The Dirichlet Problem*

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Abstract

We prove the classical result regarding the solvability of the Dirichlet problem for bounded domains with sufficiently smooth boundaries. We present the method described in [7], the main ingredient being the Fredhom alternative applied to a suitably constructed compact operator.

1 Introduction

It is well known that the Dirichlet problem has its origins in the study of physical phenomena such as the heat distribution across a certain surface, or the electrical flow trough a conductor under certain prescribed constraints. Such studies led, over the centuries, to an extraordinary evolution of the subject, motivating scientists to make fundamental advances not only in physics, but also (perhaps especially) in mathematics.

Arguably the simplest and probably one of the earliest case of a Dirichlet problem is that stated for the unit disk $D(0;1)$ in the complex plane. Those of us who have gone through a second level course of Complex Analysis, have already learned that the problem, requiring finding harmonic solutions on $D(0;1)$ matching continuously a prescribed continuous function φ on the boundary of $D(0;1)$, has always a unique solution. Moreover, the solution is explicitly computed as

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})\varphi(e^{i\theta})d\theta,$$

where $P : D(0;1) \times \partial D(0;1)$, defined as

$$P(z,w) = \frac{1 - |z|^2}{|w - s|^2},$$

is the Poisson kernel. It turned out, however, that as soon as sets more general than the disk are considered, e.g. an open bounded set in the plane, the Dirichlet problem does not always have a solution (see the example in Section 2). This observation led to more intense research that eventually produced conditions that characterize the solvability of the Dirichlet problem [2, p. 229]. Yet, a different and fundamentally important contribution is the introduction by Wiener of the notion of generalized solution for domains that are not solvable for the classical Dirichlet problem [4, pp. 1-73].

In this material we prove the solvability of the Dirichlet problem for bounded domains in $\mathbb{R}^n$ with "sufficiently" smooth boundaries. The existence is established making use of tools from the theory of compact bounded operators on Hilbert spaces, particularly the Fredholm alternative.

2 Preliminaries

Let $n \in \mathbb{N}, n \geq 1$ and let $D \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ (throughout the material, *domain* will always mean an open, connected subset of an Euclidean space of specified dimension).

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One of the possible formulations of the Dirichlet problem associated to the domain D is as follows: *given a continuous function* $\varphi : \partial D \to \mathbb{R}$*, determine whether there exists a continuous function* $u : \overline{D} \to \mathbb{R}$*, twice continuously differentiable on* D*, satisfying the following conditions,*

$$\Delta u(x) = 0, \forall x \in D, \tag{2.1}$$

$$u(x) = \varphi(x), \forall x \in \partial D, \tag{2.2}$$

where Δ is the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

and $x_1, \ldots, x_n$ represent the usual Cartesian coordinates in $\mathbb{R}^n$. If it exists, such function u is referred to as a *solution* of the given Dirichlet problem, satisfying the *boundary condition* (2.2). Also, recall that, for any open subset $D \subset \mathbb{R}^n$, a function $u \in C^2(D)$ satisfying condition (2.1) is called a *harmonic* function. So in attempting to solve the Dirichlet problem associated with a domain D, one must find a function $u : \overline{D} \to \mathbb{R}$, harmonic on D, and satisfying the boundary condition imposed by a prescribed continuous function on ∂D.

Example 2.1. It is not difficult to see that that the Dirichlet problem does not always have a solution. The simplest (and most popular) example is that of the punctured unit disk $D_0 := (D(0;1) \setminus \{0\}) \subset \mathbb{R}^2 \cong \mathbb{C}$, whose boundary is clearly given by

$$\partial D_0 = \partial D(0;1) \cup \{0\}.$$

The function $\varphi : \partial D_0 \to \mathbb{R}$ defined as $\varphi(x) = 0$ if $x \in \partial D(0;1)$ and $\varphi(0) = 1$, is continuous on ∂D_0: its domain of definition is disconnected, consisting of two components, on each of which f is constant, respectively. If u were a solution of the associated Dirichlet problem, then 0 would be a removable singularity for u, hence u would extend harmonically to the entire open disk $D(0;1)$. But by the maximum principle for harmonic functions, there cannot be a maximum of u on D; since by the boundary condition $u(x) = \varphi(x) = 0$ for all $x \in \partial D(0;1)$, we must have $u(x) \leq 0$ (in fact, $u(x) = 0$), for all $x \in D(0;1)$. Applying this to $x = 0$, we get $u(0) \leq 0$, which contradicts the fact that, by the the boundary condition on the second branch of φ, $u(0) = 1$.

