}^\infty \to \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty(S^1) \to 0. \tag{4.24}$$

The Toeplitz extension (4.23) has a grand generalization. On any closed smooth manifold M, a (scalar) pseudodifferential operator D of order zero defines a bounded linear map $D\colon L^2(M) \to L^2(M)$ and its principal symbol $\sigma(D)$ is a continuous function on $S^*(M)$, the unit cosphere bundle of M with respect to a Riemannian metric. Let $\Psi^0(M) \subset \mathcal{L}(L^2(M))$ denote the C^*-algebra generated by all pseudodifferential operators of order zero on M. We then have a short exact sequence of C^*-algebras

$$0 \to \mathcal{K}(L^2(M)) \to \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \to 0.$$

For $M = S^1$, the cosphere bundle splits as the disjoint union of two copies of S^1 and the above sequence is the direct sum of two identical copies, each of which is isomorphic to the Toeplitz extension (4.23).

Exercise 4.3.1. Give an example of a Banach algebra A and a dense subalgebra $B \subset A$ such that B is *not* stable under holomorphic functional calculus in A but for all $a \in B$, $\mathrm{sp}_B(a) = \mathrm{sp}_A(a)$.

Exercise 4.3.2 (Smooth compact operators)**.** Let $\mathcal{K}^\infty \subset \mathcal{K}(\ell^2(\mathbb{N}))$ be the algebra of infinite matrices (a_{ij}) with rapid decay coefficients. Show that $\mathcal{K}^\infty$ is stable under holomorphic functional calculus in the algebra of compact operators $\mathcal{K}$.

Exercise 4.3.3. Show that the smooth Toeplitz algebra $\mathcal{T}^\infty$ is stable under holomorphic functional calculus in $\mathcal{T}$.

Exercise 4.3.4. Show that

$$\varphi(A, B) := \mathrm{Tr}([A, B])$$

defines a cyclic 1-cocycle on $\mathcal{T}^\infty$. If f is a smooth non-vanishing function on the circle, show that

$$\mathrm{index}(T_f) = \varphi(T_f, T_{f^{-1}}).$$

4.4 A final word: basic noncommutative geometry in a nutshell

A better title for this last section would be 'This book in one diagram'. It is now time to integrate the various concepts and results we covered so far into one single big idea. In fact many aspects of noncommutative geometry that we have covered so far can be succinctly encapsulated into one commutative diagram. Rewriting Connes' index formulae (Theorem 4.2.1 and Proposition 4.2.5) in a commutative diagram, we obtain

$$\begin{array}{ccc} \mathfrak{K}^*(A) \times K_*(A) & \xrightarrow{\text{index}} & \mathbb{Z} \\ \Big\downarrow \mathrm{Ch}^* \quad \Big\downarrow \mathrm{Ch}_* & & \Big\downarrow \\ HP^*(A) \times HP_*(A) & \longrightarrow & \mathbb{C}. \end{array} \tag{4.25}$$

Furthermore the following holds:

1) A is an algebra which may very well be noncommutative.

2) $\mathfrak{K}^*(A)$ is the *set* of even resp. odd finitely summable Fredholm modules over A. It is closely related to the K-homology of A.

3) $K_*(A)$ is the algebraic K-theory of A.

4) Ch^* is Connes–Chern character in K-homology.

5) Ch_* is Connes–Chern character in K-theory.

6) $HP^*(A)$ is the periodic cyclic cohomology of A and $HP_*(A)$ is the periodic cyclic homology of A.

7) The top row is the *analytic index map*. It computes the Fredholm index of a Fredholm module twisted by a K-theory class.

8) The bottom row is the natural pairing between cyclic cohomology and homology. Once composed with vertical arrows it gives the *topological index maps*

$$[(H, F, \gamma)] \times [e] \mapsto \langle \mathrm{Ch}^0(H, F, \gamma), \mathrm{Ch}_0(e) \rangle,$$

$$[(H,\, F)] \times [u] \mapsto \langle \mathrm{Ch}^1(H,\, F),\, \mathrm{Ch}_1(u) \rangle.$$

So the commutativity of the diagram amounts to the following equality:

$$\boxed{\text{Topological Index} = \text{Analytic Index}} \tag{4.26}$$

Notice that the Atiyah–Singer index theorem amounts to an equality of the above type, where in this classical case $A = C^\infty(M)$ is a commutative algebra.

We can summarize what we have done in this book as a way of extending (4.26) beyond its classical realm of manifolds and differential operators to a noncommutative world. The following ingredients were needed:

1. What is a noncommutative space and how to construct one? This was discussed in Chapters 1 and 2. We saw that a major source of noncommutative spaces is noncommutative quotients, replacing bad quotients by groupoid algebras.

2. Cohomological apparatus. This includes noncommutative analogues of topological invariants such as K-theory, K-homology, de Rham cohomology, and Chern character maps. Cyclic cohomology and the theory of characteristic classes in noncommutative geometry as we discussed in Chapters 3 and 4 are the backbone of noncommutative geometry.

Now, the diagram (4.25) should be seen as the prototype of a series of results in noncommutative geometry that aims at expressing the analytic index by a topological formula. In the next step it would be desirable to have a *local* expression for the topological index, that is, for the Connes–Chern character Ch^i. The *local index formula* of Connes and Moscovici [54] solves this problem by replacing the characteristic classes $\mathrm{Ch}^i(H,\, F)$ by a cohomologous cyclic cocycle $\mathrm{Ch}^i(H,\, D)$. Here D is an unbounded operator that defines a refinement of the notion of Fredholm module to that of a *spectral triple*, and F is the *phase* of D.

In the above we tried to summarize the way index theory informed and influenced noncommutative geometry in its first phase of development up to the year 1985 and the publication of [39]. To gain an idea of what happened next we refer the reader to the introduction and references cited there. It suffices to say that it is the *spectral geometry* that takes the center stage now. And here I shall stop!

Appendix A

Gelfand–Naimark theorems

In Section 1.1 we defined notions of Banach and C^*-algebras and gave several examples of C^*-algebras that frequently occurs in noncommutative geometry. The classic paper of Gelfand and Naimark [82] is the birth-place of the theory of C^*-algebras. The following two results on the structure of C^*-algebras are proved in this paper.

Theorem A.1 (Gelfand–Naimark; [82])**.** a) *For any commutative C^*-algebra A with spectrum $\hat{A}$ the Gelfand transform*

$$A \to C_0(\hat{A}), \quad a \mapsto \hat{a}, \tag{A.1}$$

defines an isomorphism of C^*-algebras.

b) *Any C^*-algebra is isomorphic to a C^*-subalgebra of the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H.*

In the remainder of this appendix we sketch the proofs of statements a) and b) above. They are based on Gelfand's theory of commutative Banach algebras, and the Gelfand–Naimark–Segal (GNS) construction of representations of C^*-algebras from states, respectively.

A.1 Gelfand's theory of commutative Banach algebras

The whole theory is based on the notion of the *spectrum* of an element of a Banach algebra and the fact that the spectrum is non-empty (and compact). The notion of spectrum can be defined for elements of an arbitrary algebra and it can be easily shown that for finitely generated complex algebras the spectrum is non-empty. As is shown in [33], this latter fact leads to an easy proof of Hilbert's Nullstellensatz (over $\mathbb{C}$).

