Math 9054B/4154B, Functional Analysis
Winter 2015

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1. **Monday, January 5:** Defined complex/real inner product space (also called Hermitian/Euclidean space; or a pre-Hilbert space); Cauchy-Schwartz inequality, norm and metric structure of a pre-Hilbert space; completeness, Hilbert space; examples: $\mathbb{C}^n$, ℓ^2, $L^2[a,b], L^2(X,\mu)$.

2. **Wednesday, January 7:** Orthogonality, Pythagorean theorem, parallelogram law; main theorem on distance to closed convex sets; orthogonal complement, the orthogonal decomposition theorem, orthogonal projections.

3. **Friday, January 9:** Continuous linear functionals, Riesz representation theorem for continuous linear functionals, the dual H^* of a Hilbert space and its relation to H. A linear functional is continuous iff it is bounded, Dirichlet energy as a linear functional.

4. **Monday, January 12:** Complete orthonormal sets (Hilbert basis), existence of Hilbert basis for Hilbert spaces (using Zorn's lemma, what is Zorn's lemma?), comparing the Hilbert dimension with the vector space dimension; $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ is an orthonormal basis for $L^2[0, 2\pi]$

5. **Wednesday, January 14:** Gram-Schmidt orthonormalization process, Bessel's inequality, Parseval's identity, various equivalent formulations for an orthonormal basis.

6. **Friday, January 16:** Separable metric spaces, a Hilbert space is separable iff it has a countable basis, isomorphism theorem for Hilbert

spaces, moral: there aren't that many Hilbert spaces out there! compare with Banach spaces.

7. **Monday, January 19:** Intruduced bounded linear operators, examples: diagonal operators in l^2, (when the diagonal elements are bounded), multiplication operators on $L^2(X, \mu)$, integral operators (when the kernel is measurable and bounded).

8. **Wednesday, January 21:** Defined the adjoint of an operator using Riesz representation theorem, self-adjoint operators, projection operators, compact operator and its equivalent definitions, compact operators are automatically bounded, examples of compact operators: finite rank operators, the diagonal operator (when the diagonal elements approach to zero), The set of compact operators is closed in the space of bounded operators.

9. **Friday, January 23:** Hilbert-Schmidt operators, $Te_i = \frac{1}{i}e_i$ is a Hilbert-Schmidt operator, Hilbert-Schmidt operators are compact, an integral operator is Hilbert- Schmidt, if its kernel is in L^2, example: Voltera operator.

10. **Monday, January 26:** Norm inequalities for operators, defined $sinA$ and $cosA$ for the operator A, polarization identity, an operator on infinite dimensional spaces doesn't have necessarily an eigenvalue, example: multiplication operator by $f(x) = x$, eigenvalues of a self-adjoint operator are real, eigenspaces of a self-adjoint operator are orthogonal.

11. **Wednesday, January 28:** Eigenspaces of a compact self-adjoint operator are finite dimensional, the set of non-zero eigenvalues of a compact self-adjoint operator is countable and the only possible accumulation point is zero, either $\|T\|$ or $-\|T\|$ is an eigenvalue of the compact self-adjoint operator T.

12. **Friday, January 30:** Spectral theorem for compact self-adjoint operators: for a compact self-adjoint operator T on the Hilbert space H there exists an orthonormal basis for H consisting of eigenvalues of T, a compact operator can be injective, example: $T\boldsymbol{e}_n = \frac{1}{n}\boldsymbol{e}_n$.

13. **Monday, February 2:** Defined normal operators, commuting family of operators, a family of commuting compact self-adjoint operators can

be simoltaneously diagonalized. spectral theorem for compact normal operators.

14. **Wednesday, February 4:** Defined positive operators, all eigenvalues of a positive operator are non-negative, mini-max theorem (Raleigh-Riesz), defined $A \leq B$ if A and B are self-adjoint and $B - A \geq 0$, the set of compact operators is a closed (with norm topology) two-sided ideal in the algebra of bounded operators, defined absolute value of an operator.