In line with the above (counter)example, it is becoming apparent that the topological properties of the boundary ∂D of the domain under investigation must play an important role in ensuring the existence of a solution for the associated Dirichlet problem. Indeed, it turns out that if ∂D exhibits a satisfactory degree of "smoothness", then a solution always exists. The exact formulation of this statement and its proof are the subject of Section 3. For the moment, let us prove the following straightforward result regarding the uniqueness of such solution, in case it does exist.

Proposition 2.2. *Let* $D \subset \mathbb{R}^n$ *be a bounded domain and* $\varphi : \partial D \to \mathbb{R}$ *continuous. If there exist continuous functions* $u, v : \overline{D} \to \mathbb{R}$ *each satisfying conditions (2.1) and (2.2), then* $u \equiv v$.

Proof. Let $h = u - v$; then h is continuous on $\overline{D}$, harmonic on D and $h(x) = u(x) - v(x) = \varphi(x) - \varphi(x) = 0$, for all $x \in \partial D$. By the maximum principle for harmonic functions, we then must have $u(x) - v(x) = h(x) = 0$ for all $x \in \overline{D}$, hence $u \equiv v$. □

3 Solving the Dirichlet Problem on Domains with Smooth Boundaries

In this section we prove the existence of the solution of the Dirichlet problem for domains whose boundaries satisfy certain smoothness conditions, which we shall define in detail. The presentation of this section follows closely [7, pp. 204-206]. To simplify notations and staying in line with the above mentioned referrence, we shall consider the case $n = 3$, although the technique can be applied to arbitrary $n \geq 2$.

Let $D \subset \mathbb{R}^3$ be a bounded domain such that its boundary ∂D is a smooth (hyper)surface in $\mathbb{R}^3$. In this context, smooth means of class C^2, hence all related maps will be considered to be second order continuously differentiable. The smoothness condition ensures the existence of open sets $\{U_i \subset \mathbb{R}^3 \mid i \in I\}$, $\{V_i \subset \mathbb{R}^2 \mid i \in I\}$ and diffeomorphisms $\{\psi_i : V_i \to U_i \cap \partial D \mid i \in I\}$ such that $\{U_i \cap \partial D\}_{i \in I}$ cover the boundary of our domain[1]. By an abuse of notation let us re-denote each $U_i \cap \partial D$ by the same symbol U_i and refer to any pair (U_i, ψ_i) as a *local chart* on ∂D. Note that as a subset of $\mathbb{R}^3$, ∂D is closed and bounded, hence compact.

The main goal is to establish the existence of the solution of the Dirichlet problem (2.1) and (2.2) for a given continuous function $\varphi \in C(\partial D)$. First, define the *double layer potential kernel*, $K : (\mathbb{R}^3 \setminus \partial D) \times \partial D \to \mathbb{R}$ as

$$K(x,y) = \frac{(x-y)\cdot n(y)}{2\pi\,|x-y|^3}, \tag{3.1}$$

where $n(y)$ is the outer unit normal to ∂D at $y \in \partial D$, $\cdot$ signifies the usual Euclidean scalar product (dot product) and $|\cdot|$ is the Euclidean norm. For all $x \in D$ define

$$u(x) = \int_{\partial D} K(x,y) f(y) dS(y), \tag{3.2}$$

know as the *double layer potential*[2], where $f : \partial D \to \mathbb{R}$ is a given continuous function on the boundary and dS is the usual (Lebesgue) surface measure. Note that, since ∂D is smooth, the normal $n(y)$ varies smoothly on ∂D. It follows that $K(x,\cdot)$ is smooth on ∂D for each $x \in D$, and since by hypothesis $f \in C(\partial D)$, the above integral is well defined.

Our next task is to show that u is harmonic on D. First define the *Newton potential kernel*, $N : (\mathbb{R}^3 \setminus \partial D) \times \partial D \to \mathbb{R}$ as

$$N(x,y) = -\frac{1}{2\pi\,|x-y|},$$

which is easy to note that is smooth in each variable.