Let A be a unital complex algebra. Algebras are not assumed to be commutative, unless explicitly stated so. The *spectrum* of an element $a \in A$ is defined as follows:

$$\boxed{\operatorname{sp}(a) = \{\lambda \in \mathbb{C};\ a - \lambda 1 \text{ is } \textit{not} \text{ invertible}\}}$$

The complement of the spectrum $\mathbb{C} - \operatorname{sp}(a)$ is called the *resolvent set* of a and the function $R_a : \mathbb{C} - \operatorname{sp}(a) \to A$ sending λ to $(a - \lambda 1)^{-1}$ is the *resolvent function* of a. We should think of the spectrum as the noncommutative analogue of the set of values of a function. This is justified in the following example.

Example A.1. 1. Let $A = C(X)$ be the algebra of continuous complex-valued functions on a compact space X. For any $f \in A$,

$$\operatorname{sp}(f) = \{f(x);\ x \in X\},$$

is the range of f.

2. Let $A = M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with coefficients in $\mathbb{C}$. For any matrix $a \in A$,

$$\operatorname{sp}(a) = \{\lambda \in \mathbb{C};\ \det(a - \lambda 1) = 0\}$$

is the set of eigenvalues of a.

3. The theory of measurements in quantum mechanics postulates that when an observable, i.e., a selfadjoint operator, is measured one actually finds a point in the spectrum of the operator. This should be compared with classical measurements where one evaluates a real valued function, i.e., a classical observable, at a phase space point.

In general, the spectrum may be empty. We give two general results that guarantee the spectrum is non-empty. They are at the foundation of the Gelfand–Naimark theorem and Hilbert's Nullstellensatz. Part b) is in [33].

Theorem A.2. a) (Gelfand) *Let A be a unital Banach algebra over $\mathbb{C}$. Then for any $a \in A$, $\operatorname{sp}(a) \neq \varnothing$.*

b) *Let A be a unital algebra over $\mathbb{C}$. Assume that $\dim_{\mathbb{C}} A$ is countable. Then for any $a \in A$, $\operatorname{sp}(a) \neq \varnothing$. Furthermore, an element a is nilpotent if and only if $\operatorname{sp}(a) = \{0\}$.*

Proof. We sketch a proof of both statements. For a) assume that the spectrum of an element a is empty. Then the function

$$R : \mathbb{C} \to A, \quad \lambda \mapsto (a - \lambda 1)^{-1},$$

is holomorphic (in an extended sense), non-constant, and bounded. This is easily shown to contradict the classical Liouville theorem from complex analysis.

For b), again assume that the spectrum of a is empty. Then it is easy to see that the *uncountable* set

$$\{(a - \lambda 1)^{-1};\ \lambda \in \mathbb{C}\}$$

is linearly independent over $\mathbb{C}$. But this contradicts the fact that $\dim_{\mathbb{C}} A$ is countable.

For the second part of b), assume $\mathrm{sp}(a) = \{0\}$. Then, using a similar argument as above, we can find constants c_i, μ_i, $i = 1, \dots n$, in $\mathbb{C}$ such that $\sum_i c_i(a-\mu_i)^{-1} = 0$. It follows that a satisfies a polynomial equation. Let

$$p(a) = a^k(a - \lambda_1) \dots (a - \lambda_n) = 0$$

be the minimal polynomial of A. Then $n = 0$ since otherwise, for some i, an element $a - \lambda_i$ would be non-invertible with $\lambda_i \neq 0$ and this would contradict our assumption that $\mathrm{sp}(a) = \{0\}$. Conversely, if a is nilpotent, by a geometric series argument one can show that $(a - \lambda)$ is invertible for any $\lambda \neq 0$. □

The first part of the following corollary is known as the *Gelfand–Mazur theorem*.

Corollary A.1. *Let A be either a unital complex Banach algebra or a unital complex algebra such that $\dim_{\mathbb{C}} A$ is countable. If A is a division algebra, then $A \simeq \mathbb{C}$.*

Let A be an algebra. By a *character* of A we mean a nonzero algebra homomorphism

$$\varphi\colon A \to \mathbb{C}.$$

Note that if A is unital, then $\varphi(1) = 1$. We establish the link between characters and maximal ideals of A. For the following result A is either a commutative unital complex Banach algebra, or is a commutative unital algebra with $\dim_{\mathbb{C}} A$ countable. Using Corollary A.1, we have:

Corollary A.2. *The relation $I = \ker \varphi$ defines a one-to-one correspondence between the set of maximal ideals of A and the set of characters of A.*

Before embarking on the proof of the Gelfand–Naimark theorem, we sketch a proof of Hilbert's Nullstellensatz, following [33].

Let

$$A = \mathbb{C}[x_1, \dots, x_n]/I$$

be a finitely generated commutative *reduced algebra*. Recall that reduced means if $a^n = 0$ for some n then $a = 0$ (no nilpotent elements). Equivalently, the ideal I is radical. Let

$$V = \{z \in \mathbb{C}^n;\ p(z) = 0,\ \text{for all } p \text{ in } I\},$$

let $J(V)$ be the ideal of functions vanishing on V, and let

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/J(V)$$

be the algebra of regular functions on V. Since $I \subset J(V)$, we have an algebra homomorphism

$$\pi\colon A \to \mathbb{C}[V].$$

One of the original forms of Hilbert's Nullstellensatz states that this map is an isomorphism. It is clearly surjective. For its injectivity, let $a \in A$ and let $\pi(a) = 0$, or equivalently $\pi(a) \in J(V)$. Since a vanishes on all points of V, it follows that a is in the intersection of all maximal ideals of A. This shows that its spectrum $\mathrm{sp}(a) = \{0\}$. By Theorem A.2 b), it follows that a is nilpotent and since A is reduced, we have $a = 0$.

The rest of this section is devoted to sketching a proof of the Gelfand–Naimark theorem on the structure of commutative C^*-algebras. Let A be a unital Banach algebra. The spectrum $\mathrm{sp}(a)$ of an element $a \in A$ is a non-empty *compact* subset of $\mathbb{C}$. We only need to check the compactness. In fact the geometric series formula $(1-a)(1+a+a^2+\cdots) = 1$ shows that $1-a$ is invertible when $\|a\| < 1$. From this it easily follows that for any a and any $\lambda \in \mathrm{sp}(a)$ we have $|\lambda| \leq \|a\|$ and that the complement of $\mathrm{sp}(a)$ is an open set. So $\mathrm{sp}(a)$ is bounded and closed and hence is compact. The *spectral radius* of $a \in A$ is the number $r(a) := \max\{|\lambda|;\ \lambda \in \mathrm{sp}(a)\}$. It is given by *Beurling's formula* (cf. e.g. [122] for a proof):

$$r(a) = \mathrm{Lim}_{n\to\infty}\ \|a^n\|^{\frac{1}{n}} = \inf_{n\geq 1}\ \|a^n\|^{\frac{1}{n}}.$$

Now let A be a C^*-algebra and let $x \in A$ be *selfadjoint*. Then, using the C^*-identity, we have $\|x^2\| = \|x^*x\| = \|x\|^2$, and in general $\|x^{2^n}\| = \|x\|^{2^n}$. So, we have

$$r(x) = \mathrm{Lim}_{n\to\infty}\, \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

From this it follows that for *any* $x \in A$, we have

$$\|x\| = r(x^*x)^{\frac{1}{2}}. \tag{A.2}$$

This is indeed a remarkable result as it shows that the norm of a C^*-algebra is completely determined by its algebraic structure and therefore is *unique*. As another corollary we mention that if $f\colon A \to B$ is a C^*-map between C^*-algebras A and B, then f is automatically continuous and in fact $\|f\| \leq 1$. To prove this we can assume that A and B are unital and f is unit preserving. Then since $\mathrm{sp}(f(a)) \subset f(\mathrm{sp}(a))$, we have

$$\|f(x)\|^2 = \|f(x^*x)\| = r(f(x^*x)) \leq r(x^*x) = \|x\|^2.$$

Let A be a unital Banach algebra. It is easy to see that any character φ of A is continuous of norm 1. To prove this, note that if this is not the case then there exists an $a \in A$ with $\|a\| < 1$ and $\varphi(a) = 1$. Let $b = \sum_{n\geq 1} a^n$. Then from $a + ab = b$, we have

$$\varphi(b) = \varphi(a) + \varphi(a)\varphi(b) = 1 + \varphi(b),$$

which is impossible. Therefore $\|\varphi\| \leq 1$, and since $\varphi(1) = 1$, $\|\varphi\| = 1$. From this it follows that for any A, unital or not, and any character φ on A, $\|\varphi\| \leq 1$.