15. **Friday, February 6:** Sturm- Liouville equation:

$$-u'' + qu = \lambda u,\ u(0) = u(2\pi) = 0,\ \lambda \in \mathbb{C}$$
$$q : [0, 2\pi] \longrightarrow \mathbb{R}$$

Question: For what λ the above boundary value problem has a solution? Answer: possible values of λ's are $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \cdots \longrightarrow \infty$ (a discrete set). For each λ_i, $\exists !\ u_i$ (up to scaling); $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis for $L^2[0, 2\pi]$.

16. **Monday, February 9:** Defined the unbounded operator

$$L : C_0^2[0, 2\pi] \longrightarrow L^2[0, 2\pi]$$
$$L(u) = -u'' + qu$$

where C_0^2 is the space of C^2 functions vanishing at 0 and 2π. Proved *Rellich's compactness criterion*, since we wanted to show that the inverse of L is a compact self-adjoint operator.

17. **Wednesday, February 11:** Showed $\forall f \in C[0, 2\pi]$, $\exists ! u \in C_0^2[0, 2\pi]$ such that $u(0) = u(2\pi) = 0$ and $L(u) = f$. So

$$A : C[0, 2\pi] \longrightarrow C_0^2[0, 2\pi]$$
$$f \mapsto u$$

is a well-defined map. Infact $A = L^{-1}$. Then proving the inequality

$$\|(Af)'\|^2 + \tfrac{1}{2}\|Af\|^2 \leq \tfrac{1}{2}\|f\|^2$$

and also using Rellich's compactness criterion, we showed A is compact.

18. **Friday, February 13:** Using integration by parts and the boundary conditions we showed A is a positive selfadjoint operator. To solve the eigenvalue problem $Lu = \lambda u$ we A to both sides of this equation and obtained $Au = \lambda^{-1}u$. Finally using spectral theorem for the compact self-adjoint operators we concluded that there exists

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots \longrightarrow \infty$$

such that $-u'' + qu = \lambda u$ has a solution. Using *Wronskians* we should that all eigenvalues are *simple.* The minimax formula for eigenvalues.

19. **Monday, March 2:** Posed the question how to classify all compact normal operators? Unitariy equivalence between operators is an equivalence relation, two unitary equivalent operators have the same eigenvalues with the same multiplicities, multiplicity function for the compact normal operators, two operators are unitarily equivalent iff they have the same multiplicity function.

20. **Wednesday, March 4:** Defined normed spaces, any inner product space is a normed space, L^p-norms on $\mathbb{R}^n$, L^p-norms on functions, unit sphere, unit ball, metrics from norms, defined Banach spaces, $\mathbb{R}^n$ with L^p-norm is a Banach space, defined equivalent norms, any two norms on $\mathbb{R}^n$ are equivalent.

21. **Monday, March 9:** A norm comes from an iner product iff it stisfies the parallelogram law. Hahn-Banach theorem: Let V be a normed space over $\mathbb{C}$ and W a subspace of V. Every continuous linear functional $\phi : W \longrightarrow \mathbb{C}$ can be extended to V such that $\|\tilde{\phi}\| = \|\phi\|$, defined the dual of a Banach space, the dual of a Banach space is a banach space, defined reflexive Banach spaces, any finite dimensional normed spaceand and any Hilbert space is a reflexive space.

22. **Wednesday, March 11:** Defined Baire spaces, any complete metric space is a Baire space, any locally compact Hausdorff topological space is a Baire space, nowhere dense sets, uniform boundedness principle(Banach-Steinhous)

23. **Friday, March 13:** Open mapping theorem: every sujective bounded linear map between two Banach spaces is open, inverse of a 1-1, onto continuous linear map between two Banach spaces is continuous. Closed Graph theorem: a linear map between two Banach spaces is continuous iff its graph is closed, Hellinger-Toeplitz theorem

24. **Friday, March 20:** Defined the spectrum and the resolvent of a bounded linear operator on a Hilbert space. resolvent funcion, point spectrum, in general the point spectrum is a proper subset of the spectrum, the point spectrum of the multiplication operator (multiplication by x) is empty, while the spectrum of that is $[0, 1]$, the spectrum and the point spectrum of an operator on a finite dimensional Hilbert space are the same, the spectrum of the Voltera operator is $\{0\}$, Neumann series, the set of invertible operators is open in $\mathcal{L}(H)$, the spectrum of a bounded linear operator on a Hilbert space is a closed bounded (compact) set.