Lemma 3.1. *For any fixed $y \in \partial D$, $N(\cdot, y)$ is harmonic on $\mathbb{R}^3 \setminus \partial D$.*

Proof. Assuming w.l.o.g. $y = 0$ and ignoring the 2π factor, put $N(x) = N(x, 0)$. Let $x = (x_1, x_2, x_3)$ be the (Cartesian) coordinate expression of x and denote by ∂_i, ∂_i^2 the first and second order (unmixed) partial derivatives, respectively, with respect to the i-th coordinate. Then, for all $i \in \{1, 2, 3\}$, a direct computation gives

$$\partial_i N(x) = x_i(x_1^2 + x_2^2 + x_3^2)^{-3/2} \tag{3.3}$$

and

$$\begin{aligned}\partial_i^2 N(x) &= \partial_i[x_i(x_1^2 + x_2^2 + x_3^2)^{-3/2}] \\ &= (x_1^2 + x_2^2 + x_3^2)^{-3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{-5/2} \\ &= (x_1^2 + x_2^2 + x_3^2)^{-3/2}[1 - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{-1}].\end{aligned} \tag{3.4}$$

By adding up all identities (3.4) for $i \in \{1, 2, 3\}$, we obtain

$$\Delta N(x) = (x_1^2 + x_2^2 + x_3^2)^{-3/2}\left[3 - 3\frac{x_1^2 + x_2^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2}\right] = 0,$$

which proves the lemma. □

For fixed $x \in \mathbb{R}^3 \setminus \partial D$, let us denote by $\partial_{n(y)} N(x,y)$ the directional derivative of $N(x,\cdot)$ at the point $y \in \partial D$. A direct computation that makes use of (3.3) (for $y \neq 0$) yields,

$$\partial_{n(y)} N(x,y) = \nabla_y N(x,y) \cdot n(y) = \frac{x-y}{2\pi\,|x-y|^3} \cdot n(y) = K(x,y), \tag{3.5}$$

where ∇_y denotes the gradient of N with respect to the second variable.

[1]Note that the compatibility condition on overlapping subsets $U_i \cap U_j$, characteristic to all smooth manifolds, is automatically satisfied in this case.

[2]See the last section for a brief discussion on the significance of single and double layer potential functions in physics.

Lemma 3.2. *For any fixed $y \in \partial D$, $K(\cdot, y)$ is harmonic on $\mathbb{R}^3 \setminus \partial D$.*

Proof. Using (3.5), we have, $\Delta_x K(x, y) = \Delta_x(\nabla_y N(x, y) \cdot n(y)) = \nabla_y(\Delta_x N(x, y)) \cdot n(y) = 0 \cdot n(y) = 0$, where we used Lemma 3.1 and the fact that the partial derivatives with respect to x and y are independent, hence commute. □

Finally, since for $y \in \partial D$, $K(\cdot, y)$ is smooth on D we can differentiate with respect to x under the integral sign in (3.2). By Lemma 3.2, it follows that u is also harmonic on D.

The following result [6] [7], which we include without proof, provides the next key step toward the solution.

Proposition 3.3. *For all $x_0 \in \partial D$, the following two limits hold:*

$$\lim_{x \uparrow x_0} u(x) = -f(x_0) + \int_{\partial D} K(x_0, y) f(y) dS(y),$$

$$\lim_{x \downarrow x_0} u(x) = f(x_0) + \int_{\partial D} K(x_0, y) f(y) dS(y),$$

where $\lim\limits_{x \uparrow x_0}$, $\lim\limits_{x \downarrow x_0}$ indicate the limits when x approaches from inside or outside D, respectively.

There is an obvious problem with the above statement. The formula (3.1) defining K outside the boundary of D cannot be extended "as is" to ∂D, since $K(x, y)$ has a singularity on ∂D, whenever $x = y$. As a result, let us define $K(x, x) = 0$, whenever $x \in \partial D$, while maintaining (3.1) for all other scenarios. We still have to show, however that for any $x_0 \in \partial D$ the integral

$$\int_{\partial D} K(x_0, y) f(y) dS(y) \tag{3.6}$$

converges.