Now let us recall the definition of the *Gelfand transform* from Section 1.1. Let A be a commutative Banach algebra and let $\hat{A}$ denote the *spectrum* of A, defined as the space of characters of A. The Gelfand transform is the map $\Gamma\colon A \to C_0(\hat{A})$, defined by

$$\Gamma(a) = \hat{a}, \quad \hat{a}(\varphi) = \varphi(a),$$

for any $a \in A$ and $\varphi \in \hat{A}$. This map is obviously an algebra homomorphism. Since characters are contractive, we have $|(\Gamma(a))(\varphi)| = |\varphi(a)| \leq \|a\|$, so that $\|\Gamma\| \leq 1$, i.e., Γ is contractive.

The kernel of the Gelfand transform is called the *radical* of A. It consists of elements a whose spectral radius $r(a) = 0$, or equivalently, $\mathrm{sp}(a) = \{0\}$. Hence the radical contains all the nilpotent elements, but it may be bigger. An element a is called *quasi-nilpotent* if $\mathrm{sp}(a) = 0$. A is said to be *semi-simple* if its radical is zero, i.e., the only quasi-nilpotent elements of A is 0. Now if A is a commutative C^*-algebra and $x \in A$ is quasi-nilpotent, then x^*x is quasi-nilpotent as well and using (A.2) we see that $x = 0$. This shows that the Gelfand transform is *injective* for C^*-algebras.

We are now ready to prove the Gelfand–Naimark theorem for commutative C^*-algebras. First we need to show that Γ preserves the $*$-structure. Let $a \in A$ be a selfadjoint element of a commutative C^*-algebra A and let $\varphi \in \hat{A}$ be a character. Then $\varphi(a)$ is real. In fact since $e^{ia} := \sum_{n\geq 0} \frac{(ia)^n}{n!}$ is a unitary, we have $|e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$ and therefore $\varphi(a)$ is real. Now any $a \in A$ can be written as $a = x + iy$ with x and y selfadjoint. So we have

$$\varphi(a^*) = \varphi(x - iy) = \varphi(x) - i\varphi(y) = \overline{\varphi(x) + i\varphi(y)} = \overline{\varphi(a)}.$$

This shows that Γ is a $*$-map. We can also show that, for C^*-algebras, Γ is isometric. In fact, for any $a \in A$, we have

$$\|a\|^2 = r(a^*a) = \|\Gamma(a^*a)\|_\infty = \|\overline{\Gamma(a)}\Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2.$$

We can now prove the first Gelfand–Naimark theorem.

Theorem A.3 (Gelfand–Naimark)**.** *Let A be a commutative C^*-algebra. Then the Gelfand transform $\Gamma\colon A \to C_0(\hat{A})$ is an isomorphism of C^*-algebras.*

Proof. We prove the unital case. The non-unital case follows with minor modifications [66]. What we have shown so far is that Γ is an isometric $*$-algebra map whose image separates the points of the spectrum $\hat{A}$, and contains the constant functions. Since Γ is isometric its image is closed and thus, by the Stone–Weierstrass theorem, $\Gamma(A) = C(\hat{A})$. □

The above theorem is a landmark application of Gelfand's theory of commutative Banach algebras. While a complete classification of commutative Banach algebras seems to be impossible, this result classifies all commutative C^*-algebras. Another striking application was Gelfand's very simple proof of a classical result of Wiener to the effect that if f is a nowhere vanishing function with an absolutely convergent Fourier series then $\frac{1}{f}$ has the same property [122].

Example A.2. We give an example of a commutative Banach algebra for which the Gelfand transform is injective, in fact isometric, but not surjective. Let $A = H(D)$ be the space of continuous functions on the unit disk D which are holomorphic in the interior of the disk. With the sup-norm $\|f\| = \|f\|_\infty$ it is a Banach algebra. It is, however, not a C^*-algebra (why?). It is easy to check that $\hat{A} \simeq D$ and the Gelfand transform coincides with the embedding $H(D) \hookrightarrow C(D)$.

A.2 States and the GNS construction

Our goal in this section is to sketch a proof of the second main result of Gelfand and Naimark in [82] to the effect that any C^*-algebra can be embedded in the algebra of bounded operators on Hilbert space. The main idea of the proof is an adaptation of the idea of *left regular representation* to the context of C^*-algebras. This is the Gelfand–Naimark–Segal (GNS) construction and is based on the concept of *states* of a C^*-algebra.

The concept of state is the noncommutative analogue of Borel probability measure. A state of a unital C^*-algebra A is a positive normalized linear functional $\varphi\colon A \to \mathbb{C}$, namely φ is $\mathbb{C}$-linear and satisfies

$$\varphi(a^*a) \geq 0 \quad \text{for all } a \in A \quad \text{and} \quad \varphi(1) = 1.$$

The quantum mechanical intuition behind this concept is important too. We can think of a C^*-algebra as the algebra of operators generated by the set of observables of a quantum system. In fact one of the postulates of quantum mechanics dictates that an observable, e.g. energy or angular momentum, of a quantum system is a (selfadjoint unbounded) operator on the Hilbert space while the states of the system are represented by rays (or unit vectors, up to phase) in the Hilbert space. In the abstract formulation of quantum mechanics we abandon the idea of Hilbert space and instead start with a C^*-algebra while states are defined as above. The *expectation value* of an element (an 'observable') $a \in A$, when the system is in the state φ, is defined by $\varphi(a)$. This terminology is also directly related and motivated by the notion of states in classical statistical mechanics, where one abandons the idea of describing the state of a system by a point in the phase space. Instead, the only reasonable question to ask is the probability of finding the system within a certain region in the phase space. This probability is of course given by a probability measure μ. Then the expected value of an observable $f\colon M \to \mathbb{R}$, if the system is in the state μ, is $\int f\, d\mu$.

Similarly, in quantum statistical mechanics [20] the idea of describing the quantum states of a system by a vector (or ray) in Hilbert space is extended and instead of a *pure state* one uses a *mixed state* or a *density matrix*, i.e., a trace class positive operator p with $\mathrm{Tr}(p) = 1$. The expectation value of an observable a, if the system is in the state p, is given by $\mathrm{Tr}(ap)$. As we shall see later these are exactly the (normal) states of the C^*-algebra $\mathcal{L}(H)$. We go back to mathematics now.

A *positive linear functional* on a C^*-algebra A is a $\mathbb{C}$-linear map $\varphi\colon A \to \mathbb{C}$

such that for all a in A,

$$\varphi(a^*a) \geq 0.$$

Equivalently $\varphi(x) \geq 0$ for any positive element $x \in A$. (By definition, x is called *positive* if it is selfadjoint and $\text{sp}(a) \subset \mathbb{R}$.) Positive functionals are automatically bounded. A *state* on A is a positive linear functional φ with $\|\varphi\| = 1$. It can be shown that if A is unital then this last condition is equivalent to $\varphi(1) = 1$.