25. **Monday, March 23:** The spectrum of a bounded linear operator on a Hilbert space is a nonempty set, spectral mapping theorem for polynomials, spectral radius, the spectral radius of an operator can be obtained using the norm of powers of that operator.

26. **Wednesday, March 25:** Proved the *spectral radius formula*:

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

    Bounded below operaors, invertiblity criterion: T is invertible iff T and T^* are bounded below.

27. **Friday, March 27:** Approximate eigenvalue, T is normal iff

$$\|Tx\| = \|T^*x\|, \forall x \in H$$

    and in that case any element of the spectrum is an approximate eigenvalue, the spectrum of a unitary operator is a subset of the unit circle, the spectrum of a self-adjoint operator is a subset of $\mathbb{R}$, for a self-adjoint operator T, either $\|T\|$ or $\|T^*\|$ is in the spectrum, functional calculus for polynomials, Stone-Weierstrass theorem, functional calculus for continuous functions.

28. **Monday, March 30:** Defined continuous spectrum, singular spectrum, point spectrum of the backward shift operator is the open unit disc and its continuous spectrum is the unit circle, the point spectrum of the unilateral shift operator is the empty set and its continuous spectrum is the closed unit disc, the spectrum of the discrete Laplacian on $\mathbb{Z}$ is $[-2, 0]$, σ-algebra, measure, measurable space, measure space, $L^2(X, F, \mu)$ is a Hilbert space, multiplication operators M_ϕ, $M_\phi^* = M_{\bar{\phi}}$, The multiplication operator is a normal operator, unitarily equivalent operators, Is any self-adjoint or normal operator unitarily equivalent to a multiplication operator? Yes! this is the multiplication operator version of the spectral theorem, Riesz representation theorem, cyclic vectors for an operator.

29. **Monday, April 6:** Any self-adjoint operator with a cyclic vector is unitarily equivalent to a multiplication operator, an operator on a finite dimensional Hilbert space has a cyclic vector iff its spectrum is simple i.e. all eigenvalues of that are simple, if T is a self-adjoint operator on the Hilbert space H, then we can decompose H into invariant subspaces such that restriction of T to each of them has a cyclic vector and this fact can be used to prove the multiplication operator version of the spectral theorem, Borel subsets, Borel measure, for any monotone function α on $\mathbb{R}$ there exist a borel measure μ_α on $\mathbb{R}$ such that

$$\mu_\alpha(a, b) = \alpha(b^+) - \alpha(a^-)$$

For

$$\alpha(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

μ_α would be the Dirac measure, pure points (atoms) of a measure space, the set of all pure points is at most countable, pure point part of a measure, absolutely continuous part of a measure, singular part of a measure, the Radon-Nikodym theorem, any Borel measure can be decomposed into sum of its pure point part, absolutely continuous part and singular part.

30. **Wendsday, April 8:** Defined pure point vectors, absolouely continuous vectors and singular vectors for a self-adjoint operator T on the Hilbert space H, denoted the set of these vectors respectively by H_{pp}, H_{ac} and H_{sing}, these spaces are invariant under T, so one can define subsets

$\sigma_{pp}(T)$, $\sigma_{ac}(T)$ and $\sigma_{sing}(T)$ of the spectrum of T. These subsets are not necessarily disjoint but we have

$$\sigma(T) = \sigma_{pp}(T) \cup \sigma_{ac}(T) \cup \sigma_{sing}(T).$$

Computed the spectrum of the discrete Laplacian on $\mathbb{Z}$; briefly discussed the 10 martini problem.