Lemma 3.4. *There exist a constant $c > 0$ such that*

$$|(x - y) \cdot n(y)| \leq c\, |x - y|^2,$$

for all $x, y \in \partial D$.

Proof. Let $x \in \partial D$ and (U, ψ) a local chart such that $x \in U$. By slightly "shrinking" U and considering its closure, we may consider that U is in fact compact while $x \in \operatorname{int} U$. Let $\delta = d(x, \partial U) > 0$. If $y \in \partial D$ is such that $y \notin U$, then $|x - y| > \delta$. By the Cauchy-Schwarz inequality

$$|(x - y) \cdot n(y)| \leq |x - y|\, |n(y)| = |x - y| \leq \frac{|x - y|^2}{\delta}.$$

Putting $c = 1/\delta$ we get the desired result. Suppose now $y \in U$, so there exist $\xi, \eta \in V := \psi^{-1}(U) \subset \mathbb{R}^2$ such that $x = \psi(\xi)$, $y = \psi(\eta)$. Note that, since U is compact and ψ^{-1} is continuous, V is also compact. Let $\psi = (\psi_1, \psi_2, \psi_3)$, $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ be the corresponding coordinate expressions. By Taylor's formula of second order with remainder, we get for all $i \in \{1, 2, 3\}$,

$$\psi_i(\xi) - \psi_i(\eta) = \partial_1 \psi_i(\eta)(\xi_1 - \eta_1) + \partial_2 \psi_i(\eta)(\xi_2 - \eta_2) + \frac{1}{2} \sum_{j,k=1}^{2} \partial^2_{j,k} \psi_i(\alpha_i)(\xi_j - \eta_j)(\xi_k - \eta_k),$$

where α_i are some points on the segment $\xi - \eta$ (as per Taylor's theorem). So, in vector form,

$$\begin{aligned} \psi(\xi) - \psi(\eta) =& \partial_1 \psi(\eta)(\xi_1 - \eta_1) + \partial_2 \psi(\eta)(\xi_2 - \eta_2) \\ &+ \frac{1}{2} \sum_{j,k=1}^{2} \left[\partial^2_{j,k} \psi_1(\alpha_1),\ \partial^2_{j,k} \psi_2(\alpha_2),\ \partial^2_{j,k} \psi_3(\alpha_3)\right]^T (\xi_j - \eta_j)(\xi_k - \eta_k), \end{aligned}$$

where $\partial_i\psi(\eta)$ are the coordinate tangent vectors at $y=\psi(\eta)\in\partial D$. Since these vectors are orthogonal to $n(y)$, we obtain

$$\begin{aligned}|(x-y)\cdot n(y)| &= |\psi(\xi)-\psi(\eta)| \\ &= \frac{1}{2}\left|\sum_{j,k=1}^{2}\left[\partial^2_{j,k}\psi_1(\alpha_1),\ \partial^2_{j,k}\psi_2(\alpha_2),\ \partial^2_{j,k}\psi_3(\alpha_3)\right]^T(\xi_j-\eta_j)(\xi_k-\eta_k)\right| \\ &\le \frac{1}{2}\sum_{i=1}^{3}\left(\sum_{k,l=1}^{2}\left[\partial^2_{j,k}\psi_i(\alpha_i)(\xi_j-\eta_j)(\xi_k-\eta_k)\right]^2\right)^{1/2} \\ &\le \frac{1}{2}\max_{i,k,l}\{\sup_{w\in V}\partial^2_{k,l}\psi_i(w)\}\sum_{i=1}^{3}\left(\sum_{k,l=1}^{2}(\xi_j-\eta_j)^2(\xi_k-\eta_k)^2\right)^{1/2},\end{aligned} \tag{3.7}$$

where we used that $\sup_{w\in V}\left(\partial^2_{k,l}\psi_i(w)\right)<\infty$ since V is compact and all second order partial derivatives are continuous. Finalizing, $\sum_{k,l=1}^{2}(\xi_j-\eta_j)^2(\xi_k-\eta_k)^2=[(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2]^2=|\xi-\eta|^4$. Replacing in (3.7) and putting $c=\frac{3}{2}\max_{i,k,l}\{\sup_{w\in V}\partial^2_{k,l}\psi_i(w)\}$, the result follows. □

The next required result [6] we give without proof.