If φ_1 and φ_2 are states then for any $t \in [0,1]$, $t\varphi_1 + (1-t)\varphi_2$ is a state as well. Thus the set of states of A, denoted by $\mathcal{S}(A)$, forms a convex subset of the unit ball of A^*. The extreme points of $\mathcal{S}(A)$ are called *pure states*. Other states are *mixed states*. A state φ is said to be *faithful* if $\varphi(aa^*) = 0$ implies $a = 0$. A *tracial state* is a state which is a trace at the same time, i.e., $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.

Example A.3. 1. States are noncommutative analogues of probability measures. This idea is corroborated by the Riesz representation theorem: For a locally compact Hausdorff space X there is a one-to-one correspondence between states on $C_0(X)$ and *regular* Borel probability measures on X. To such a probability measure μ is associated the state φ defined by

$$\varphi(f) = \int_X f \, d\mu.$$

φ is a pure state if and only if $\mu = \delta_x$ is a Dirac measure for a point $x \in X$.

2. Let $A = M_n(\mathbb{C})$ and $p \in A$ be a positive matrix with $\text{tr}(p) = 1$. (Such matrices, and their infinite dimensional analogues, are called *density matrices* or *density operators* in quantum statistical mechanics.) Then

$$\varphi(a) = \text{tr}(ap) \tag{A.3}$$

defines a state on A. It is easy to see that all states on $M_n(\mathbb{C})$ are obtained this way. φ is pure if and only p is a rank one projection.

3. Let $A = \mathcal{L}(H)$ be the algebra of bounded operators on the Hilbert space H and let $p \in \mathcal{L}(H)$ be a *density operator*, namely a positive trace class operator with $\text{Tr}(p) = 1$. Then (A.3) defines a state on $\mathcal{L}(H)$, and again all *normal* states on $\mathcal{L}(H)$ are of this form [15].

4. Let $\pi \colon A \to \mathcal{L}(H)$ be a *representation* of a unital C^*-algebra A on the Hilbert space H. This simply means that π is a morphism of unital C^*-algebras. Let $x \in H$ be a vector of length one. Then

$$\varphi(a) = \langle \pi(a)x, \, x \rangle$$

defines a state on A, called a *vector state*. In the following we show that, conversely, any state on A is a vector state with respect to a suitable representation, called the GNS (Gelfand–Naimark–Segal) representation.

5. For a final example let $A = C_r^*G$ be the reduced group C^*-algebra of a *discrete group* G. In Example 1.1.6 we defined a trace on A by

$$\tau(a) = \langle a\delta_e, \, \delta_e \rangle, \quad a \in C_r^*G.$$

Clearly τ is a vector state, is faithful, and since it is a trace it is called a *tracial state*. Notice that not all states have the trace property. For example on $M_n(\mathbb{C})$ the only tracial state is the normalized canonical trace $\frac{1}{n}\mathrm{tr}$ corresponding to $p = \frac{1}{n}I$ in (A.3).

Let φ be a positive linear functional on A. Then

$$\langle a, \, b \rangle = \varphi(b^*a)$$

is a semi-definite sesquilinear form on A. Thus it satisfies the *Cauchy–Schwarz* inequality: for all a, b,

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Let

$$N = \{a \in A; \, \varphi(a^*a) = 0\}.$$

It is easy to see, using the above Cauchy–Schwarz inequality, that N is a *closed left ideal* of A. So the following positive-definite inner product is well defined on the quotient space A/N:

$$\langle a + N, \, b + N \rangle := \langle a, \, b \rangle.$$

Let H_φ denote the Hilbert space completion of A/N under the above inner product. The *left regular representation* $A \times A \to A$, $(a, b) \mapsto ab$ of A on itself induces a bounded linear map $A \times A/N \to A/N, (a, b + N) \mapsto ab + N$. We denote its unique extension to H_φ by

$$\pi_\varphi \colon A \to \mathcal{L}(H_\varphi).$$

The representation (π_φ, H_φ) is called the GNS representation defined by the state φ. The state φ can be recovered from the representation (π_φ, H_φ) as a vector state as follows. Let $\Omega := \pi_\varphi(1)$. Then for all a in A,

$$\varphi(a) = \langle \pi_\varphi(a)(\Omega), \, \Omega \rangle.$$

It is natural to ask if the GNS construction has a universal property. In fact it does: For any other representation $\pi \colon A \to \mathcal{L}(H)$ such that $\varphi(a) = \langle \pi(a)(v), \, v \rangle$ for some unit vector $v \in H$, there is a unique isometry $V \colon H_\varphi \to H$ such that $\pi(a) = V\pi_\varphi(a)V^*$ for all $a \in A$.

The representation (π_φ, H_φ) may not be *faithful*. It can be shown that it is irreducible if and only if φ is a pure state [66]. To construct a faithful representation, and hence an embedding of A into the algebra of bounded operators on a Hilbert space, one first shows that there are enough pure states on A. The proof of the following result is based on the Hahn–Banach and Krein–Milman theorems.

Lemma A.1. *For any selfadjoint element a of A, there exists a pure state φ on A such that $|\varphi(a)| = \|a\|$.*

Using the GNS representation associated to φ, we can then construct, for any $a \in A$, an irreducible representation π of A such that $\|\pi(a)\| = |\varphi(a)| = \|a\|$.

We can now prove the second theorem of Gelfand and Naimark.

Theorem A.4 (Gelfand–Naimark)**.** *Every C^*-algebra is isomorphic to a C^*-subalgebra of the algebra of bounded operators on a Hilbert space.*

Proof. Let $\pi = \sum_{\varphi \in \mathcal{S}(A)} \pi_\varphi$ denote the direct sum of all GNS representations for all states of A. By the above remark π is faithful. □

Example A.4. We give a couple of simple examples of GNS representations. Let $A = C(X)$ and φ be the state defined by a probability measure μ on a compact space X. Then $H_\varphi \simeq L^2(X, \mu)$ and the GNS representation is the representation of $C(X)$ as multiplication operators on $L^2(X, \mu)$. For example if $\mu = \delta_x$ is the Dirac mass at $x \in X$ then $H_\varphi \simeq \mathbb{C}$ and $\pi_\varphi(f) = f(x)$ for all $f \in C(X)$.

For a simple noncommutative example let $A = M_n(\mathbb{C})$ and let $\varphi(a) = \mathrm{Tr}(ap)$, where p is a rank one projection in A. Then $H_\varphi \simeq \mathbb{C}^n$ and the GNS representation is the standard representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$. If on the other extreme we let $\varphi(a) = \frac{1}{n}\,\mathrm{Tr}(a)$, then $H_\varphi = M_n(\mathbb{C})$ where the inner product is the Hilbert–Schmidt product $\langle a, b\rangle = \frac{1}{n}\,\mathrm{Tr}(b^*a)$. The action of A is simply by left multiplication (left regular representation).

In the remainder of this section we look at the Kubo–Martin–Schwinger (KMS) equilibrium condition for states and some of its consequences. KMS states replace the Gibbs equilibrium states for interacting systems with infinite number of degrees of freedom. See [20] for an introduction to quantum statistical mechanics; see also [52] for relations between quantum statistical mechanics, number theory and noncommutative geometry. For relations with Tomita–Takesaki theory and Connes' classification of factors the best reference is [41].