Lemma 3.5. *With the same settings as above, there exists a constant $\gamma>0$ such that*

$$\sup_{x\in\partial D}\int_{B(x;\delta)\cap\partial D}|x-y|^{-1}\,dS(y)\le\gamma\delta,$$

for all $\delta>0$, where $B(x;\delta)$ denotes the open ball in $\mathbb{R}^3$ centered at x and of radius δ.

Proposition 3.6. *For all $x_0\in\partial D$, the integral (3.6) converges.*

Proof.

$$\begin{aligned}\left|\int_{\partial D}K(x_0,y)f(y)dS(y)\right| &\le \int_{\partial D}|K(x_0,y)|\,|f(y)|\,dS(y) \\ &\le \|f\|_\infty\int_{\partial D}|K(x_0,y)|\,dS(y),\end{aligned}$$

where $\|f\|_\infty=\sup_{\partial D}|f|<\infty$, since ∂D is compact in $\mathbb{R}^3$ and f is continuous. By Lemma 3.4,

$$\left|\int_{\partial D}K(x_0,y)f(y)dS(y)\right|\le c\,\|f\|_\infty\int_{\partial D}|x_0-y|^{-1}\,dS(y)$$

Let $\delta>0$ be such that $B_\delta:=B(x_0;\delta)\cap\partial D\neq\emptyset$. Then, we may write $\partial D=B_\delta\cup\overline{\partial D\setminus B_\delta}$, where the union is disjoint, hence

$$\int_{\partial D}|x_0-y|^{-1}\,dS(y)=\int_{B_\delta}|x_0-y|^{-1}\,dS(y)+\int_{\overline{\partial D\setminus B_\delta}}|x_0-y|^{-1}\,dS(y).$$

The second integral exists and is clearly finite, since $\overline{\partial D\setminus B_\delta}$ is compact and $|x_0-y|^{-1}$ is well defined and continuous on $\overline{\partial D\setminus B_\delta}$. On the other hand, for the first integral, by Lemma 3.5 we obtain

$$\int_{B_\delta}|x_0-y|^{-1}\,dS(y)\le\sup_{x\in\partial D}\int_{B_\delta}|x-y|^{-1}\,dS(y)\le\gamma\delta,$$

which completes the proof. □

Lemma 3.7. *For fixed $f \in C(\partial D)$, the function $x \mapsto F(x) := \int_{\partial D} K(x,y)f(y)dS(y)$ is continuous on ∂D.*

Proof. For $x, x_0 \in \partial D$ we clearly have

$$|F(x) - F(x_0)| \le \|f\|_\infty \int_{\partial D} |K(x,y) - K(x_0,y)|\, dS(y). \tag{3.8}$$

For any $\rho > 0$ put $B_\rho = B(x_0;\rho) \cap \partial D$, as in the proof of Proposition 3.6. Then we can write

$$\int_{\partial D} |K(x,y) - K(x_0,y)|\, dS(y) = \int_{\overline{\partial D \setminus B_\rho}} |K(x,y) - K(x_0,y)|\, dS(y) + \int_{B_\rho} |K(x,y) - K(x_0,y)|\, dS(y). \tag{3.9}$$

The first integral in the above sum, tends to zero as $x \to x_0$, since K is continuous and with no singularities on $\overline{\partial D \setminus B_\rho}$. Before analyzing the second integral, note first that

$$\begin{aligned} |K(x,y) - K(x_0,y)| &\le |K(x,y)| + |K(x_0,y)| \\ &= \frac{|(x-y)n(y)|}{|x-y|^3} + \frac{|(x_0-y)n(y)|}{|x_0-y|^3} \\ &\le c\left(\frac{1}{|x-y|} + \frac{1}{|x_0-y|}\right), \end{aligned}$$

for all $y \in \partial D \setminus \{x_0, x\}$, where the last inequality follows from Lemma 3.4. This implies that

$$\int_{B_\rho} |K(x,y) - K(x_0,y)|\, dS(y) \le c\left(\int_{B_\rho} \frac{dS(y)}{|x-y|} + \int_{B_\rho} \frac{dS(y)}{|x_0-y|}\right). \tag{3.10}$$