Let (A, G, σ) be a C^*-dynamical system consisting of a C^*-algebra A, a locally compact group G and a continuous action

$$\sigma\colon G \to \mathrm{Aut}(A)$$

of G on A. Of particular interest is when $G = \mathbb{R}$. Then the dynamical system represents a quantum mechanical system evolving in time. For example, by Stone's theorem one knows that continuous one-parameter groups of automorphisms of $A = \mathcal{L}(\mathcal{H})$ are of the form

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where H, the *Hamiltonian* of the system, is a selfadjoint, in general unbounded, operator on $\mathcal{H}$. Assuming the operator $e^{-\beta H}$ is trace class, the corresponding *Gibbs equilibrium state* at inverse temperature $\beta = \frac{1}{kT} > 0$ is the state

$$\varphi(a) = \frac{1}{Z(\beta)}\ \mathrm{Tr}(a e^{-\beta H}), \tag{A.4}$$

where the *partition function* Z is defined by

$$Z(\beta) = \operatorname{Tr}(e^{-\beta H}).$$

According to Feynman, formula (A.4) for the Gibbs equilibrium state (and its classical analogue) is the apex of statistical mechanics. It should however be added that (A.4) is not powerful enough to deal with systems with infinite number of degrees of freedom (cf. [74], [20], [41] for explanations and examples), and in general it should be replaced by the KMS equilibrium condition.

Let (A, σ_t) be a C^*-dynamical system evolving in time. A state $\varphi\colon A \to \mathbb{C}$ is called a *KMS state at inverse temperature* $\beta > 0$ if for all $a, b \in A$ there exists a function $F_{a,b}(z)$ which is continuous and bounded on the closed strip $0 \leq \operatorname{Im} z \leq \beta$ in the complex plane and holomorphic in the interior such that for all $t \in \mathbb{R}$,

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

Let $\mathcal{A} \subset A$ denote the set of *analytic vectors* of σ_t consisting of those elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to a holomorphic function on $\mathbb{C}$. One shows that $\mathcal{A}$ is a dense $*$-subalgebra of A. Now the KMS condition is equivalent to a *twisted trace property* for φ: for all analytic vectors $a, b \in \mathcal{A}$ we have

$$\varphi(ba) = \varphi(a\sigma_{i\beta}(b)).$$

Notice that the automorphism $\sigma_{i\beta}$ obtained by analytically continuing σ_t to imaginary time (in fact imaginary temperature!) is only densely defined.

Example A.5. Any Gibbs state is a KMS state as can be easily checked.

Example A.6 (Hecke algebras [18], [52])**.** A subgroup Γ_0 of a group Γ is called *almost normal* if every left coset $\gamma\Gamma_0$ is a *finite* union of right cosets. In this case we say (Γ, Γ_0) is a *Hecke pair*. Let $L(\gamma)$ denote the number of distinct right cosets $\Gamma_0\gamma_i$ in the decomposition

$$\gamma\Gamma_0 = \bigcup_i \Gamma_0\gamma_i,$$

and let $R(\gamma) = L(\gamma^{-1})$.

The rational *Hecke algebra* $\mathcal{A}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ of a Hecke pair (Γ, Γ_0) consists of functions with finite support

$$f\colon \Gamma_0 \setminus \Gamma \to \mathbb{Q}$$

which are right Γ_0-invariant, i.e., $f(\gamma\gamma_0) = f(\gamma)$ for all $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma_0$. Under the convolution product

$$(f_1 * f_2)(\gamma) := \sum_{\Gamma_0\setminus\Gamma} f_1(\gamma\gamma_1^{-1}) f_2(\gamma_1),$$

$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is an associative unital algebra. Its complexification

$$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

is a $*$-algebra with an involution given by

$$f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

Notice that if Γ_0 is normal in Γ then one obtains the group algebra of the quotient group Γ/Γ_0. We refer to [18], [52] for the C^*-completion of $\mathcal{A}_{\mathbb{C}}$, which is similar to the C^*-completion of group algebras. There is a one-parameter group of automorphisms of this Hecke algebra (and its C^*-completion) defined by

$$(\sigma_t f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma).$$

Let P^+ denote the subgroup of the "$ax+b$" group with $a > 0$. The corresponding C^*-algebra for the Hecke pair (Γ_0, Γ) where $\Gamma = P^+_{\mathbb{Q}}$ and $\Gamma_0 = P^+_{\mathbb{Z}}$ is the Bost-Connes C^*-algebra. One feature of these systems is that their partition functions are expressible in terms of zeta and L-functions of number fields.

Given a normal and faithful state φ on a C^*-algebra A one may ask if there is a one-parameter group of automorphisms of A for which φ is a KMS state at inverse temperature $\beta = 1$. Thanks to Tomita's theory (cf. [41], [15]) one knows that the answer is positive if A is a von Neumann algebra, which we will denote by M now. The corresponding automorphism group σ_t^{φ}, called the *modular automorphism group*, is uniquely defined subject to the condition $\varphi\sigma_t^{\varphi} = \varphi$ for all $t \in \mathbb{R}$.

A von Neumann algebra typically carries many states. One of the first achievements of Connes, which set his grand classification program of von Neumann algebras in motion, was his proof that the modular automorphism group is unique up to inner automorphisms. More precisely, for any other state ψ on M there is a continuous map u from $\mathbb{R}$ to the group of unitaries of M such that

$$\sigma_t^{\varphi}(x) = u_t \sigma_t^{\psi}(x) u_t^{-1} \quad \text{and} \quad u_{t+s} = u_t \sigma_s^{\varphi} u_s.$$

It follows that the modular automorphism group is independent, up to inner automorphisms, of the state (or weight) and if $\mathrm{Out}(M)$ denotes the quotient of the group of automorphisms of M by inner automorphisms, any von Neumann algebra has a god-given dynamical system

$$\sigma \colon \mathbb{R} \to \mathrm{Out}(M)$$

attached to it. This is a purely non-abelian phenomenon, as the modular automorphism group is trivial for abelian von Neumann algebras as well for type II factors. For type III factors it turns out that the modular automorphism group possesses a complete set of invariants for the isomorphism type of the algebra in the injective case. This is the beginning of Connes' grand classification theorems for von Neumann algebras, for which we refer the reader to his book [41] and references therein.

Exercise A.1. 1) Show that if a is nilpotent then $\mathrm{sp}(a) = \{0\}$.

2) Show that

$$\mathrm{sp}(ab) \setminus \{0\} = \mathrm{sp}(ba) \setminus \{0\}.$$

Appendix B

Compact operators, Fredholm operators, and abstract index theory

The theory of operators on Hilbert space is essential for noncommutative geometry. Operator theory is the backbone of von Neumann and C^*-algebras and these are natural playgrounds for noncommutative measure theory and topology. We saw, for example, that K-homology has a natural formulation in operator theoretic terms using compact and Fredholm operators and it is this formulation that lends itself to generalization to the noncommutative setup. Similarly, the more refined aspects of noncommutative geometry, like noncommutative metric and Riemannian geometry, can only be formulated through spectral invariants of operators on Hilbert space.

We assume that the reader is familiar with concepts of Hilbert space, bounded operators on Hilbert space, and basic spectral theory as can be found in the first chapters of, e.g. [122], [153], [70]. A good reference for ideals of compact operators is [165]. For basic Fredholm theory and abstract index theory we recommend [70], [122]. In this section H will always stand for a Hilbert space over the complex numbers and $\mathcal{L}(H)$ for the algebra of bounded linear operators on H. The adjoint of an operator T shall be denoted by T^*.