We can apply Lemma 3.5 to the second integral on the right to get a suitable bound, but cannot do so on the first integral, since $B_\rho = B(x_0;\rho) \cap \partial D$ is not centered at x. To address this, consider $|x - x_0| < \rho/2$, put $B'_{3\rho/2} = B(x;3\rho/2) \cap \partial D$, which is centered at x, and note that $B_\rho \subset B'_{3\rho/2}$. Since $1/|x-y|$ is nonnegative,

$$\int_{B_\rho} \frac{dS(y)}{|x-y|} \le \int_{B'_{3\rho/2}} \frac{dS(y)}{|x-y|},$$

which combined with 3.10 gives

$$\int_{B_\rho} |K(x,y) - K(x_0,y)|\, dS(y) \le c\left(\int_{B'_{3\rho/2}} \frac{dS(y)}{|x-y|} + \int_{B_\rho} \frac{dS(y)}{|x_0-y|}\right).$$

We can now apply Lemma 3.5, separately to each integral on the right to obtain

$$\int_{B_\rho} |K(x,y) - K(x_0,y)|\, dS(y) \le c\,(\gamma 3\rho/2 + \gamma\rho) = \frac{5}{2}c\gamma\rho, \tag{3.11}$$

for all $x \in \partial D$ satisfying $|x - x_0| < \rho/2$. Lastly, putting $\rho = \frac{2}{5c\gamma}\varepsilon$, for arbitrary $\varepsilon > 0$, the inequality (3.11) implies that for all $|x - x_0| < \rho/2 = \frac{1}{5c\gamma}\varepsilon$, we have

$$\int_{B_\rho} |K(x,y) - K(x_0,y)|\, dS(y) < \varepsilon,$$

which shows that the second integral in (3.9) can be made arbitrarily small. In combination with (3.8), this proves the lemma. □

It follows from Proposition 3.3 that we are seeking a solution $f \in C(\partial D)$ of the following integral equation,

$$\varphi(x) = -f(x) + \int_{\partial D} K(x,y)f(y)dS(y), \tag{3.12}$$

where $\varphi \in C(\partial D)$ is the function prescribing the boundary condition (2.2). Let $T : C(\partial D) \to C(\partial D)$ be the integral operator given as

$$Tf(x) = \int_{\partial D} K(x,y) f(y) dS(y), \tag{3.13}$$

which according to Lemma 3.7 is well defined. Here, $C(\partial D)$ is considered with the usual L^2 Hilbert space structure (as a dense subspace of $L^2(\partial D)$). So, the question becomes whether the equation

$$(T - I)f = \varphi \tag{3.14}$$

has a solution in $C(\partial D)$ or not, where I is the identity operator on $C(\partial D)$. However, in order to apply the Fredholm alternative, we need to ensure that T is bounded and compact.

Remark 3.8. It follows from the proof of Proposition 3.6 that, for all $x \in \partial D$, $\int_{\partial D} |K(x,y)|\, dS(y) \leq c\gamma\delta < \infty$ and since $x \mapsto \int_{\partial D} |K(x,y)|\, dS(y)$ is continuous and the boundary ∂D is compact, we have

$$\gamma_0 := \sup_{x\in\partial D} \int_{\partial D} |K(x,y)|\, dS(y) < \infty, \tag{3.15}$$

Note that with a minor modification of the same proof, we can also prove that,

$$\gamma_1 := \sup_{y\in\partial D} \int_{\partial D} |K(x,y)|\, dS(x) < \infty. \tag{3.16}$$

Proposition 3.9. *The operator T defined in (3.13) is bounded.*

Proof. Let $f \in C(\partial D)$ and fix $x \in \partial D$. Then, using Hölder's inequality,

$$\begin{aligned} |Tf(x)| &= \left| \int_{\partial D} K(x,y)^{(1/2+1/2)} f(y) dS(y) \right| \\ &\leq \left(\int_{\partial D} |K(x,y|\, dS(y) \right)^{1/2} \left(\int_{\partial D} |K(x,y)|\, |f(y)|^2\, dS(y) \right)^{1/2} \\ &\leq \sqrt{\gamma_0} \left(\int_{\partial D} |K(x,y)|\, |f(y)|^2\, dS(y) \right)^{1/2}. \end{aligned} \tag{3.17}$$