Our first task in this section is to introduce several classes of ideals in $\mathcal{L}(H)$, most notably ideals of compact operators and the Schatten ideals. Let $\mathcal{F}(H)$ denote the set of *finite rank* operators on H, i.e., operators whose range is finite dimensional. $\mathcal{F}(H)$ is clearly a two-sided $*$-ideal in $\mathcal{L}(H)$ and in fact it is easy to show that it is the smallest proper two-sided ideal in $\mathcal{L}(H)$.

Let

$$\mathcal{K}(H) := \overline{\mathcal{F}(H)}$$

be the norm closure of $\mathcal{F}(H)$. It is clearly a norm closed two-sided $*$-ideal in $\mathcal{L}(H)$. An operator T is called *compact* if $T \in \mathcal{K}(H)$. Let H_1 denote the closed unit ball of H. It can be shown that an operator $T \in \mathcal{L}(H)$ is compact if and only if the norm closure $\overline{T(H_1)}$ is a compact subset of H in norm topology. It follows that the range of a compact operator can never contain a closed infinite dimensional subspace. The spectrum of a compact operator is a countable subset of $\mathbb{C}$ with 0 as its only possible limit point. Any nonzero point in the spectrum is an eigenvalue whose corresponding eigenspace is finite dimensional. For a compact operator T, let

$$\mu_1(T) \geq \mu_2(T) \geq \mu_3(T) \geq \cdots$$

denote the sequence of *singular values* of T. By definition, $\mu_n(T)$ is the n-th eigenvalue of $|T| := (T^*T)^{\frac{1}{2}}$, the absolute value of T,

It can be shown that if H is separable and infinite dimensional, which is the case in almost all examples, then $\mathcal{K}(H)$ is the unique proper and closed two-sided ideal of $\mathcal{L}(H)$. In this case it is also the largest proper two-sided ideal of $\mathcal{L}(H)$. Thus for any other two sided *operator ideal* $\mathcal{J}$ we have

$$\mathcal{F}(H) \subset \mathcal{J} \subset \mathcal{K}(H).$$

An interesting point of view, advocated by Connes and of fundamental importance for noncommutative geometry [41], is that compact operators are the true counterparts of infinitesimals in noncommutative geometry. If we regard $\mathcal{L}(H)$ as a replacement for $\mathbb{C}$ in noncommutative geometry (as in going from *c-numbers* to *q-numbers* in quantum mechanics), then compact operators should be regarded as infinitesimals. Classically, an infinitesimal is a 'number' whose absolute value is less than any positive number! The following lemma shows that the norm of a compact operator can be made as small as we wish, provided we stay away from a finite dimensional subspace:

Lemma B.1. *Let T be a compact operator. For any $\varepsilon > 0$ there is a finite dimensional subspace $V \subset H$ such that $\|PTP\| < \varepsilon$, where P is the orthogonal projection onto the orthogonal complement of V.*

The first thorough study of the ideal structure of $\mathcal{L}(H)$ was done by Calkin [28]. Among the two-sided ideals of $\mathcal{L}(H)$, and perhaps the most important ones for noncommutative geometry, are the *Schatten ideals*, and ideals related to the Dixmier trace [41]. Let us recall the definition of the former class of ideals next.

A compact operator $T \in \mathcal{K}(H)$ is called a *trace class* operator if

$$\sum_{n=1}^{\infty} \mu_n(T) < \infty.$$

Let e_n, $n \geq 1$, be an orthonormal basis of H. It is easy to see that if T is trace class then

$$\operatorname{Tr}(T) := \sum_i \langle Te_i, e_i \rangle$$

is finite and is independent of the choice of basis. We denote the set of trace class operators by $\mathcal{L}^1(H)$. It is a two sided $*$-ideal in $\mathcal{L}(H)$. Using the definition of the trace Tr, it is easy to check that if A and B are both trace class, then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{B.1}$$

What is much less obvious though, and that is what we actually used in Chapter 4, is that if both AB and BA are trace class then (B.1) still holds. A proof of this can be given using *Lidski's theorem*. This theorem is one of the hardest facts to establish about trace class operators (cf. [165] for a proof).

Theorem B.1 (Lidski's theorem)**.** *If A is a trace class operator then*

$$\mathrm{Tr}(A) = \sum_i^{\infty} \lambda_i,$$

where the summation is over the set of eigenvalues of A.

Now since for any two operators A and B, AB and BA have the same spectrum (and spectral multiplicity) except for 0 (cf. Exercise A.1) we obtain the

Corollary B.1. *Assume A and B are bounded operators such that AB and BA are both trace class. Then* (B.1) *holds.*

Next we define the class of *Schatten-p* ideals for $p \in [1, \infty)$ by

$$\mathcal{L}^p(H) := \{T \in \mathcal{L}(H);\ |T|^p \in \mathcal{L}^1(H)\}.$$

Thus $T \in \mathcal{L}^p(H)$ if and only if

$$\sum_{n=1}^{\infty} \mu_n(T)^p < \infty.$$

It is clear that if $p \leq q$ then $\mathcal{L}^p(H) \subset \mathcal{L}^q(H)$. The Schatten p-norm is defined by

$$\|T\|_p^p = \sum_{n=1}^{\infty} \mu_n(T)^p.$$

Proposition B.1. 1) *$\mathcal{L}^p(H)$ is a two-sided ideal of $\mathcal{L}(H)$.*

2) (Hölder inequality) *Let $p, q, r \in [1, \infty]$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For any $S \in \mathcal{L}^p(H)$ and $T \in \mathcal{L}^q(H)$, we have $ST \in \mathcal{L}^r(H)$ and*

$$\|ST\|_r \leq \|S\|_p \|T\|_q.$$

In particular if $A_i \in \mathcal{L}^n(H)$ for $i = 1, 2, \ldots, n$, then their product $A_1 A_2 \ldots A_n$ is in $\mathcal{L}^1(H)$.

Example B.1. 1. Let us fix an orthonormal basis e_n, $n = 0, 1, 2, \ldots$, in H. The diagonal operator defined by $Te_n = \lambda_n e_n$, $n \geq 0$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$. It is in $\mathcal{L}^p(H)$ if and only if $\sum_i |\lambda_i|^p < \infty$. By the spectral theorem for compact operators, every selfadjoint compact operator is unitarily equivalent to a diagonal operator as above.

2. Integral operators with L^2 kernels provide typical examples of operators in $\mathcal{L}^2(H)$, the class of *Hilbert–Schmidt operators*. Let K be a complex-valued square integrable function on $X \times X$ where (X, μ) is a measure space. Then the operator T_K on $L^2(X, \mu)$ defined by

$$(T_K f)(x) = \int_X K(x, y) f(y) \, d\mu$$

is a Hilbert–Schmidt (in particular compact) operator, with

$$\|T_K\|_2^2 = \|K\|_2^2 = \int_X \int_X |K(x, y)|^2 \, dx dy.$$

Under suitable conditions, e.g. when X is compact Hausdorff and the kernel K is continuous, T_K is a trace class operator and

$$\mathrm{Tr}(T_K) = \int_X K(x, x) \, dx.$$

In the remainder of this section we shall recall some basic definitions and facts about Fredholm operators and index. A bounded linear operator $T \colon H_1 \to H_2$ between two Hilbert spaces is called a *Fredholm operator* if its kernel and cokernel are both finite dimensional:

$$\dim \ker(T) < \infty, \quad \dim \mathrm{coker}(T) < \infty.$$

The *index* of a Fredholm operator is the integer

$$\begin{aligned} \mathrm{index}(T) &:= \dim \ker(T) - \dim \mathrm{coker}(T) \\ &= \dim \ker(T) - \dim \ker(T^*). \end{aligned}$$

We list some of the standard properties of Fredholm operators and the index that are frequently used in noncommutative geometry:

1. (Atkinson's theorem) A bounded operator $T \colon H_1 \to H_2$ is Fredholm if and only if it is invertible modulo compact operators, that is, if there exists an operator $S \colon H_2 \to H_1$ such that $1 - ST$ and $1 - TS$ are compact operators on H_1 and H_2 respectively. S is called a *parametrix* for T. It can also be shown that T is Fredholm if and only if it is invertible module finite rank operators.