Now,

$$\begin{aligned} \|Tf\|_2 &= \left(\int_{\partial D} |Tf(x)|^2\, dS(x) \right)^{1/2} \\ &\leq \sqrt{\gamma_0} \int_{\partial D} \left(\int_{\partial D} |K(x,y)|\, |f(y)|^2\, dS(y) \right) dS(x) \\ &= \sqrt{\gamma_0} \int_{\partial D} |f(y)|^2 \left(\int_{\partial D} |K(x,y)|\, dS(x) \right) dS(y) \\ &\leq \sqrt{\gamma_0\gamma_1} \|f\|_2, \end{aligned}$$

which proves the statement. ☐

Proposition 3.10. *The operator T defined in (3.13) is compact.*

Proof. For $\delta > 0$, let $K_\delta := \dfrac{(x-y)\cdot n(y)}{|x-y|^3 + \delta}$ be defined for all $x, y \in \overline{D}$. Note that K_δ is well defined and continuous on $\overline{D}$. It is shown [7, pp.198-199] by making use of the Arzela-Ascoli theorem that by the compactness of $\overline{D}$ and continuity of K, the integral operator T_δ having K_δ as its kernel is compact, for all $\delta > 0$. By a known result [7, p.200], if we show that $\lim_{\delta\to 0} \|T_\delta - T\| = 0$, where the convergence takes place in the operator norm topology on $\mathscr{L}(C(\partial D))$, then T must also be compact.

For $f \in C(\partial D)$,

$$\begin{aligned}\|(T-T_\delta)f\|_2 &= \int_{\partial D} |Tf(x) - T_\delta f(x)|^2\, dS(x) \\ &\le \|f\|_\infty^2 \int_{\partial D}\left(\int_{\partial D} |K(x,y) - K_\delta(x,y)|\, dS(y)\right)^2 dS(x).\end{aligned} \tag{3.18}$$

As in the proof of Proposition 3.6, let $\varepsilon > 0$ be such that $B_\varepsilon := B(x;\delta) \cap \partial D \neq \emptyset$, for a given $x \in \partial D$. So, $\partial D = B_\varepsilon \cup \overline{\partial D \setminus B_\varepsilon}$, where the union is disjoint. It follows that the most inner integral on the last term of (3.18) is equal to the sum of the integrals (of the same function) on B_ε and $\overline{\partial D \setminus B_\varepsilon}$, respectively. For $\delta > 0$, chose $\varepsilon < \delta/\gamma$, where γ is as in Lemma 3.5. By the same lemma, it follows that

$$\int_{B_\varepsilon} |K(x,y) - K_\delta(x,y)|\, dS(y) < \delta. \tag{3.19}$$

On the other hand, for all $y \in \overline{\partial D \setminus B_\varepsilon}$,

$$\begin{aligned}|K(x,y) - K_\delta(x,y)|\, dS(y) &= (x-y)\cdot n(y) \frac{\delta}{|x-y|^3\, |x-y|^3 + \delta} \\ &\le \max_{y \in \overline{\partial D \setminus B_\varepsilon}} \{(x-y)\cdot n(y)\} \frac{\delta}{\varepsilon^3(\varepsilon^3+\delta)^3}\end{aligned}, \tag{3.20}$$

which clearly converges to 0 as $\delta \to 0$. By (3.18), (3.19) and (3.20), $\lim_{\delta\to 0} \|(T-T_\delta)f\|_2 = 0$, for all $f \in C(\partial D)$, hence $\lim_{\delta \to 0} \|T - T_\delta\| = 0$. □

Now, we show that equation (3.14) can always be solved, thus ending our search for the solution of the initial Dirichlet problem. Since T is bounded and compact, by the Fredholm alternative, two cases are possible: either $T - I$ is not injective, which implies the existence of a function $g \in C(\partial D)$, not identically zero, such that

$$Tg = g \tag{3.21}$$

(i.e. $\lambda = 1$ is an eigenvalue of $T - I$), or $T - I$ is surjective, which is exactly what we would like to show. Assume, then, that the first case is possible. By replacing g in (3.12), we get

$$u(x) = -g(x) + Tg(x) = 0, \tag{3.22}$$

for all $x \in \partial D$. By the maximum principle, it follows that $u(x) = 0$ for all $x \in \overline{D}$. Note that, although u was defined initially only on $\overline{D}$, the kernel K is defined on the entire space $\mathbb{R}^3$ (see paragraph immediately following Proposition 3.3), hence we can extend u by the same formula (3.2) on $\mathbb{R}^3 \setminus \overline{D}$. With some additional effort[3], it can be shown that, under the given scenario, u also vanishes on $\mathbb{R}^3 \setminus \overline{D}$, which by the second identity of Proposition 3.3, leads to $2g(x) = 0$, for all $x \in \partial D$, in contradiction with the initial assumption that g is not the zero function.

4 On the Significance of the Single and Double Layer Potential Functions in Electrostatics

If a electric charge is placed at a point $y \in \mathbb{R}^3$, it is a direct consequence of Coulomb's law that the electric field at another point $x \in \mathbb{R}^3$ is given by

$$E(x) = \frac{x-y}{|x-y|^3}.$$

It is also known that, in the case of stationary charges, the electric force, hence the associated force field E, is conservative, i.e. it is the gradient of a scalar field $u : \mathbb{R}^3 \to \mathbb{R}$, called the *electric potential*:

[3] In [7, p. 205], it is said that since the derivative of u in the direction of the outer normal vanishes on ∂D, then an integration by parts implies that $u \equiv 0$. We were not able to see why this is the case. A more detailed proof of the same fact can be found in [6, p. 104-106]

$E = -\nabla_x u$. It is immediate to see that, in our example, the potential is given by $u(x) = \dfrac{1}{|x-y|}$. Now, if instead of the point-charge we have a bounded domain $D \Subset \mathbb{R}^3$ with smooth boundary ∂D, such that its boundary is electrically charged according to a charge distribution $\rho : \partial D \to \mathbb{R}$, then the electric potential is given by the surface integral

$$u(x) = \int_{\partial D} \frac{1}{|x-y|}\rho(y)dS(y)$$

and it is called the *single layer potential* corresponding to the charge distribution ρ, the word "layer" referring of course to the electrically charged boundary ∂D. We recognize the (integral) kernel of u to be (up to a constant factor) $N(x, y)$, the Newton potential kernel we defined in the preceding section[4].

Next, consider that in addition to our bounded domain set up as above, there exists a smooth, closed surface Σ situated outside of D at a constant distance $h > 0$ from ∂D, i.e.

$$\Sigma = \{y + hn(y) \mid y \in \partial D\},$$

where $n(y)$ is the outer unit normal at $y \in \partial D$. Assume that Σ is also electrically charged, but with the exact opposite charge as ∂D[5]. In fact, suppose that the electrical charge on ∂D is given by $\dfrac{\rho}{h}$ and that on Σ, by $-\dfrac{\rho}{h}$ (the role of dividing by h will become apparent right away). Then, the total generated electric field is the sum of the electric fields generated by each of the two charged "layers" ∂D and Σ, its potential being given by

$$u(x) = \int_{\partial D} \left(\frac{1}{|x-y|} - \frac{1}{|x-y-hn(y)|} \right) \frac{\rho(y)}{h} dS(y). \tag{4.1}$$

But,

$$\lim_{h \to 0} \frac{1}{h} \left(\frac{1}{|x-y|} - \frac{1}{|x-y-hn(y)|} \right) = \partial_{n(y)} \left(\frac{1}{|x-y|} \right) = \partial_{n(y)} N(x, y),$$

is the directional derivative of the Newton potential at y (for some fixed x) in the direction of the outer normal $n(y)$. By (3.5) we know that the double layer potential kernel defined in Section 3 satisfies $K(x, y) = \partial_{n(y)} N(x, y)$ so, replacing in (4.1) we obtain

$$u(x) = \int_{\partial D} K(x, y)\rho(y)dS(y), \tag{4.2}$$

i.e. exactly the double layer potential as it was defined by (3.2) in Section 3.

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[4] In fact, the same kernel formula is also used to define the gravitational potential which in turn defines the Newtonian (non-relativistic) gravitational field.

[5] Such setting defines what in physics is called an electric *dipole*