Let $\mathcal{C} := \mathcal{L}(H)/\mathcal{K}(H)$ denote the *Calkin algebra* and $\pi \colon \mathcal{L}(H) \to \mathcal{C}$ be the quotient map. (By general C^*-algebra theory, a quotient of a C^*-algebra by a closed two sided $*$-ideal is a C^*-algebra in a natural way). Thus Atkinson's theorem can be reformulated as saying that an operator T is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra. This for example immediately implies that Fredholm operators form an open subset of $\mathcal{L}(H)$ which is invariant under compact perturbations.

2. If T_1 and T_2 are Fredholm operators then $T_1 T_2$ is also a Fredholm operator and

$$\mathrm{index}(T_1 T_2) = \mathrm{index}(T_1) + \mathrm{index}(T_2).$$

3. The Fredholm index is stable under compact perturbations: if K is a compact operator and T is Fredholm, then $T + K$ is Fredholm and

$$\mathrm{index}(T + K) = \mathrm{index}(T).$$

4. The Fredholm index is a *homotopy invariant*: if T_t, $t \in [0,1]$ is a norm continuous family of Fredholm operators then

$$\mathrm{index}(T_0) = \mathrm{index}(T_1).$$

It is this homotopy invariance, or continuity, of the index that makes it computable and extremely useful. Note that $\dim \ker T_t$ can have jump discontinuities.

5. Let $\mathrm{Fred}(H)$ denote the set of Fredholm operators on a separable infinite dimensional Hilbert space. It is an open subset of $\mathcal{L}(H)$ and the *index map*

$$\mathrm{index}\colon \mathrm{Fred}(H) \to \mathbb{Z}$$

induces a one-to-one correspondence between the connected components of $\mathrm{Fred}(H)$ and $\mathbb{Z}$.

6. $\mathrm{Fred}(H)$ is a *classifying space* for K-theory. More precisely, by a theorem of Atiyah and Jänich, for any compact Hausdorff space X, we have a canonical isomorphism of abelian groups

$$K^0(X) \simeq [X, \mathrm{Fred}(H)],$$

where $[X, \mathrm{Fred}(H)]$ is the set of homotopy classes of norm continuous maps from $X \to \mathrm{Fred}(H)$. Thus continuous families of Fredholm operators on X, up to homotopy, gives the K-theory of X.

7. (Calderón's formula [27]) Let $P\colon H_1 \to H_2$ be a Fredholm operator and let $Q\colon H_2 \to H_1$ be a parametrix for P. Assume that for some positive integer n, $(1 - PQ)^n$ and $(1 - QP)^n$ are both trace class operators. Then we have

$$\mathrm{index}(P) = \mathrm{Tr}(1 - QP)^n - \mathrm{Tr}(1 - PQ)^n. \tag{B.2}$$

Here is an alternative formulation of the above result. Let $H = H_1 \oplus H_2$. It is a *super Hilbert space* with even and odd parts given by H_1 and H_2. Let $F = \left(\begin{smallmatrix} 0 & Q \\ P & 0 \end{smallmatrix}\right)$ and γ be the corresponding grading operator. Then we have

$$\mathrm{index}(P) = \mathrm{Tr}_s(1 - F^2)^n, \tag{B.3}$$

where Tr_s is the *supertrace* of trace class operators, defined by $\mathrm{Tr}_s(X) = \mathrm{Tr}(\gamma X)$.

Example B.2. We give a few examples of Fredholm operators and Fredholm index.

1. Any operator $T: H_1 \to H_2$ where both H_1 and H_2 are finite dimensional is Fredholm. Its index is independent of T and is given by

$$\text{index}\,(T) = \dim(H_1) - \dim(H_2).$$

If only one of H_1 or H_2 is finite dimensional then no T can be Fredholm. This shows that the class of Fredholm operators and index is a purely infinite dimensional phenomenon and are only interesting when both H_1 and H_2 are infinite dimensional.

2. Let us fix an orthonormal basis e_n, $n = 0, 1, 2, \ldots,$ for H. The *unilateral shift* operator is defined by

$$T(e_i) = e_{i+1}, \quad i \geq 0.$$

It is easy to see that T is injective and its range is the closed subspace spanned by $e_i, i \geq 1$. Thus T is a Fredholm operator with $\text{index}(T) = -1$. Its adjoint T^* (the backward shift) has index $+1$. Their powers T^m and T^{*m} are m-step forward and backward shifts, respectively, with $\text{index}(T^m) = m$ and $\text{index}(T^{*m}) = -m$.

3. We saw in Section 4.2 that for any odd Fredholm modules (H, F) over an algebra A and an invertible element $U \in A$ the operator $PUP: PH \to PH$ is a Fredholm operator, where $P = \frac{1+F}{2}$ is the projection onto the 1-eigenspace of F. Similarly for an even Fredholm module (H, F, γ) over A and an idempotent $e \in A$, the operator $F_e^+: (eFe)^+: e^+H_+ \to e^-H_-$ is Fredholm.

4. Elliptic differential operators acting on smooth sections of vector bundles over closed manifolds define Fredholm operators on the corresponding Sobolev spaces of sections. Computing the index of such Fredholm operators is what the index theorem of Atiyah–Singer achieves. Let M be a smooth manifold and let E and F be smooth complex vector bundles on M. Let

$$D: C^\infty(E) \to C^\infty(F)$$

be a liner differential operator. This means that D is a $\mathbb{C}$-linear map which is locally expressible by an $m \times n$ matrix of differential operators. This matrix of course depends on the choice of local coordinates on M and local frames for E and F. The *principal symbol* of D is defined by replacing differentiation by covectors in the leading order terms D. The resulting 'matrix-valued function on the cotangent bundle'

$$\sigma_D \in C^\infty(\text{Hom}(\pi^*E, \pi^*F))$$

can be shown to be invariantly defined. Here $\pi: T^*M \to M$ is the natural projection map of the cotangent bundle. A differential operator D is called elliptic if for all $x \in M$ and all nonzero $\xi \in T_x^*M$, the principal symbol $\sigma_D(x, \xi)$ is an invertible matrix.

Let $W^s(E)$ denote the Sobolev space of sections of E (roughly speaking, it consists of sections whose 'derivatives of order s' are square integrable). The main results of the theory of linear elliptic PDE's show that for each $s \in \mathbb{R}$, D has a unique extension to a bounded and Fredholm operator $D\colon W^s(E) \to W^{s-n}(F)$ between Sobolev spaces (n is the order of the differential operator D). Moreover the Fredholm index of D is independent of s and coincides with the index defined using smooth sections.

Exercise B.1. Any invertible operator is clearly Fredholm and its index is zero. Thus any compact perturbation of a an invertible operator is Fredholm and its index is zero. Is it true that any Fredholm operator with zero index is a compact perturbation of an invertible operator?

Exercise B.2. Prove Calderón's formula (B.2) for $n = 1$.

Exercise B.3. Formula (B.3) relates the Fredholm index with the operator trace. Here is a similar formula. Let H be a $\mathbb{Z}_2$-graded Hilbert space and let D be an unbounded *odd* selfadjoint operator on H such that e^{-tD^2} is a trace class operator for all $t > 0$. Show that $\mathrm{index}(D) := \dim\ker(D^+) - \dim\ker(D^-)$ is well defined and is given by the *McKean–Singer formula*

$$\mathrm{index}(D) = \mathrm{Tr}_s(e^{-tD^2}) \quad \text{for all } t > 0.$$

Appendix C

Projective modules

Let A be a unital algebra over a commutative ring k and let $\mathcal{M}_A$ denote the category of right A-modules. We assume that our modules are *unitary* in the sense that the unit of the algebra acts as the identity on the module. A morphism of this category is a right A-module map $f\colon M \to N$, i.e., f is additive and $f(ma) = f(m)a$ for all a in A and m in M.

A *free* module, indexed by a set I, is a module of the type

$$M = A^I = \bigoplus_I A,$$

where the action of A is by componentwise right multiplication. Equivalently, M is free if and only if there are elements $m_i \in M$, $i \in I$, such that any $m \in M$ can be uniquely expressed as a finite sum $m = \sum_i m_i a_i$. A module M is called *finite* (= *finitely generated*) if there are elements $m_1, m_2, \ldots, m_k$ in M such that every element of $m \in M$ can be expressed as $m = m_1 a_1 + \cdots + m_k a_k$, for some $a_i \in A$. Equivalently, M is finite if there is a surjective A-module map $A^k \to M$ for some integer k.

Free modules correspond to trivial vector bundles. To obtain a more interesting class of modules we consider the class of projective modules. A module P is called *projective* if it is a direct summand of a free module, that is, if there exists a module Q such that

$$P \oplus Q \simeq A^I.$$

A module is said to be *finite projective* (= *finitely generated projective*), if it is both finitely generated and projective.

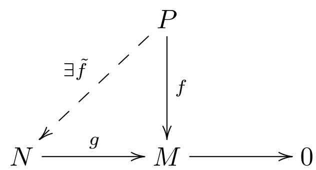
Lemma C.1. *Let P be an A-module. The following conditions on P are equivalent:*

1. *P is projective.*
2. *Any surjection*

$$M \xrightarrow{f} P \to 0,$$

splits in the category of A-modules.

3. *For all A-modules N and M and morphisms f, g with g surjective in the following diagram, there exists a morphism $\tilde{f}$ such that the following diagram commutes:*



We say that $\tilde{f}$ is a lifting of f along g.

4. *The functor*

$$\mathrm{Hom}_A(P, -) \colon \mathcal{M}_A \to \mathcal{M}_k$$

is exact in the sense that for any short exact sequence of A-modules

$$0 \to R \to S \to T \to 0,$$

the sequence of k-modules

$$0 \to \mathrm{Hom}_A(P, R) \to \mathrm{Hom}_A(P, S) \to \mathrm{Hom}_A(P, T) \to 0$$

is exact.

Example C.1. We give a few examples of projective modules:

1. Free modules are projective.
2. If A is a division ring, then any A-module is free, hence projective.
3. $M = \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$, is not projective as a $\mathbb{Z}$-module.
4. A direct sum $P = \bigoplus_i P_i$ of modules is projective if and only if each summand P_i is projective.

We refer to Section 1.3 for more examples of finite projective modules and methods of constructing them, e.g. via idempotents in matrix algebras.

Appendix D

Equivalence of categories

There are at least two ways to compare two categories with each other: isomorphism and equivalence. *Isomorphism* of categories is a very strong requirement and is hardly useful. *Equivalence* of categories, on the other hand, is a much more flexible concept and is very useful.

Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if there is a functor $F\colon \mathcal{A} \to \mathcal{B}$ and a functor $G\colon \mathcal{B} \to \mathcal{A}$, called a *quasi-inverse* of F, such that

$$F \circ G \simeq 1_{\mathcal{B}} \quad \text{and} \quad G \circ F \simeq 1_{\mathcal{A}},$$

where $\simeq$ means isomorphism, or natural equivalence, of functors. This means for every $X \in \operatorname{obj} \mathcal{A}$, $Y \in \operatorname{obj} \mathcal{B}$,

$$FG(Y) \sim Y \quad \text{and} \quad GF(X) \sim X,$$

where $\sim$ denotes isomorphism of objects.

If $F \circ G = 1_{\mathcal{B}}$ and $G \circ F = 1_{\mathcal{A}}$ (*equality* of functors), then we say that the categories $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, and F is an isomorphism.

The *opposite*, or *dual*, of a category $\mathcal{A}$, is a category denoted by $\mathcal{A}^{\mathrm{op}}$. It has the same class of objects as $\mathcal{A}$ and its morphisms are given by $\operatorname{Hom}_{\mathcal{A}^{\mathrm{op}}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(Y, X)$. Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *antiequivalent* if the *opposite category* $\mathcal{A}^{\mathrm{op}}$ is equivalent to $\mathcal{B}$.

Note that a functor $F\colon A \to B$ is an isomorphism if and only if $F\colon \operatorname{obj} \mathcal{A} \to \operatorname{obj} \mathcal{B}$ is one-to-one, onto and F is *full and faithful* in the sense that for all $X, Y \in \operatorname{obj} \mathcal{A}$,

$$F\colon \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is one-to-one (faithful) and onto (full).

It is easy to see that an equivalence $F\colon \mathcal{A} \to \mathcal{B}$ is full and faithful, but it may not be one-to-one, or onto on the class of objects. As a result an equivalence may have many quasi-inverses. The following concept clarifies the situation with objects of equivalent categories.

A subcategory $\mathcal{A}'$ of a category $\mathcal{A}$ is called *skeletal* if 1) the embedding $\mathcal{A}' \to \mathcal{A}$ is full, i.e., if

$$\mathrm{Hom}_{\mathcal{A}'}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$$

for all $X, Y \in \mathrm{obj}\,\mathcal{A}'$, and 2) for any object $X \in \mathrm{obj}\,\mathcal{A}$, there is a unique object $X' \in \mathrm{obj}\,\mathcal{A}'$ isomorphic to X. Any skeleton of $\mathcal{A}$ is equivalent to $\mathcal{A}$ and it is not difficult to see that two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if they have isomorphic skeletal subcategories $\mathcal{A}'$ and $\mathcal{B}'$.

In some examples, like the Gelfand–Naimark theorem, there is a canonical choice for a quasi-inverse for a given equivalence functor F ($F = C_0$ and G is the spectrum functor). There are instances, however, like the Serre–Swan theorem, where there is no canonical choice for a quasi-inverse. The following proposition gives a necessary and sufficient condition for a functor F to be an equivalence of categories. We leave its simple proof to the reader.

Proposition D.1. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if and only if*

a) F is full and faithful, and

b) any object $Y \in \mathrm{obj}\,\mathcal{B}$ is isomorphic to an object of the form $F(X)$, for some $X \in \mathrm{obj}\,A$.

A functor satisfying condition b) in the above proposition is called *essentially surjective.*

Exercise D.1. Show that the category of finite dimensional vector spaces over a field is equivalent to its opposite category, but the category of *all* vector spaces over a field is *not* equivalent to its opposite. There is a similar problem with the category of finite sets and the category of all sets.

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